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Julia set of the function $z \exp(z+\mu)$

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Introduction

Let f_μ be an entire transcendental function $z \mapsto z\exp(z+\mu)$, where μ is a complex parameter. Put $f_\mu^n = f_\mu \circ f_\mu^{n-1}$ for a positive integer n, where f_μ^0 means the identity mapping of the complex plane $\mathbb{C}$. The Julia set J_μ of f_μ is defined as the set of all points on $\mathbb{C}$, in any neighbourhood of every point of which the sequence $\{f_\mu^n\}_{n=0}^{\infty}$ does not form a normal family.

Baker [1] proved the following theorem.

Theorem There exists a real value of the parameter μ such that the Julia set J_μ of f_μ coincides with $\mathbb{C}$.

Jang [4] proved the following result by studying Baker's argument in detail : There are infinitely many positive real values of μ with the property $J_\mu = \mathbb{C}$.

In this article, we study the distribution of values of μ stated in the above result of Jang. Noting another result $J_\mu \neq \mathbb{C}$ $(-\infty < \mu < 2)$ of Jang [4], we restrict the parameter μ to the real value not less than 1.

§1 Values μ_n and $\mu^{(n)}$ of the parameter μ

Obviously the set of singular values of $f : z \mapsto z \exp(z + \mu)$ consists of two values $z = 0$ and $z = f_\mu(-1)$. The point $z = 0$ is the only one finite transcendental singularity of the inverse function f_μ^{-1} of f_μ and this is fixed by f_μ. The point $z = f_\mu(-1)$ is the only one finite algebraic singularity of f_μ^{-1}.

For a fixed value μ of the parameter, we put

$$s_0(\mu) = -1 \quad \text{and} \quad s_n(\mu) = f_\mu(s_{n-1}(\mu)), \qquad n \geqq 1.$$

The sequence $\{s_n(\mu)\}_{n=1}^{\infty}$ is the so-called orbit of the critical value $z = f_\mu(-1)$ of f_μ under the iteration of f_μ. The behaviour of this orbit plays a very important role in the study of the bifurcation of Julia sets J_μ. So, first we state some properties of $s_n(\mu)$.

Since the parameter μ is real, every $s_n(\mu)$ is negative and we have

$$(1) \qquad s_n(\mu) = s_k(\mu) \exp \psi_{k,n-k}(\mu), \qquad 0 \leqq k \leqq n-1,$$

where

$$(2) \qquad \psi_{k,\ell}(\mu) = \sum_{j=k}^{k+\ell-1} (s_j(\mu) + \mu), \qquad \ell \geqq 1.$$

For an arbitrary real constant α, we see

$$(3) \qquad \lim_{\mu \to \infty} (s_1(\mu) + \alpha\mu) = -\infty.$$

As Jang [4] showed, (3) implies

$$(4) \qquad \lim_{\mu \to \infty} s_n(\mu) = 0, \qquad n \geqq 2.$$

Evidently we see

$$(5) \qquad \mu \leqq - s_1(\mu) = \exp(-1 + \mu),$$

where the equality holds only for $\mu = 1$. In other words, the equation $s_1(\mu) + \mu = 0$ in the unknown μ has the only one root $\mu_1 = 1$. We see also that the equation $s_1(\mu) + 1 = 0$ has the only one root $\mu^{(1)} = 1$. A simple calculation shows that $s_2(\mu) + \mu = 0$ has the only one root $\mu_2 = 1$ in the interval $1 \leqq \mu < \infty$ and that $s_2(\mu) + \mu$ is positive for $\mu > \mu_2$. It is also easy to see that the equation $\psi_{0,2}(\mu) = -1 + s_1(\mu) + 2\mu = 0$ has two root $\mu = 1$ and $\mu = \mu^{(2)}$ (> 1) and $\psi_{0,2}(\mu)$ is positive in the interval $1 < \mu < \mu^{(2)}$ and is negative in the intervals $0 < \mu < 1$ and $\mu^{(2)} < \mu < \infty$. Since we see

$$\psi_{0,2}(1 + \log 3) = -4 + 2(1 + \log 3) > 0,$$

the equation $s_2(\mu) + 1 = -\exp \psi_{0,2}(\mu) + 1 = 0$ has the greatest root $\mu^{(2)}$ greater than $1 + \log 3$.

For completeness of our discussion, we recall Jang's argument in [4] under a slight improvement. Since $s_2(\mu^{(2)}) + 1 = 0$, (5) implies

$$s_3(\mu^{(2)}) + \mu^{(2)} = s_1(\mu^{(2)}) + \mu^{(2)} < 0.$$

Hence (4) gives us the existence of the greatest root $\mu = \mu_3$ $(> \mu^{(2)})$ of the equation $s_3(\mu) + \mu = 0$. Clearly $s_3(\mu) + \mu$ is positive for $\mu > \mu_3$. Since $s_3(\mu_3) = -\mu_3 < -\mu^{(2)} < -(1 + \log 3)$, the equality (4) shows the existence of the greatest root $\mu^{(3)}$ $(> \mu_3)$ of the equation $s_3(\mu) + 1 = 0$. Obviously $s_3(\mu) + 1$ is positive for $\mu > \mu^{(3)}$.

We use $\mu^{(3)}$ instead of $\mu^{(2)}$ in the above observation and see the existence of the greatest root μ_4 $(> \mu^{(3)})$ of the equation $s_4(\mu) + \mu = 0$ and the existence of the greatest root $\mu^{(4)}$ $(> \mu_4)$ of the equation $s_4(\mu) + 1 = 0$. It is easy to check that $s_4(\mu) + \mu$ is positive for $\mu > \mu_4$ and $s_4(\mu) + 1$ is also positive for $\mu > \mu^{(4)}$.

Repeating the above procedure, we have a sequence of infinitely many values μ_n and $\mu^{(n)}$ of the parameter μ such that

$$\begin{aligned} &1 = \mu_1 = \mu^{(1)} = \mu_2 < 1 + \log 3 < \\ &\mu^{(2)} < \mu_3 < \mu^{(3)} < \cdots < \mu_n < \mu^{(n)} < \mu_{n+1} < \mu^{(n+1)} < \cdots, \end{aligned} \tag{6}$$

where

$$\begin{cases} s_n(\mu_n) + \mu_n = 0, & n \geqq 1, \\ s_n(\mu) + \mu > 0 \quad \text{for } \mu > \mu_n, & n \geqq 2 \end{cases} \tag{7}$$

and

$$\begin{aligned} &s_n(\mu^{(n)}) + 1 = 0, & n \geqq 1, \\ &s_n(\mu) + 1 > 0 \quad \text{for } \mu > \mu^{(n)}, & n \geqq 2. \end{aligned} \tag{8}$$

Remark Jang [4] states only that, for $n \geqq 3$, the equation $s_n(\mu) + \mu = 0$ has a root μ_n $(> \mu^{(n-1)})$ (not necessarily the greatest) and that the equation $s_n(\mu) + 1 = 0$ has a root $\mu^{(n)}$ $(> \mu_n)$ (not necessarily the greatest).

§ 2 Distribution of the seuqence $\{\mu_n\}_{n=1}^{\infty}$

First we prove the following proposition.

Proposition 1 For values $\mu^{(n)} (n \geq 2)$ of the parameter μ, the n points $s_k(\mu^{(n)})$, $0 \leqq k \leqq n-1$, are mutually distinct and are super-attractive n-th periodic points of $f_{\mu^{(n)}}$. Therefore, the Julia set of $f_{\mu^{(n)}}$ does not coincide with $\mathbb{C}$.

Proof Suppose that there are integers k and ℓ $(0 \leqq k < \ell \leqq n-1)$ with the property $s_k(\mu^{(n)}) = s_\ell(\mu^{(n)})$. Clearly $s_k(\mu^{(n)}) = s_{k+q(\ell-k)}(\mu^{(n)})$ for any non-negative integer q. There is a positive integer p satisfying $k + p(\ell - k) \leqq n < k + (p+1)(\ell - k)$. The sequence $\{s_j(\mu^{(n)})\}_{j=k+p(\ell-k)}^{k+(p+1)(\ell-k)}$ containing $s_n(\mu^{(n)})$ coincides with the sequence $\{s_j(\mu^{(n)})\}_{j=k}^{\ell}$ and this shows the existence of such a j $(k \leqq j < \ell)$ that $s_j(\mu^{(n)}) = s_n(\mu^{(n)})$. This contradicts (8). Thus n points $s_k(\mu^{(n)})$ $(0 \leqq k \leqq n-1)$ are mutually distinct. Since $f'_{\mu^{(n)}}(-1) = 0$, it is easy to see that these n points are super-attractive n-th periodic points of $f_{\mu^{(n)}}$.

On the value μ_n $(n \geqq 3)$ of the parameter μ, we can see that the point $s_n(\mu_n)$ is a repulsive fixed point of $f = f_{\mu_n}$. To see this, we note (7) and (6) and have

$$f(s_n(\mu_n)) = f(-\mu_n) = -\mu_n$$

and

$$f'(s_n(\mu_n)) = f'(-\mu_n) = -\mu_n + 1 < -\log 3.$$

Thus $s_n(\mu_n)$ is a repulsive fixed point of f. Hence, as Jang stated in [4], Baker's argument in [1], which was used to prove the theorem stated in the introduction of this article, leads us to the following result of Jang stated also in the introduction : The Julia set of f_{μ_n} $(n \geq 3)$ coincides with $\mathbb{C}$. This is also proved in the following way. By Eremenko-Lyubich's theorem [2], the function f_{μ_n} has no wandering domain and no Baker domain. Hence Sullivan's argument [5] implies $J_{\mu_n} = \mathbb{C}$.

Now we prove the following theorem.

Theorem 2 $\quad \lim_{n\to\infty} \mu^{(n)} = \lim_{n\to\infty} \mu_n = \infty$.

Proof By (6), it suffices to show $\lim_{n\to\infty} \mu^{(n)} = \infty$. Since the sequence $\{\mu^{(n)}\}_{n=1}^{\infty}$ is increasing, we see the existence of $\mu^{(\infty)} = \lim_{n\to\infty} \mu^{(n)} \leqq \infty$. Assume $\mu^{(\infty)} < \infty$. Cearly we have $1 + \log 3 < \mu^{(\infty)}$ by (6) and $-1 < s_n(\mu^{(\infty)}) < 0$ $(n \geqq 2)$ by (8). Hence we have

$$s_{n+1}(\mu^{(\infty)})/s_n(\mu^{(\infty)}) = \exp(s_n(\mu^{(\infty)}) + \mu^{(\infty)})$$

$$> \exp(-1 + \mu^{(\infty)}) > 3$$

for every n $(\geqq 2)$, which implies

$$-1 < s_{n+1}(\mu^{(\infty)}) < 3^{n-1} s_2(\mu^{(\infty)}).$$

The right hand side of this tends to $-\infty$, as n tends to infinity. This is a contradiction. Hence $\mu^{(\infty)}$ must be infinity.

The above theorem can also be deduced from the following proposition.

Proposition 3 $\mu^{(n)} > 1 + \log(n+1)$ for $n \geqq 2$.

Proof In the case $n = 2$, we have seen $1 + \log 3 < \mu^{(2)}$ in (6). Hereafter, we consider the case $n \geqq 3$.

Put $y_1 = y_1(\mu) = -s_1(\mu)$, $y_2 = y_2(\mu) = \psi_{0,n}(\mu) - s_1(\mu)$ and $y_3 = y_3(\mu) = -(n-1) + n\mu$. We see easily that the equation $y_1 = y_3$ has two roots $\mu = 1$ and $\mu = \mu_*$ (> 1) and that $y_1 < y_3$ if and only if μ is in the open interval $1 < \mu < \mu_*$.

In the case $\mu_* \leqq \mu^{(n-1)}$, (6) implies $\mu_* < \mu^{(n)}$.

Consider the contrary case $\mu^{(n-1)} < \mu_*$. In this case, (6) and (8) give us $s_k(\mu) + 1 > 0$ in $\mu > \mu^{(n-1)}$ for $2 \leqq k \leqq n - 1$. Hence we have

$$y_2 - y_3 = \sum_{j=0}^{n-1} (s_j(\mu) + \mu) - s_1(\mu) + (n-1) - n\mu > 0$$

for $\mu > \mu^{(n-1)}$. As was seen already, we have $y_1 < y_3$ in the interval $\mu^{(n-1)} < \mu < \mu_*$. Hence we see $y_1 < y_2$ in this interval. On the other hand, (3) and (4) imply

$$\lim_{\mu\to\infty} (y_2 - y_1) = \lim_{\mu\to\infty} \psi_{0,n}(\mu) = -\infty .$$

Since $y_2(\mu_*) - y_1(\mu_*) = y_2(\mu_*) - y_3(\mu_*)$ is positive, the equation $y_1 - y_2 = 0$ has a root greater than μ_*. As $\mu^{(n)}$ is the greatest root of $s_n(\mu) + 1 = 0$ and of $\psi_{0,n}(\mu) = y_2 - y_1 = 0$, we see $\mu_* < \mu^{(n)}$.

Thus we have always $\mu_* < \mu^{(n)}$. On the other hand, we have

$$\begin{aligned} y_1(1 + \log(n+1)) &= n + 1 < 1 + n\log(n+1) \\ &= y_3(1 + \log(n+1)), \end{aligned}$$

which implies $1 + \log(n+1) < \mu_*$. Therefore, we have

$$1 + \log(n+1) < \mu^{(n)}$$

for $n \geq 3$. This is the required.

Remark By more careful observation, we can see

$$\mu^{(n)} > \begin{cases} 1 + \log(2n+1) & n \geq 4, \\ 1 + \log(3n+1), & n \geq 9, \\ 1 + \log(4n+1), & n \geq 20 \end{cases}$$

and so on. The proofs of these may be omitted here.

We have also the following proposition.

Proposition 4 $\qquad \mu^{(3)} > 3.$

Proof A direct calculation gives us

$$-74/10 < s_1(3) = -\exp 2 < -7.$$

Hence we see

$$s_2(3) = -\exp(5 + s_1(3)) > -\exp(-2) > -1/7$$

and

$$\begin{aligned} s_3(3) &= -\exp(8 + s_1(3) + s_2(3)) \\ &< -\exp(8 - 74/10 - 1/7) < -1. \end{aligned}$$

Since the value $\mu^{(3)}$ is the greatest root of $s_3(\mu) + 1 = 0$, we have $\mu^{(3)} > 3$ by (4).

Remark According to Sagawa, $\mu^{(3)}$ lies between 31/10 and 32/10.

§ 3 Repulsive periodic points of f_μ for some values of μ

In the preceding section, we were concerned with the values μ_n of the parameter μ, each of which is the greatest root of the equation $\psi_{n,1}(\mu) = s_n(\mu) + \mu = 0$. In this section, we are concerned with the greatest root of the equation $\psi_{n,k}(\mu) = 0$ for $n \geqq 3$ and $k \geqq 2$. We see easily by (1) that, for this greatest root μ of $\psi_{n,k}(\mu) = 0$, $s_{n+k}(\mu)$ is equal to $s_n(\mu)$ so that $s_n(\mu)$ is a periodic point of f_μ.

Under the conditions $n \geqq 3$ and $k \geqq 2$, we see $\mu^{(n+k-2)} \geqq \mu^{(3)}$ by (6). If μ is not less than $\mu^{(n+k-2)}$, we see $s_{n+k-2}(\mu) + 1 \geqq 0$ and $-1 < s_j(\mu) < 0$ for $2 \leqq j \leqq n + k - 3$. Those are conclusions from (8). Hence we have

$$\begin{aligned} s_{n+k-3}(\mu) &= s_{n+k-2}(\mu) \exp(-s_{n+k-3}(\mu) - \mu) \\ &> s_{n+k-2}(\mu) \exp(1 - \mu) > -\exp(1-\mu) \end{aligned}$$

for $\mu \geqq \mu^{(n+k-2)}$. Similarly, for $2 \leqq j \leqq n + k - 4$, we have

$$\begin{aligned} s_j(\mu) &> s_{j+1}(\mu) \exp(1-\mu) \\ &> s_{n+k-3}(\mu) \exp((n+k-3-j)(1-\mu)) \\ &> -\exp((n+k-2-j)(1-\mu)) \end{aligned}$$

for $\mu \geqq \mu^{(n+k-2)}$. Therefore, for $2 \leqq p \leqq n + k - 3$ and for $\mu \geqq \mu^{(n+k-2)}$, we have

$$\begin{aligned} \sum_{j=p}^{n+k-3} s_j(\mu) &> -\sum_{j=p}^{n+k-3} \exp((n+k-2-j)(1-\mu)) \\ &> -1/(\exp(\mu - 1) - 1). \end{aligned}$$

Proposition 4 and (6) imply

$$\sum_{j=p}^{n+k-3} s_j(\mu) > -1/((\exp 2) - 1) > -1/6$$

for $2 \leqq p \leqq n + k - 3$ and $\mu \geqq \mu^{(n+k-2)}$. Hence we see

$$\psi_{0,n+k-2}(\mu^{(n+k-2)}) - \sum_{j=0}^{1} (s_j(\mu^{(n+k-2)}) + \mu^{(n+k-2)}) - (k - 2)\mu^{(n+k-2)}$$
$$= \sum_{j=2}^{n+k-3} s_j(\mu^{(n+k-2)}) + (n - 2)\mu^{(n+k-2)} > 0.$$

Here we recall $\mu^{(n+k-2)}$ is a root of $s_{n+k-2}(\mu) + 1 = 0$, that is, a root of $\psi_{0,n+k-2}(\mu) = 0$. Hence the above inequality shows

$$\sum_{j=0}^{1} s_j(\mu^{(n+k-2)}) + k\mu^{(n+k-2)} < 0. \tag{9}$$

Now we can prove the following proposition.

Proposition 5 For $n \geqq 3$ and $k \geqq 2$, the equation $\psi_{n,k}(\mu) = 0$ has the greatest root $\mu = \mu_{n,k}$, and $\psi_{n,k}(\mu)$ is positive for $\mu > \mu_{n,k}$. In addition, the inequalities $\mu^{(n+k-2)} < \mu_{n,k} < \mu^{(n+k-1)}$ hold.

Proof The inequality (9) shows

$$\psi_{n,k}(\mu^{(n+k-2)}) = \sum_{j=n}^{n+k-1} (s_j(\mu^{(n+k-2)}) + \mu^{(n+k-2)})$$
$$< s_{n+k-2}(\mu^{(n+k-2)}) + s_{n+k-1}(\mu^{(n+k-2)}) + k\mu^{(n+k-2)}$$
$$= s_0(\mu^{(n+k-2)}) + s_1(\mu^{(n+k-2)}) + k\mu^{(n+k-2)} < 0$$

by virtue of $s_j(\mu) < 0$ and of $s_{n+k-2}(\mu^{(n+k-2)}) = -1 = s_0(\mu^{(n+k-2)})$. On the other hand, for $\mu \geqq \mu^{(n+k-1)}$, we see

$$\psi_{n,k}(\mu) = \sum_{j=n}^{n+k-1} (s_j(\mu) + \mu) > -k + k\mu > 0 \tag{10}$$

by (8) and (6). Hence there is the greatest root $\mu_{n,k}$ of the equation $\psi_{n,k}(\mu) = 0$ such that $\mu^{(n+k-2)} < \mu_{n,k} < \mu^{(n+k-1)}$. Thus we have our proposition.

Using this proposition, we prove the following proposition.

Proposition 6 For $n \geq 3$ and $k \geq 2$, the points $s_j(\mu_{n,k})$ $(n \leq j \leq n+k-1)$ are mutually distinct k-th periodic points of $f_{\mu_{n,k}}$.

Proof For simplicity, put $\mu = \mu_{n,k}$ and $f = f_\mu$. As was stated at the beginning of this section, $s_n(\mu)$ is equal to $s_{n+k}(\mu)$. So, it suffices to prove $s_{n+j}(\mu) \neq s_{n+l}(\mu)$ for $0 \leq j < l \leq k-1$.

Assume $s_{n+j}(\mu) = s_{n+l}(\mu)$ for $0 \leq j < l \leq k-1$. Then we see

$$s_{n+j}(\mu) = s_{n+l}(\mu) = f^{l-j}(s_{n+j}(\mu)) = s_{n+j}(\mu) \exp \psi_{n+j,l-j}(\mu),$$

which shows $\psi_{n+j,l-j}(\mu) = 0$. Proposition 5 shows that the greatest root of the equation $\psi_{n+j,l-j}(\mu) = 0$ lies beween $\mu^{(n+l-2)}$ and $\mu^{(n+l-1)}$. So we have $\mu < \mu^{(n+l-1)} \leq \mu^{(n+k-2)}$. Since $\mu = \mu_{n,k}$ is greater than $\mu^{(n+k-2)}$ by Proposition 5, we have a contradiction. Therefore, we see $s_{n+j}(\mu) \neq s_{n+l}(\mu)$ for $0 \leq j < l \leq k-1$ and we have our proposition.

Proposition 7 For $n \geq 3$ and $k \geq 2$, the values $\mu_{n,k}$ in Proposition 5 satisfy the following :

$$\mu^{(n+k-2)} < \mu_{3,n+k-3} < \mu_{4,n+k-4} < \cdots < \mu_{n+k-2,2} < \mu_{n+k-1} < \mu^{(n+k-1)}.$$

Proof First, as was stated in Proposition 5, we have

$$\psi_{n,k}(\mu_{n,k}) = \sum_{j=n}^{n+k-1} (s_j(\mu_{n,k}) + \mu_{n,k}) = 0.$$

Hence we see

$$\psi_{n+1,k-1}(\mu_{n,k}) = \sum_{j=n+1}^{n+k-1} (s_j(\mu_{n,k}) + \mu_{n,k})$$

$$= \psi_{n,k}(\mu_{n,k}) - s_n(\mu_{n,k}) - \mu_{n,k}$$

$$= - s_n(\mu_{n,k}) - \mu_{n,k} .$$

By Proposition 5 and (6), we see $\mu^{(n)} \leq \mu^{(n+k-2)} < \mu_{n,k}$, which shows $s_n(\mu_{n,k}) + 1 > 0$. Hence (6) leads us to

$$\psi_{n+1,k-1}(\mu_{n,k}) = - s_n(\mu_{n,k}) - \mu_{n,k} < 1 - \mu_{n,k} < 0.$$

Therefore, we see by Proposition 5 that the greatest root $\mu_{n+1,k-1}$ of the equation $\psi_{n+1,k-1}(\mu) = 0$ is greater than $\mu_{n,k}$. From this observation, we have

$$\mu^{(n+k-2)} < \mu_{3,n+k-3} < \mu_{4,n+k-4} < \cdots < \mu_{n+k-2,2} < \mu^{(n+k-1)}.$$

Furthermore, since μ_{n+k-1} is the greatest root of the equation $\psi_{n+k-1,1}(\mu) = s_{n+k-1}(\mu) + \mu = 0$, we may put $\mu_{n+k-1} = \mu_{n+k-1,1}$ in the notation used in Proposition 5. So,similarly to the above, we see easily $\mu_{n+k-2,2} < \mu_{n+k-1} < \mu^{(n+k-1)}$. Thus we have our proposition.

Now we prove the following theorem.

Theorem 8 Assume $n \geq 3$ and $k \geq 2$. Then, for the values $\mu_{n,k}$ of the parameter μ obtained in Proposition 5, the Julia set of $f_{\mu_{n,k}}$ coincides with $\mathbb{C}$.

Proof Proposition 6 shows that k-th periodic points $s_j(\mu_{n,k})$ $(n \leqq j \leqq n+k-1)$ of $f = f_{\mu_{n,k}}$ are mutually distinct. Suppose that there is a j $(n \leqq j \leqq n+k-1)$ with the property $s_j(\mu_{n,k}) = -1$. This means that the point -1 is a k-th periodic point of f and we have $s_k(\mu_{n,k}) = f^k(-1) = -1$. This and (8) imply $\mu_{n,k} \leqq \mu^{(k)}$. Proposition 5 leads us to a contradiction. Hence every point $s_j(\mu_{n,k})$ $(n \leqq j \leqq n+k-1)$ is different from -1. The equation $z \exp(z+\mu) = s_1(\mu) = -\exp(-1+\mu)$ has the only one real root $z = -1$ and hence the sequence $\{s_j(\mu_{n,k})\}_{j=n}^{n+k-1}$ does not contain $s_1(\mu_{n,k})$. Therefore, the critical point $s_1(\mu_{n,k})$ of f is a preperiodic point of f. In the same way as was stated after Proposition 1, Eremenko-Lyubich's theorem [2] and Sullivan's argument [4] give us the desired.

Remark In Fagella [3], we can find discussions about the same problem as ours.

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