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An introduction to the piecewise algebraic curve

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Abstract

In this paper, we review the recent development of our research on piecewise algebraic curves.

Keywords piecewise algebraic curve, spline function, conformality condition.

1 Introduction

Let us recall the formulation of splines at first. Let D be a bounded polygonal domain of R^2 and we partition D with irreducible algebraic curves into cells $\Delta_i, i = 1, \ldots, N$. The partition is denoted by Δ. A function $f(x)$ defined on D is a spline function if $f(x) \in C^r(D)$ and $f(x)|_{\Delta_i} = p_i \in P_k$, which is expressed for short as follows:

$$f(x) \in S_n^r(D, \Delta).$$

In [1] R. H. Wang got the following basic results:

Let Δ_i and Δ_j be two adjacent cells with partitioning curve $l_{ij} = 0$. $f(x) \in c^r(\Delta_i \bigcup \Delta_j)$ if and only if

$$p_i - p_j = l_{ij}^{r+1} * q_{ij}.$$

where $q_{ij} \in P_{k-(\mu+1)d_{ij}}$ is called a smooth cofactor of the partitioning curve l_{ij} and d_{ij} is the degree of l_{ij}.

Further, $f(x) \in S_n^r(D, \Delta)$, if and only if there exists a smooth cofactor on each interior partitioning curve and

$$\sum_{l_{ij} \in L_k} l_{ij}^{r+1} * q_{ij} \equiv 0.$$

where L_k is the set of partitioning curves sharing the same interior vertex.

Algebraic curve Γ is defined as follows

$$(*) \qquad \Gamma = \{(x, y) | p(x, y) = 0, p \in P\}.$$

The so-called piecewise algebraic curve is defined by using the piecewise polynomial or polynomial spline function $s(x, y)$ to replace the polynomial $p(x, y)$ in $(*)$, we have

$$\Gamma = \{(x, y) \ s(x, y) = 0\}.$$

Let $\Gamma : s(x,y) = 0$ and $\gamma : t(x,y) = 0$ be two piecewise algebraic curves. γ is called a local branch of Γ, if there exists a union of cells in Δ

$$\Omega = \bigcup \delta_i$$

such that γ is a branch of Γ on Ω .

Why do we have to study piecewise algebraic curves? Let us consider the following interpolation problem: Let $d = dim S_k^\mu(\Delta)$. How can we choose a set of knots $K = \{(x_i, y_i)\}_{i=1}^d$ such that for any given values $z_1, \ldots, z_d$, there exists a unique $s \in S_k^\mu(\Delta)$ satisfying

$$s(x_i, y_i) = z_i, i = 1, \ldots, d$$

According to the theory on bivariate spline mentioned above, the interpolation problem is a linear algebraic problem. Therefore there is a unique solution if and only if the linear homogeneous equations

$$s(x_i, y_i) = 0, i = 1, \ldots, d$$

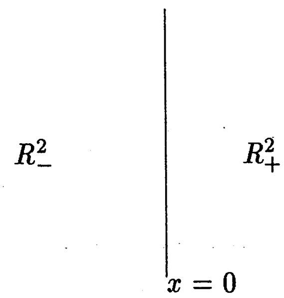
has only a trivial solution, that is, if and only if K does not lie on any piecewise algebraic curve $\Gamma : s(x,y) = 0, s \in S_k^\mu(\Delta)$. Denote by $p_i(x,y) \in P_k$ the polynomial defined by $s(x,y) \in S_k^\mu(\Delta)$ on Δ_i. Because there is the possibility that

$$\{(x,y) | p_i(x,y) = s|_{\Delta_i} = 0\} \bigcap \overline{\Delta_i} = \emptyset$$

it is difficult to derive the piecewise algebraic curve.

2 Some Examples

Example 1 $\quad D = R^2, \Delta : x = 0$, 2 cells



$$R^2_- = \{(x,y) \in R^2 : x < 0\}$$
$$R^2_+ = \{(x,y) \in R^2 : x \geq 0\}$$

Define $s \in S_1^0(\Delta)$ as follows

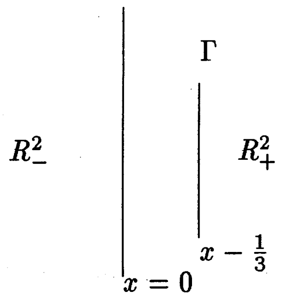
$$s(x,y) = \begin{cases} x - 1 & (x,y) \in R^2_-, \\ -x - 1 & (x,y) \in R^2_+. \end{cases}$$

The piecewise algebraic curve $\Gamma : s(x,y) = 0$ is empty.

Example 2 $s \in S_1^0(\Delta)$ is defined by

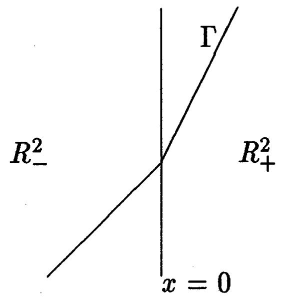
$$s(x,y) = \begin{cases} x - 1 & (x,y) \in R_-^2, \\ 3x - 1 & (x,y) \in R_+^2. \end{cases}$$

The piecewise algebraic curve $\Gamma : s(x,y) = 0$ is $s - \frac{1}{3} = 0$.



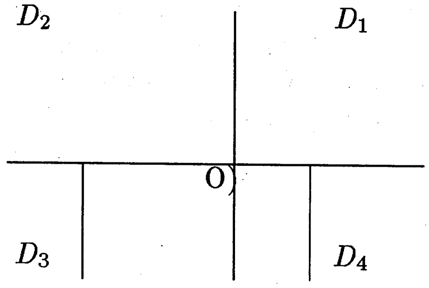
Example 3 $s \in S_1^0(\Delta)$ is defined as follows

$$s(x,y) = \begin{cases} x - y & (x,y) \in R_-^2, \\ 2x - y & (x,y) \in R_+^2 \end{cases}$$



Example 4 $D = R^2, \Delta : x = 0, y = 0$, $s \in S_2^1(\Delta)$ is defined as follows

$$s(x,y) = \begin{cases} 3x^2 + 3y^2 - 1 & (x,y) \in D_1, \\ x^2 + 3y^2 - 1 & (x,y) \in D_2, \\ x^2 - 1 & (x,y) \in D_3, \\ 3x^2 - 1 & (x,y) \in D_4. \end{cases}$$

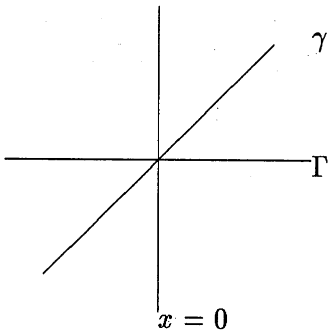


Example 5 $\quad D = R^2, \Delta : x = 0$

$$\Gamma : s(x,y) = xy - y^2 - yx_+ = 0, s \in S_2^0(\Delta)$$

$$\gamma : t(x,y) = x - y = 0, t \in S_1(\Delta)$$

γ is a local branch of Γ on R^2.



3 Intersection of piecewise algebraic curves

Denote by $Inter(\Gamma_1, \Gamma_2)$ the intersection set of the two piecewise algebraic curves Γ_1 : $s_1(x,y) = 0$ and $\Gamma_2 : s_2(x,y) = 0$. The number

$$BN(m_1, r_1; m_2, r_2)$$

$$:= max\{Card\, Inter(\Gamma_1, \Gamma_2) < \infty; \Gamma_i : s_i(x,y) = 0, s_i \in S_{m_i}^{r_i}(\Delta), i = 1,2\}$$

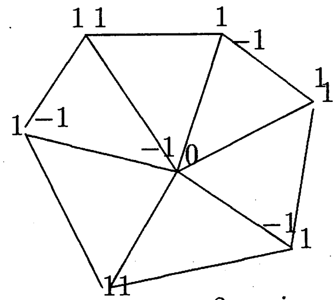
is called the Bezout number of $S_{m_1}^{r_1}$ and $S_{m_2}^{r_2}$. It is obvious that

$$BN(m_1, r_1; m_2, r_2) \leq N m_1 m_2,$$

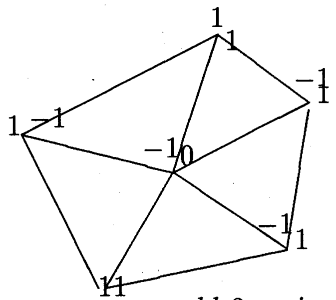
where N is the number of cells in Δ.

X.Q.Shi and R.H.Wang[3] discussed the Bezout number of $S^0_m(\Delta)$ and $S^0_n(\Delta)$.We find that the Bezout number $BN(m,0;n,0)$ depends on some property of the triangulation Δ.

A triangulation Δ is called to be 2-signs, if one can mark -1 or 1 on each vertex of Δ such that the numbers marked on 3 vertices of any cell in Δ are not the same one. A triangulation Δ is called to be 3-signs, if one can mark -1,0 or 1 on each vertex of Δ such that the numbers marked on 3 vertices of any cell in Δ are totally different.

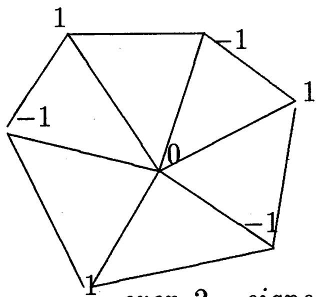


even, 2 − signs
3 − signs

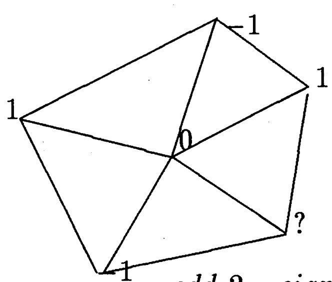


odd, 2 − signs
Itisnot3 − signs

Let v be an interior vertex of Δ. Denote by $d(v)$ the number of boundary vertices of the star $st(v)$. $d(v)$ is called the degree of v. An interior vertex is called to be even(odd) if $d(v)$ is even(odd). A triangulation Δ is called to be even, if all of its interior vertices are even.



even, 2 − signs
3 − signs



odd, 2 − signs
not3 − signs

Proposition The even triangulation of a simple connected domain is of 3-signs.

X.Q.Shi and R.H.Wang [3] proved

Theorem 1 If Δ is a triangulation of a simple connected domain, then

$$\begin{array}{lll} 1^o & BN(1,0;1,0) = t, & \text{if}\Delta\text{is even;} \\ 2^o & BN(1,0;1,0) \le T - [(V_{odd}+2)/3], & \text{otherwise,} \end{array}$$

where T is the number of cells in Δ, V_{odd} is the number of odd vertices of Δ, and $[x]$ denotes the maximum integer less than or equal to x.

Denote by $\delta = [v_1, v_2, v_3]$ the triangle with vertices v_1, v_2 and v_3. Let $f, g \in S^0_1(\Delta)$, and

$$f_i = f(v_i), g_i = g(v_i), i = 1,2,3.$$

Then the piecewise algebraic curves $f = 0$ and $g = 0$ can be represented on δ as follows

$$f_1u_1 + f_2u_2 + f_3u_3 = 0, \tag{1}$$

and

$$g_1u_1 + g_2u_2 + g_3u_3 = 0 \tag{2}$$

respectively, where (u_1, u_2, u_3) are the barycentric coordinates of a point v with respect to the triangle δ. Suppose that (u_1^*, u_2^*, u_3^*) are the barycentric coordinates of the intersection point of $f_1u_1 + f_2u_2 + f_3u_3 = 0$ and $g_1u_1 + g_2u_2 + g_3u_3 = 0$, then

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = u_1^* \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} + u_2^* \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} + u_3^* \begin{pmatrix} f_3 \\ g_3 \end{pmatrix} \tag{3}$$

$$u_i^* \geq 0, u_1^* + u_2^* + u_3^* = 1$$

Lemma 1 Suppose (1) and (2) have only one intersection point p.Then the point p is an interior point of the triangle $\delta = [v_1, v_2, v_3]$, if and only if the origin $(0,0)$ is an interior point of triangle $\delta^* = [\omega_1, \omega_2, \omega_3]$, where $\omega_i, i = 1,2,3$ are defined by

$$\omega_i = (f_i, g_i),$$

Note: $u_i^* > 0, i = 1,2,3$.

Lemma 2 Let v be an interior vertex of the triangulation Δ, and $f = 0, g = 0$ have only finite intersection points on $st(v)$, where $f, g \in S_1^0(\Delta)$. Then $f = 0$ and $g = 0$ have at most N intersection points:

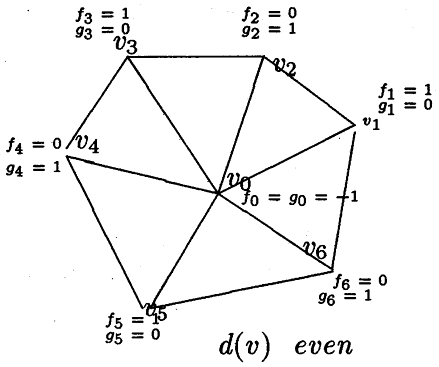
$$N = \begin{cases} d(v), & \text{if } d(v) \text{ is even,} \\ d(v) - 1, & \text{if } d(v) \text{ is odd.} \end{cases}$$

Proof. Assume $d(v) = 2m$, and $v_0, v_1, ..., v_{2m}$ are the vertices of $st(v)$, where $v_0 = v$. Let ω_0, ω_1 and ω_2 be some points on R^2 such that the origin is an interior point of the triangle $[\omega_0, \omega_1, \omega_2]$, for example,

$$\omega_0 = (-1,-1), \omega_1 = (1,0), \omega_2 = (0,1).$$

Now we define two piecewie linear curves $f = 0, g = 0$ on $st(v)$ by using the following values

$$(f(v_0), g(v_0)) = \omega_0 = (-1,-1),$$
$$(f(v_{2i-1}), g(v_{2i-1})) = \omega_1 = (1,0), (i = 1, ..., m)$$
$$(f(v_{2i}), g(v_{2i})) = \omega_2 = (0,1).$$



$d(v)$ *even*

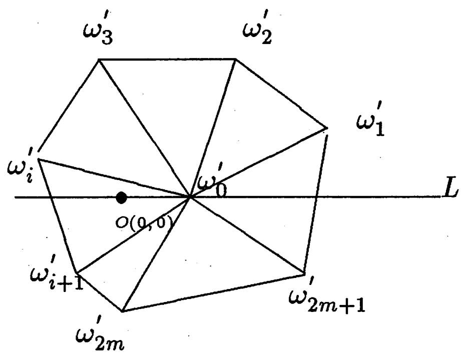
Because the origin is inside the triangle $[\omega_0, \omega_1, \omega_2]$, By Lemma 1,$N = d(v)$.

Now assume $d(v) = 2m + 1$. For two piecewise linear curves $f = 0$ and $g = 0$, suppose

$$\omega_0' = (f(v_0), f(v_0)), \omega_i' = (f(v_i), g(v_i)),$$

$$i = 1, ..., 2m + 1, v_0 = v$$

Lemma 1 shows that $f = 0$ and $g = 0$ have an intersection point inside the triangle $[v_0, v_i, v_{i+1}]$ if and only if the origin is inside the triangle $[\omega_0', \omega_i', \omega_{i+1}']$ $(i = 1, ..., 2m+1, \omega_1' = \omega_{2m+2}')$.



$d(v)$ *odd*

By joining the origin and ω_0', we obtain a straight line L. According to Lemma 1, two piecewise algebraic curves have a unique intersection point inside the triangle $[\omega_0', \omega_i', \omega_{i+1}']$, if and only if the vertices ω_i' and ω_{i+1}' are located at two different sides of the straight line L. So it is obvious that

$$N \leq d(v) - 1,$$

where $d(v)$ is odd. Moreover, if we take

$$\omega_0' = (-1, -1), \omega_{2i}' = (0, 1), \omega_{2i+1}' = (1, 0),$$

$i = 1, ..., m$, then $f = 0$ and $g = 0$ have $2m = d(v) - 1$ intersection points.

The proof of Theorem 1:

Let $f,g \in S_1^0(\Delta)$ be defined by

$$(f(v),g(v)) = \omega_i, v \in \Delta,$$

where v is marked by $i, i = -1,0,1,$

$$\omega_{-1} = (-1,-1), \omega_0 = (1,0) \text{ and } \omega_1 = (0,1).$$

According to Lemma 1, the piecewise linear curves $f = 0$ and $g = 0$ have just an intersection point in each triangle of Δ, i.e. if Δ is even, then

$$BN(1,0;1,0) = T.$$

Similarly, we can prove 2° in Theorem 1.

Note: One can find some triangulations satisfying

$$BN(1,0;1,0) = T - [(V_{odd}+2)/3].$$

Lemma 3 If the triangulation δ is of 2-signs, then

$$BN(1,0;2,0) = 2T.$$

where T is the number of triangles in Δ.

Let $f \in S_1^0(\Delta)$ be defined as follow

$$f(v) = \begin{cases} 1 & \text{if } v \in \Delta \text{ is marked by 1} \\ -1 & \text{if } v \in \Delta \text{ is marked by -1} \end{cases} \tag{4}$$

Assuming that $\delta = [v_1, v_2, v_3] \in \Delta$ is a triangle, and $f(u_1,u_2,u_3) = f|_\delta = u_1 + u_2 - u_3$, where$(u_1,u_2,u_3)$ are the barycentric coordinates of $(x,y) \in \delta$ with respect to δ.

Define $g(x,y) \in S_2^0(\Delta)$ by using the following way

$$\begin{aligned} g(x,y)|_\delta &= g(u_1,u_2,u_3) \\ &= u_1^2 + u_2^2 + u_3^3 - \tfrac{3}{2}(u_1u_2 + u_2u_3 + u_3u_1) \end{aligned} \tag{5}$$

for any $\delta \in \Delta$.

It is no difficult to check that the piecewise algebraic curves $f(u_1,u_2,u_3) = 0$ and $g(u_1,u_2,u_3) = 0$ have two intersection points in δ. So

$$BN(1,0;2,0) = 2T.$$

Lemma 4 If the triangulation Δ is of 2-signs, then

$$BN(1,0;3,0) = 3T,$$

where T is the number of triangles in Δ.

Proof. Let $f \in S_1^0(\Delta)$ be defined as in Lemma 3, and let

$$\begin{aligned} g(u_1,u_2,u_3) &= u_1^3 + u_2^3 + u_3^3 + au_1^2u_2 + u_2^2u_3 + u_3^2u_1 \\ &+ bu_1u_2^2 + u_2u_3^2 + u_3u_1^2 + u_1u_2u_3. \end{aligned} \tag{6}$$

To find the conditions such that $f(u_1, u_2, u_3) = 0$ and $g(u_1, u_2, u_3) = 0$ have 3 intersection points in the triangle δ, take $u_1 + u_2 = u_3 = \frac{1}{2}$ and consider

$$g(u_1, u_2, \frac{1}{2}) = 0, u_1 + u_2 = \frac{1}{2}.$$

If there are 3 real constants α_1, α_2 and α_3 such that

$$\begin{aligned} g(u_1, u_2, \tfrac{1}{2}) &= g(u_1, u_2, u_1 + u_2) \\ &= u_1^3 + u_2^3 + 2(u_1 + u_2)^3 + a u_1^2 u_2 \\ &+ (u_1^2 + u_2^2)(u_1 + u_2) + b u_1 u_2^2 + u_1 u_2 (u_1 + u_2) \\ &= 4u_1^3 + 4u_2^3 + (8 + a) u_1^2 u_2 + (8 + b) u_1 u_2^2 \\ &= 4(u_1 + \alpha_1 u_2)(u_1 + \alpha_2 u_2)(u_1 + \alpha_3 u_2). \end{aligned}$$

then

$$\begin{cases} a & = 4(\alpha_1 + \alpha_2 + \alpha_3) - 8, \\ b & = 4(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1) - 8, \\ \alpha_1\alpha_2\alpha_3 & = 1. \end{cases} \tag{7}$$

Choose $\alpha_1, \alpha_2, \alpha_3 > 0$ satisfying (7). one can obtain a special $g(u_1, u_2, u_3)$ by (7) such that $g(u_1, u_2, u_3) = 0$ and $f(u_1, u_2, u_3) = 0$ have 3 intersection points in the interval $u_1 \in (0, \frac{1}{2}), u_1 + u_2 = \frac{1}{2}$.

This shows that $f(x, y) = 0$ and $\overline{g}(x, y) = 0$ have $3T$ intersection points, where $\overline{g}(x, y)$ is defined by

$$\overline{g}(x, y)|_\delta = g(u_1, u_2, u_3), \forall \delta \in \Delta. \tag{8}$$

Theorem 2 If Δ is a 2-signs triangulation, and $max\{m, n\} \geq 2$, then the Bezout number of the spaces $S_m^0(\Delta)$ and $S_n^0(\Delta)$ is mnT, i.e.

$$BN(m, 0; n, 0) = nmT,$$

where T is the number of triangles in Δ.

Proof. Let $f(x.y)$ and $g(x, y)$ be defined by (4) and (5), $\overline{g}(x, y)$ be defined as (8), and $m \geq n$.

If $m \geq 2$ is even, we define

$$f_1 = f^n, g_1 = g^{m/2},$$

then

$$f_1(x, y) \in S_n^0(\Delta), g_1(x, y) \in S_m^0(\Delta),$$

moreover, the piecewise algebraic curves $f_1(x, y) = 0, g_1(x, y) = 0$ have mnT intersection points in Δ . This means that

$$BN(m, 0; n, 0) = nmT.$$

If $m \geq 3$ is odd, we define

$$f_1 = f^n, g_1 = g^{\frac{m-3}{2}} \overline{g},$$

then theorem 2 can be also proved.

It seems that the Bezout number depends on whether the triangulation is of 2-signs or not. Based on many examples, however, the following conjecture may be right.

Conjecture Any triangulation is of 2-signs.

For the $c^1 - smoothness$ cases

Let $s_i \in S^1_{m_i}(\Delta), i = 1, 2$ be two bivariate splines. We are going to consider the problem on intersection of two piecewise curves $s_1(x, y) = 0$ and $s_2(x, y) = 0$.

A partition Δ of D is called a proper partition, if all angles of the intersection determined by any two adjacent edges of Δ are less than $\pi/2$.

By using the resultant of two bivariate splines s_1 and s_2 with respect to ρ in the polar coordinate(ρ, θ), R.H.Wang and G.H.Zhao[4] proved

Theorem 3 Let $\Gamma_i : s_i(x, y) = 0, i = 1, 2$ be two piecewise algebraic curves, where $s_i \in S^1_{m_i}(\Delta), i = 1, 2$. For any given interior vertex of Δ, the cardinality Λ of the intersection set δ of Γ_1 and Γ_2 on $st(v)$ is upper bounded by

$$n_i(m_1 m_2 - 1) + 1$$

except that the cardinality of δ is infinite, where n_i is the number of edges passing through v.

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