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Augmenting Edge-Connectivity and Vertex-Connectivity Simultaneously

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Abstract Given an undirected multigraph $G=(V,E)$ and requirement functions $\{r_\lambda(x,y)\in Z^+|$ $x,y\in V\}$ and $\{r_\kappa(x,y)\in Z^+|x,y\in V\}$ (where Z^+ is the set of nonnegative integers), the edge and vertex-connectivities augmentation problem asks to augment G by adding the smallest number of new edges to G so that for every $x,y\in V$, the edge-connectivity and vertex-connectivity between x and y are at least $r_\lambda(x,y)$ and $r_\kappa(x,y)$, respectively in the resulting graph G'. In this paper, we show that if $r_\kappa(x,y)=2$ holds for every $x,y\in V$, then the problem can be solved in polynomial time.

1 Introduction

Let $G=(V,E)$ stand for an undirected multigraph with a set V of *vertices* and a set E of *edges*, where an edge with end vertices u and v is denoted by (u,v). A singleton set $\{x\}$ may be simply denoted by x. For two disjoint subsets of vertices X, $Y\subset V$, we denote by $E_G(X,Y)$ the set of edges, one of whose end vertices is in X and the other is in Y, and also denote $c_G(X,Y)=|E_G(X,Y)|$. In particular, $E_G(u,v)$ implies the set of edges with end vertices u and v. We denote $n=|V|$ and $e=|E|$. For a subset $V'\subseteq V$ in G, $G-V'$ denotes the subgraph induced by $V-V'$. A *cut* is defined as a subset X of V with $\emptyset\neq X\neq V$, and the *size* of a cut X is denoted by $c_G(X,V-X)$, which may also be written as $c_G(X)$. A cut with the minimum size is called a (*global*) *minimum cut*, and its size, denoted by $\lambda(G)$, is called the *edge-connectivity* of G. The local edge-connectivity $\lambda_G(x,y)$ for two vertices $x,y\in V$ is defined to be the minimum size of a cut in G that separates x and y (i.e., x and y belong to different sides of X and $V-X$), or equivalently the maximum number of edge-disjoint path between x and y by Menger's theorem [4].

For a subset X of V, $\{v\in V-X\mid(u,v)\in E$ for some $u\in X\}$ is called the *neighbor set* of X, denoted by $\Gamma_G(X)$. Let $p(G)$ denote the number of components in G. A *separator* of G is defined as a cut S of V such that $p(G-S)>p(G)$ holds and no $S'\subset S$ has this property. A separator always exists, unless G contains the complete graph K_n. If G does not contain K_n, then a separator of the minimum size is called a (*global*) *minimum separator*, and its size, denoted by $\kappa(G)$, is called the *vertex-connectivity* of G. If G contains the complete graph K_n, we define $\kappa(G)=n-1$. The local vertex-connectivity $\kappa_G(x,y)$ for two vertices $x,y\in V$ is defined to be the number of internally-disjoint paths between x and y in G. For any separator S, there is the component X of G such that $X\supseteq S$, and we call the components in $G[X]-S$ *the S-components*. Let

$$\beta(G)=\max\{p(G-S)|S \text{ is a minimum separator in } G\}. \quad (1.1)$$

A cut $T\subset V$ is called *tight* if $\Gamma_G(T)$ is a minimum separator in G and no $T'\subset T$ has this property (hence, $G[T]$ induces a connected graph). Let $t(G)$ denotes the maximum number of pairwise disjoint tight sets in G.

In this paper, for a given function $a:\binom{V}{2}\to R^+$ (resp., $b:\binom{V}{2}\to R^+$), where R^+ denotes the set of nonnegative real numbers, we call G *a-edge-connected* (resp., *b-vertex-connected*) if $\lambda_G(x,y)\geq a(x,y)$ (resp., $\kappa_G(x,y)\geq b(x,y)$) holds for every $x,y\in V$. Given a multigraph $G=(V,E)$ and a requirement function $r_\lambda:\binom{V}{2}\to Z^+$, (resp., a requirement function $r_\kappa:\binom{V}{2}\to Z^+$), where Z^+ denotes the set of nonnegative integers, the *edge-connectivity augmentation problem*, (resp., the *vertex-connectivity augmentation problem*) asks to augment G by adding the smallest number of new edges so that the resulting graph G' becomes r_λ-edge-connected (resp., r_κ-vertex-connected). When the requirement function r_λ (resp., r_κ) satisfies $r_\lambda(x,y)=k\in Z^+$ for all $x,y\in V$ (resp., $r_\kappa(x,y)=\ell\in Z^+$ for all $x,y\in V$), this problem is called the global k-edge-connectivity problem (resp., the global ℓ-vertex-connectivity problem).

Watanabe and Nakamura [16] first proved that the global k-edge-connectivity augmentation problem can be solved in polynomial time for any given integer k. Their algorithm increases edge-connectivity one by one, each time augmenting edges on the basis of structural information of the current G. Currently, $O(e+k^2n\log n)$ time algorithm due to Gabow [6] and $\tilde{O}(n^3)$ time randomized algorithm due to Benczúr [1], whose deterministic running time is $O(n^4)$, are the fastest among existing algorithms. Different from the approach by Watanabe and Nakamura, Cai and Sun [2] first pointed out that the augmentation problem for a given k can be directly solved by applying the Mader's edge-splitting theorem. Based on this, Frank [5] gave an $O(n^5)$ time augmentation algorithm. Afterwards, Gabow [7] and Nagamochi and Ibaraki [14] improved it to $O(mn^2 \log(n^2/m))$ and $O(n^2(m + n \log n))$, respectively. Recently, Nagamochi and Ibaraki [15] gave an $O(n(m + n \log n) \log n)$ time algorithm. For a general requirement function r_λ, Frank [5] showed that the edge-connectivity augmentation problem can be solved in polynomial time by using Mader's edge-splitting theorem, and recently the time complexity was improved by Gabow [7] to $O(n^3 m \log(n^2/m))$.

As to the vertex-connectivity augmentation problem, the problem of adding the minimum number of new edges to a k-vertex-connected graph to make it $(k+1)$-vertex-connected has been studied by several researchers. It is easy to see that $M(G) = \max\{\beta(G) - 1, \lceil t(G)/2 \rceil\}$ provides a lower bound on the optimal value to this problem. Eswaran and Tarjan [3] proved that the vertex-connectivity augmentation problem can be solved by adding $M(G)$ edges to G for $k = 1$. Watanabe and Nakamura [17] stated the same result for $k = 2$. However, $M(G)$ can be smaller than the optimal value for general $k \geq 3$. Recently Jordán presented an $O(n^5)$ time approximation algorithm for this problem [11, 12]. The difference between the number of new edges added by his algorithm and the optimal value is at most $(k-2)/2$.

It is known that if the requirement function r_κ satisfies $r_\kappa(x, y) = k$ for all $x, y \in V$, where $k \in \{2, 3, 4\}$, then the global k-vertex-connectivity augmentation problem can be solved in polynomial time due to [3, 9], [17, 8], [10], where an input graph G may not be $(k-1)$-vertex-connected. However, whether there is an polynomial time algorithm for the global vertex-connectivity augmentation problem for an arbitrary k is an open question (even if G is $(k-1)$-vertex-connected).

In this paper, we consider the problem of augmenting the edge-connectivity and the vertex-connectivity of a given graph G simultaneously by adding the smallest number of new edges. For two given functions $a : \binom{V}{2} \to R^+$ and $b : \binom{V}{2} \to R^+$, we say that G is (a, b)*-connected* if G is a-edge-connected and b-vertex-connected.

Given a multigraph $G = (V, E)$, and two requirement functions r_λ: $\binom{V}{2} \to Z^+$ and r_κ: $\binom{V}{2} \to Z^+$, the *edge-and-vertex-connectivity augmentation problem*, denoted by EVAP(r_λ, r_κ), asks to augment G by adding the smallest number of new edges to G so that the resulting graph G' becomes (r_λ, r_κ)-connected. Without loss of generality, $r_\lambda(x, y) \geq r_\kappa(x, y)$ is assumed for all $x, y \in V$, since if a graph is r_κ-vertex-connected then it is r_κ-edge-connected. Clearly, EVAP(r_λ, r_κ) contains the edge-connectivity augmentation problem and the vertex-connectivity augmentation problem as its special cases.

When the requirement function r_κ satisfies $r_\kappa(x, y) = \ell \in Z^+$ for all $x, y \in V$, this problem is denoted by EVAP(r_λ, ℓ), if no confusion arises. In this paper, we consider this problem in case $r_\kappa(x, y) = 2$ holds for every $x, y \in V$ (but $r_\lambda(x, y)$ are arbitrary). We first present a lower bound on the number of edges that is necessary to make a given graph G $(r_\lambda, 2)$-connected. We then show that this problem can be solved in polynomial time, by actually presenting a polynomial time algorithm that adds a new edge set whose size is equal to this lower bound.

In Section 2, after introducing basic definitions and the concept of edge-splitting, we derive a lower bound on the number of edges that are necessary to make a given graph G (r_λ, r_κ)-connected. In Section 3, we outline our algorithm for making a given graph G $(r_\lambda, 2)$-connected by adding a new edge set whose size is equal to the above lower bound. In Sections 4 – 7, we prove the correctness of each step in our algorithm.

2 Preliminaries

2.1 Definitions

For a multigraph $G = (V, E)$, its vertex set V and edge set E may be denoted by $V[G]$ and $E[G]$, respectively. For a subset $V' \subseteq V$ (resp., $E' \subseteq E$) in G, $G[V']$ (resp., $G[E']$) denotes the subgraph induced by V' (resp., E'). For $V' \subset V$ (resp., $E' \subseteq E$) in G, we denote $G[V - V']$ (resp., $G[E - E']$) simply by $G - V'$ (resp., $G - E'$). For an edge set F with $F \cap E = \emptyset$, we denote $G = (V, E \cup F)$ by $G + F$. A *partition* $X_1, \cdots, X_t$ of vertex set V means a family of nonempty disjoint subsets of V whose union is V, and a *subpartition* of V means a partition of a subset of V.

We say that a cut X *separates* two disjoint subsets Y and Y' of V if $Y \subseteq X$ and $Y' \subseteq V - X$ (or $Y \subseteq V - X$ and $Y' \subseteq X$) hold. In particular, a cut X separates x and y if $x \in X$ and $y \in V - X$ (or $x \in V - X$ and $y \in X$) hold. A cut X *crosses* another cut Y if none of subsets $X \cap Y$, $X - Y$, $Y - X$ and $V - (X \cup Y)$ is empty. We say that a separator $S \subset V$ separates two disjoint subsets Y and Y' of $V - S$ if no two vertices $x \in Y$ and $y \in Y'$ are connected in $G - S$. In particular, a separator S separates vertices x and y in $V - S$ if x and y are contained in different components of $G - S$.

2.2 Edge-Splitting

In this section, we introduce an operation of transforming a graph, called *edge-splitting*, which is helpful to solve the edge-connectivity augmentation problem.

Given a multigraph $G = (V, E)$, a designated vertex $s \in V$, vertices $u, v \in \Gamma_G(s)$ (possibly $u = v$) and a nonnegative integer $\delta \le \min\{c_G(s,u), c_G(s,v)\}$, we construct graph $G' = (V, E')$ from G by deleting δ edges from $E_G(s,u)$ and $E_G(s,v)$, respectively, and adding new δ edges to $E_G(u,v)$:

$c_{G'}(s,u) := c_G(s,u) - \delta,$
$c_{G'}(s,v) := c_G(s,v) - \delta,$
$c_{G'}(u,v) := c_G(u,v) + \delta,$
$c_{G'}(x,y) := c_G(x,y)$ for all other pairs $x, y \in V$.

In case of $u = v$, we interpret that $c_{G'}(s,u) := c_G(s,u) - 2\delta, c_{G'}(u,u) := c_G(u,u) + 2\delta$, and $c_{G'}(x,y) := c_G(x,y)$ for all other pairs $x, y \in V$, where an integer δ is chosen so as to satisfy $0 \le \delta \le \frac{1}{2}c_G(s,u)$. We say that G' is obtained from G by *splitting* δ pair of edges (s,u) and (s,v) (or by splitting (s,u) and (s,v) by size δ), and denote the resulting graph G' by $G/(u,v;\delta)$. A sequence of splittings is *complete* if the resulting graph G' does not have any neighbor of s.

The following theorem is proven by Mader [13].

Theorem 2.1 [13] *Let $G = (V, E)$ be a multigraph with a designated vertex $s \in V$ with $c_G(s) \ne 1, 3$ and $\lambda_G(x,y) \ge 2$ for all pairs $x, y \in V$. Then for any edge $(s,u) \in E$ there is an edge $(s,v) \in E$ such that $\lambda_{G/(u,v;1)}(x,y) = \lambda_G(x,y)$ holds for all pairs $x, y \in V - s$.* □

This says that if $c_G(s)$ is even, there always exists a complete splitting at s such that the resulting graph G' satisfies $\lambda_{G'-s}(x,y) = \lambda_G(x,y)$ for every pair of $x, y \in V - s$.

2.3 Lower Bound

In this section, we consider problem EVAP(r_λ, r_κ), and give a lower bound on the number of edges that is necessary to make a graph G (r_λ, r_κ)-connected, where r_λ and r_κ are given requirement functions. Define

$r_\lambda(X) \equiv \max\{r_\lambda(u,v) \mid u \in X, v \in V - X\}$
for each cut X,
$r_\kappa(X) \equiv \max\{r_\kappa(u,v) \mid u \in X, v \in V - X - \Gamma_G(X)\}$
for each cut X with $V - X - \Gamma_G(X) \ne \emptyset$, where see Section 1 for the definition of $\Gamma_G(X)$. To make a graph G r_λ-edge-connected, it is necessary to add

(1) at least $r_\lambda(X) - c_G(X)$ edges between X and $V - X$ for each cut X.

Also, to make a graph G r_κ-vertex-connected, it is necessary to add

(2) at least $r_\kappa(X) - |\Gamma_G(X)|$ edges between X and $V - X - \Gamma_G(X)$ for each cut X with $V - X - \Gamma_G(X) \ne \emptyset$.

For a separator S of G, let $T_1, \cdots, T_q$ denote all components of $G - S$. Now we consider a graph $H_S = (\{T_1, \cdots, T_q\}, \mathcal{E})$ in which we regard each T_i as one vertex of H_S and the edge set $\mathcal{E}$ is defined as follows:

$$(T_i, T_j) \in \mathcal{E} \longleftrightarrow \begin{array}{l}\text{There is a pair of vertices}\\ x \in T_i \text{ and } y \in T_j \text{ with}\\ r_\kappa(x,y) \ge |S| + 1.\end{array}$$

In a r_κ-vertex-connected graph, any pair of vertices $x, y \in V$ with $r_\kappa(x,y) \ge |S| + 1$ cannot be separated by such separator S. Hence if there is a pair of vertices $x \in T_i$ and $y \in T_j$ with $r_\kappa(x,y) \ge |S| + 1$, then we must add at least one edge between T_i and T_j (i.e., the number of S-components must become at most $p(H_S)$), in order to make G r_κ-vertex-connected. Therefore in this case, it is necessary to add

(3) at least $p(G - S) - p(H_S)$ edges to connect components of $G - S$ for a separator S.

(See Section 1 for the definition of $p(G - S)$.) Now define $\delta(G) = \max\{p(G - S) - p(H_S) \mid S$ is a separator in $G\}$.

Given a subpartition $\{X_1, \cdots, X_p, X_{p+1}, \cdots, X_q\}$ of V such that $q \ge p \ge 0$ and $V - X_i - \Gamma_G(X_i) \ne \emptyset$ $(i = p+1, \cdots, q)$, we need to add $\max\{r_\lambda(X_i) - c_G(X_i), 0\}$ edges for each X_i, $i = 1, \cdots, p$, and to add $\max\{r_\kappa(X_i) - |\Gamma_G(X_i)|, 0\}$ edges for each X_i, $i = p+1, \cdots, q$, based on observations (1) and (2). Now note that adding one edge to G can contribute to the requirements of at most two X_i. Therefore, we need to add $\lceil \alpha(G)/2 \rceil$ new edges to make G (r_λ, r_κ)-edge-connected, where

$$\alpha(G) = \max\left\{\sum_{i=1}^{p}(r_\lambda(X_i) - c_G(X_i)) + \sum_{i=p+1}^{q}(r_\kappa(X_i) - |\Gamma_G(X_i)|)\right\}, \tag{2.1}$$

and the max is taken over all subpartitions $\{X_1, \cdots, X_p, X_{p+1}, \cdots, X_q\}$ of V such that $q \ge p \ge 0$ and $V - X_i - \Gamma_G(X_i) \ne \emptyset$, $i = p+1, \cdots, q$. On the other hand, from observation (3), to make G r_κ-vertex-connected, at least $\max\{p(G - S) - p(H_S) \mid S$ is a separator in $G\}$ new edges are necessarily added to G. Consequently, we have the next lemma.

Lemma 2.1 (Lower Bound) *To make a given graph G (r_λ, r_κ)-connected, at least*

$$\gamma(G) \equiv \max\{\lceil \alpha(G)/2 \rceil, \delta(G)\}$$

new edges must be added. □

Now we specialize this lower bound to problem EVAP(r_λ,2) based on which we give a polynomial time algorithm for solving EVAP(r_λ,2) in the next section.

In problem EVAP(r_λ,2), we can assume $r_\lambda(x,y) \ge r_\kappa(x,y) = 2$ for all $x, y \in V$. Now the $\alpha(G)$ in (2.1) can be simplified to

$$\alpha(G) = \max\left\{\sum_{i=1}^{p}(r_\lambda(X_i) - c_G(X_i)) + \sum_{i=p+1}^{q}(2 - |\Gamma_G(X_i)|)\right\}, \tag{2.2}$$

where the maximization is taken over all subpartitions $\{X_1, \cdots, X_p, X_{p+1}, \cdots, X_q\}$ of V such that $q \geq p \geq 0$ and $V - X_i - \Gamma_G(X_i) \neq \emptyset$ for $i = p+1, \cdots, q$.

Also we specialize the second lower bound $\delta(G)$. Now, to derive $\delta(G)$, the maximization is taken over all separators S that satisfy $|S| \leq 1$, since each pair of vertices $x, y \in V$ satisfy $r_\kappa(x,y) = 2$. Note that $p(H_S) = 1$ holds for any separator S with $|S| \leq 1$, since any pair of S-components T_i and T_j has a pair of vertices $x \in T_i$ and $y \in T_j$ where $r_\kappa(x,y) = 2 > |S|$. Hence this lower bound can be rewritten by

$$\max\{p(G - S) - 1 \mid S \text{ is a separator with } |S| \leq 1\}. \tag{2.3}$$

A vertex v is called a *cut vertex* in $G = (V, E)$ if $S = \{v\}$ is a minimum separator in G. If G has a cut vertex $v \in V$, then $p(G - v) > p(G)$ holds from the definition of a separator; otherwise $p(G - v) = p(G)$ holds for all $v \in V$. Hence the lower bound in (2.3) can be simplified to

$$\max_{v \in V}\{p(G - v) - 1\}.$$

Also note that if $\kappa(G) \leq 1$ holds, then (1.1) in Section 1 satisfies $\beta(G) = \max_{v\in V}\{p(G - v)\}$ and the lower bound in (2.3) can be simplified to $\beta(G) - 1$. In case of $\kappa(G) \geq 2$, the lower bound in (2.3) is not defined but $\max_{v\in V}\{p(G - v) - 1\} = 0$ holds. Therefore, in Problem EVAP(r_λ, 2), we can define the lower bound in (2.3) by $\max_{v\in V}\{p(G - v) - 1\}$ without confusion. This means that we can define

$$\beta(G) = \max_{v \in V}\{p(G - v)\}. \tag{2.4}$$

and the lower bound in (2.3) becomes

$$\beta(G) - 1.$$

Now define $\gamma(G) = \max\{\lceil\alpha(G)/2\rceil, \beta(G) - 1\}$. From the above discussion, a set of new edges gives an optimal solution to EVAP(r_λ,2) if its size is equal to $\gamma(G)$ and the graph obtained by adding $\gamma(G)$ edges to G is (r_λ, 2)-connected. We now show that this is always possible, by presenting a polynomial time algorithm in the next section for making G (r_λ, 2)-connected by adding $\gamma(G)$ new edges.

Lemma 2.2 *If $\kappa(G) = 1$ (i.e., G is connected and has a cut vertex), then any two tight sets X and Y in G are disjoint.* □

3 A Polynomial Time Algorithm for EVAP(r_λ,2)

We now present a polynomial time algorithm, based on the argument in the previous section. Call an edge $e = (u, u')$ *admissible* with respect to a vertex v, if v is a cut vertex such that $v \neq u, u'$ and $p(G - v) = p((G - e) - v)$. For a subset F of edges in a graph G, we say that two edge $e_1 = (u_1, w_1)$ and $e_2 = (u_2, w_2)$ are *switched* in F if we delete e_1 and e_2 from F, and add edges (u_1, u_2) and (w_1, w_2) to F. Our algorithm for solving the EVAP(r_λ, 2), denoted by Algorithm EVA(r_λ, 2), consists of the following four major steps.

Algorithm EVA(r_λ, 2)

Input: An undirected multigraph $G = (V, E)$, and a requirement function $\{r_\lambda(x,y) \in Z^+ \mid x, y \in V\}$.

Output: An undirected multigraph $G^* = G + F$ with $\lambda_{G^*}(x,y) \geq r_\lambda(x,y)$ for every $x, y \in V$ and $\kappa(G^*) \geq 2$ where the size of new edge set F is the minimum.

Step I. (Addition of vertex s and associated edges):

After adding a new vertex s, add a set F' of a sufficiently large number of edges between s and V so that the resulting graph $G' = (V \cup \{s\}, E \cup F')$ satisfies

$$c_{G'}(X) \geq r_\lambda(X) \tag{3.1}$$

for all X with $\emptyset \neq X \subset V$,

$$|\Gamma_{G'}(X \cup s)| \geq 2 \tag{3.2}$$

for all X with $\emptyset \neq X \subset V$ and $V - X - \Gamma_{G'}(X) \neq \emptyset$. (This can be done for example by adding $\max\{r_\lambda(x,y) \mid x, y \in V\}$ edges between s and each vertex $v \in V$.)

Next, to make F' minimal we discard new edges in F', one by one, as long as (3.1) and (3.2) remain valid. Denote the resulting set of new edges by F_1 and the resulting graph by $G_1 = (V \cup \{s\}, E \cup F_1)$, where $F_1 = E_{G_1}(s, V)$.

Clearly, these operations can be performed in polynomial time. We claim the next.

Remark: Note that if the original graph G is not connected, then $\kappa_{G_1}(x,y) \geq 2$ cannot be attained for some $x, y \in V$, since a subset $X \subset V$ which induces a component $G[X]$ of G satisfies $\Gamma_{G_1}(X) = \emptyset$ or $\{s\}$, and hence $\kappa_{G_1}(x,y) \leq 1$ for $x \in X$ and $y \in V - X$.

Property 3.1 *In the above step, it is possible to choose a subset F_1 for which $|F_1| = \alpha(G)$ holds.* □

Step II. (Edge-splitting): If $c_{G_1}(s)$ is odd, then we add one edge (s,w) to G by choosing vertex an arbitrary $w \in V$ which is not a cut vertex in G.

Next we find a complete edge-splitting at s in $G_1 = (V \cup \{s\}, E \cup F_1)$ which preserves condition (3.1) (i.e., the r_λ-edge-connectivity). By Mader's theorem, there always exists such a complete edge-splitting at s, and it can be computed in polynomial time. Let $G_2 = (V, E \cup F_2)$ denote the graph obtained by such a complete edge-splitting, ignoring the isolated vertex s. The next is immediate from Mader's theorem.

Property 3.2 *There is a complete edge-splitting at s of G_1, so that the resulting graph G_2 is r_λ-edge-connected.* □

If G_2 is also 2-vertex-connected, then we are done because $|F_2| = |F_1|/2 = \lceil \alpha(G)/2 \rceil$ implies that G_2 is optimally augmented by lower bound $\lceil \alpha(G)/2 \rceil$. Otherwise, go to Step III.

Step III. (Switching edges): Now G_2 has cut vertices. Then, by property (3.2) for G_1, G_2 satisfies

$$G_2[X \cup \{v\}] \text{ contains at least one edge in } F_2 \text{ for any cut vertex } v \text{ and its } v\text{-component } X. \quad (3.3)$$

Property 3.3 *Assume that G_2 has an admissible edge $e_1 \in F_2$ with respect to a cut vertex v. Let X be a v-component with $e_1 \notin E[G_2[X \cup \{v\}]]$, and e_2 be chosen arbitrarily from $F_2 \cap E[G_2[X \cup \{v\}]]$. Then switching e_1 and e_2 decreases the number of v-components in G_2 at least by one while preserving the r_λ-edge-connectivity. Moreover, the resulting graph G_2' from switching e_1 and e_2 still satisfies (3.3), and $\kappa_{G_2'}(x,y) \geq 2$ holds for any pair of vertices x and y with $\kappa_{G_2}(x,y) \geq 2$.* □

Property 3.4 *If G_2 has two cut vertices v_1 and v_2, then there are v_1-component X_1 and v_2-component X_2 such that $X_1 \cap X_2 = \emptyset$. Let edge e_1 be arbitrarily chosen from $F_2 \cap E[G_2[X_1 \cup \{v\}]]$. Then e_1 is admissible with respect to v_2.* □

Based on Property 3.3, Step III repeats switching pairs of edges in F_2 until the resulting graph has no admissible edge in F_2.

Let $G_3 = (V, E \cup F_3)$ be the resulting graph obtained by such a sequence of switching edges in F_2, where F_3 denotes the final F_2. Then Property 3.4 implies that, if there are at least two cut vertices, then G_3 has an admissible edge in F_3, which is a contradiction. Hence G_3 has the following property.

Property 3.5 *G_3 has at most one cut vertex.* □

If G_3 has no cut vertex, then we are done, since $|F_3| = \lceil \alpha(G)/2 \rceil$ implies that G_3 is optimally augmented. Otherwise, go to Step IV.

Step IV. (Edge augmentation): Now G_3 has exactly one cut vertex v. Then G_3 and v satisfy the following property.

Property 3.6 *For the graph G_3 and its cut vertex v, it holds $p(G_3 - v) = p(G - v) - \lceil \alpha(G)/2 \rceil$.* □

Now let $T_1, \cdots, T_q$ be all v-components in G_3, where $q = p(G_3 - v)$. We can make G_3 2-vertex-connected by adding one edge between T_i and T_{i+1} for each $i = 1, \cdots, q-1$ (i.e., $p(G_3 - v) - 1$ edges in total). Let F_4 denote a set of these $p(G_3 - v) - 1$ edges added. Note that $p(G_3 - v) = p(G - v) - \lceil \alpha(G)/2 \rceil \leq \beta(G) - \lceil \alpha(G)/2 \rceil$ holds from Property 3.6 and $\beta(G) \geq p(G - v)$ (see (2.4)). Also note that $|F_4| + |F_3| = p(G_3 - v) - 1 + \lceil \alpha(G)/2 \rceil \geq \beta(G) - 1$ holds since $\beta(G) - 1$ is a lower bound on the number of edges that must be added to make G $(r_\lambda, 2)$-connected. These imply $|F_4| = \beta(G) - 1 - \lceil \alpha(G)/2 \rceil$. Therefore we have the following property.

Property 3.7 *There is a set of $\beta(G) - 1 - \lceil \alpha(G)/2 \rceil$ new edges F_4 obtained for G_3 such that the resulting graph $G_4 = (V, E \cup F_3 \cup F_4)$ is 2-vertex-connected.*

Finally, we are done since $|F_3| + |F_4| = \beta(G) - 1$ implies that G_4 is optimally augmented by lower bound $\beta(G) - 1$.

We shall explain in the subsequent sections that the required properties (summarized as Properties 3.1 – 3.7) always hold. Together with these proofs, this algorithm establishes the next theorem, which is the main goal of this thesis.

Theorem 3.1 *Given a requirement function $\{r_\lambda(x, y) \in Z^+ \mid x, y \in V\}$, a multigraph G can be made $(r_\lambda, 2)$-connected by adding $\gamma(G) = \max\ \{\lceil \alpha(G)/2 \rceil, \beta(G) - 1\}$ new edges in $O(n^3 m \log \frac{n^2}{m})$ time.* □

4 Correctness of Step I

We give a proof of Property 3.1 in order to prove the correctness of Step I.

Proof of Property 3.1: It is clear that $\lambda_{G_1}(x,y) \geq r_\lambda(x,y) \geq 2$ holds for all $x, y \in V$ by (3.1).

First, we show $|F_1| \geq \alpha(G)$. Let $\mathcal{F}^* = \{X_1^*, \cdots, X_p^*, X_{p+1}^*, \cdots, X_q^*\}$ be a subpartition of V

with $V - X_i^* - \Gamma_{G_1}(X_i^*) \neq \emptyset$ for $i = p+1, \cdots, q$ that attains the maximum of (2.2); i.e., $\alpha(G) = \sum_{i=1}^{p}(r_\lambda(X_i^*) - c_G(X_i^*)) + \sum_{i=p+1}^{q}(2 - |\Gamma_G(X_i^*)|)$. If $|F_1| < \alpha(G)$ holds, then there must be at least one cut $X_i^* \in \mathcal{F}^*$ that violates (3.1) or (3.2), contradicting construction of G_1.

Now we prove the converse, $|F_1| \leq \alpha(G)$, through five claims.

A cut $X \subset V$ is called *critical* in G_1 if $s \in \Gamma_{G_1}(X)$ holds and the removal of any edge $e \in E_{G_1}(s, X)$ violates (3.1) or (3.2). Clearly, a subset $X \subset V$ with $s \in \Gamma_{G_1}(X)$ is critical if and only if X satisfies at least one of the following conditions:

(1) $c_{G_1}(X) = r_\lambda(X)$.
(2) $c_{G_1}(s, X) = 1, |\Gamma_{G_1}(X) - s| = 1$, and $V - X - \Gamma_{G_1}(X) \neq \emptyset$.
(3) $\Gamma_{G_1}(X) = \{s\}$, $|\Gamma_{G_1}(s) \cap X| = 2$, and there is a vertex $v \in \Gamma_{G_1}(s) \cap X$ with $c_{G_1}(s, v) = 1$.

We call a critical cut X *v-minimal* if $v \in \Gamma_{G_1}(s) \cap X$ and there is no critical cut X' with $\{v\} \subseteq X' \subset X$. A subset X is called *critical of type* (1) (resp., (2), (3)) if it satisfies (1) (resp., (2), (3)).

We will prove that G_1 has a set of critical cuts $X_1, \cdots, X_q$ only of type (1) and (2) such that

$$\begin{array}{l} X_i \cap X_j = \emptyset,\ 1 \leq i < j \leq q \text{ and} \\ \Gamma_{G_1}(s) \subseteq X_1 \cup \cdots \cup X_q, \end{array} \quad (4.1)$$

This implies that

$$|F_1| = \left\{\sum_{i=1}^{p}(r_\lambda(X_i) - c_G(X_i)) + \sum_{i=p+1}^{q}(2 - |\Gamma_G(X_i)|)\right\}$$

where $X_i, i = 1, \cdots, p$ is of type (1) and $X_i, i = p+1, \cdots, q$ is of type (2), from which $|F_1| \leq \alpha(G)$ by definition of $\alpha(G)$.

Claim 4.1 *Any critical cut X of type (3) is also critical of type (1).* □

By this claim, we can regard critical cuts of type (3) as those of type (1). The next property is known in [5].

Claim 4.2 *Let X and Y be critical cuts of type (1) in G_1. Then at least one of the following statements holds.*
(i) Both $X \cap Y$ and $X \cup Y$ are critical.
(ii) Both $X - Y$ and $Y - X$ are critical, and $c_{G_1}(X \cap Y, (V \cup \{s\}) - (X \cup Y)) = 0$. □

An analogous property holds for type (2) critical cuts.

Claim 4.3 *Let X and Y be critical cuts of type (2). If Y is v-minimal for some $v \in V - X$, then they do not cross each other.* □

Claim 4.4 *Let X be a critical cut of type (1), and Y be a critical cut of type (2) such that $\Gamma_{G_1}(s) \cap (Y - X) \neq \emptyset$. If X and Y cross each other, then $c_{G_1}(X \cap Y, s) = 0$ holds and cut $Y - X$ is critical of type (1).* □

Now we are ready to prove that G_1 has a set of critical cuts $X_1, \cdots, X_q$ that satisfies (4.1). Let $N_1 \subseteq \Gamma_{G_1}(s)$ be the set of neighbors u of s such that there is a critical cut X of type (1) with $u \in X$. Let us choose a critical cut X_u of type (1) with $u \in X_u$ for each $u \in N_1$ so that $\sum_{X \in \{X_u | u \in N_1\}} |X|$ is minimized. Denote such a set $\{X_u \mid u \in N_1\}$ by $\mathcal{F}_1$. For $N_2 = \Gamma_{G_1}(s) - N_1$, we choose a u-minimal critical cut X_u for each $u \in N_2$, and let $\mathcal{F}_2 = \{X_u \mid u \in N_2\}$. Then we claim the next.

Claim 4.5 *$\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ consists of disjoint critical cuts whose union contains $\Gamma_{G_1}(s)$.*

Proof. Let $\mathcal{F}_1 = \{X_1, \cdots, X_p\}$ and $\mathcal{F}_2 = \{X_{p+1}, \cdots, X_q\}$ with each $\emptyset \neq X_i \subset V$. Clearly, $\Gamma_{G_1}(s) \subseteq \cup_{X_i \in \mathcal{F}} X_i$ holds from construction of $\mathcal{F}$.

We show that X_i and X_j are pairwise disjoint for each $X_i, X_j \in \mathcal{F}_1$. Assume that $\mathcal{F}_1$ contains X_i and X_j which are not pairwise disjoint. Note that $X_i \subset X_j$ does not hold from construction of $\mathcal{F}_1$. If X_i and X_j cross each other, then Claim 4.2 implies that at least one of the following statements holds:
(i) Both $X_i \cap X_j$ and $X_i \cup X_j$ are critical.
(ii) Both $X_i - X_j$ and $X_j - X_i$ are critical, and $c_{G_1}(X \cap Y, (V \cup \{s\}) - (X \cup Y)) = 0$.
If the statement (i) holds, then $\mathcal{F}_1' = (\mathcal{F}_1 - X_i - X_j) \cup \{X_i \cup X_j\}$ would satisfy $N_1 \subseteq \mathcal{F}_1'$ and $\sum_{X \in \mathcal{F}_1'} |X| < \sum_{X \in \mathcal{F}_1} |X|$, contradicting the minimality of $\sum_{X \in \mathcal{F}_1} |X|$. If the statement (ii) holds, then $\mathcal{F}_1' = (\mathcal{F}_1 - X_i - X_j) \cup \{X_i - X_j, X_j - X_i\}$ satisfies $\sum_{X \in \mathcal{F}_1'} |X| < \sum_{X \in \mathcal{F}_1} |X|$ and $N_1 \subseteq \mathcal{F}_1'$ (by $c_{G_1}(X \cap Y, (V \cup \{s\}) - (X \cup Y)) = 0$). This again contradicts the minimality of $\sum_{X \in \mathcal{F}_1} |X|$. Therefore X_i and X_j are pairwise disjoint for each $X_i, X_j \in \mathcal{F}_1$.

Claim 4.3 implies that X_i and X_j are pairwise disjoint for each $X_i, X_j \in \mathcal{F}_2$.

Finally, we show that X_i and X_j are pairwise disjoint for each $X_i \in \mathcal{F}_1$ and $X_j \in \mathcal{F}_2$. Note that $\Gamma_{G_1}(s) \cap (X_j - X_i) \neq \emptyset$ holds from definition of N_1. Then $X_j \subset X_i$ does not hold. Also note that $X_i \subset X_j$ does not hold, otherwise $\Gamma_{G_1}(s) \cap X_i \neq \emptyset$ and $\Gamma_{G_1}(s) \cap (X_j - X_i) \neq \emptyset$ imply $c_{G_1}(X_j, s) \geq c_{G_1}(X_i, s) + 1 \geq 2$, contradicting that X_j is of type (2). Assume that X_i and X_j cross each other. Now $\Gamma_{G_1}(s) \cap (X_j - X_i) \neq \emptyset$ holds. Therefore Claim 4.4 implies that $c_{G_1}(s, X_i \cap X_j) = 0$ holds and $X_j - X_i$ is a critical cut of type (1). This implies that any vertex in X_j cannot belong to N_2, contradicting $X_j \in \mathcal{F}_2$. □

Clearly $\mathcal{F}$ is a subpartition of V by Claim 4.5. Since $\Gamma_{G_1}(s) \subseteq X_1 \cup \cdots \cup X_q$ with $X_i \in \mathcal{F}$ holds, it holds

$$|F_1| = \sum_{i=1}^{p}(r_\lambda(X_i) - c_G(X_i)) + \sum_{i=p+1}^{q} (2 - |\Gamma_G(X_i)|),$$

for $\mathcal{F}_1 = \{X_1, \cdots, X_p\}$ and $\mathcal{F}_2 = \{X_{p+1}, \cdots, X_q\}$. From definition of $\alpha(G)$, we have $|F_1| \le \alpha(G)$. □

5 Correctness of Step II

Let $G_1 = (V \cup \{s\}, E \cup F_1)$ be the graph obtained from a given graph $G = (V, E)$ after Step I. In this section, we describe about the correctness of Property 3.2 and the purpose of operations in case where $c_{G_1}(s)$ is odd.

In Step II, a graph $G_2 = (V, E \cup F_2)$ is constructed from G_1 by a complete edge-splitting at s. Then the correctness of Property 3.2 is immediate from Mader's theorem (see Theorem 2.1).

In this step, a non cut vertex w is chosen when we add an extra edge (s, w) to G_1 if $c_{G_1}(s)$ is odd. Such choice of w will be used for the correctness of Step IV in Section 7 (i.e., by this choice of w, we will be able to make G $(r_\lambda, 2)$-connected by adding $\beta(G) - 1$ new edges in case of $\beta(G) - 1 > \lceil \alpha(G)/2 \rceil$).

6 Correctness of Step III

Let $G_2 = (V, E \cup F_2)$ be the graph obtained in Step II. Now G_2 is 2-edge-connected but has cut vertices.

In order to justify Step III, we now prove Property 3.3 in Step III.

Proof of Property 3.3: We prove Property 3.3 via two claims.

Claim 6.1 *Let $v \in V$ denote a cut vertex in G_2. Assume that a v-component T contains an admissible edge $e = (u, u')$ with respect to v. Then $G_2[T] - e$ contains a path P between u and u'.* □

Claim 6.2 *Let $e_1 = (u_1, w_1)$ and $e_2 = (u_2, w_2)$ be the edges in the statement of Property 3.3. Then the graph $G_2' = (V, E \cup F_2')$ obtained by switching e_1 and e_2, where $F_2' = F_2 \cup \{ (u_1, u_2), (w_1, w_2)\} - \{e_1, e_2\}$, satisfies followings:*
(i) $\lambda_{G_2'}(x, y) \ge r_\lambda(x, y)$ for every $x, y \in V$.
(ii) $p(G_2' - v) < p(G_2 - v)$.
(iii) $\kappa_{G_2'}(x, y) \ge 2$ holds for every pair of vertices x and y that satisfies $\kappa_{G_2}(x, y) \ge 2$.
(The statements (ii) and (iii) and Lemma 2.2 imply that switching e_1 and e_2 decreases the number $t(G_2)$ of tight sets in G_2 by at least one if e_1 or e_2 is contained in a tight set in G_2.)

Proof. (i) We assume that there is a cut X such that $c_{G_2'}(X) \le r_\lambda(X) - 1$ holds. Note that $c_{G_2}(X) \le c_{G_2'}(X)$ holds if cut X does not separate $\{u_1, u_2\}$ and $\{w_1, w_2\}$ in G_2'. Since $c_{G_2}(X) \ge r_\lambda(X)$ originally holds, cut X separates $\{u_1, u_2\}$ and $\{w_1, w_2\}$ and hence $c_{G_2'}(X) = c_{G_2}(X) - 2$ holds. Since the cut X crosses both v-components T_1 and T_2 in G_2, either $G_2[X]$ or $G_2[V - X]$ consists of at least two components. Without loss of generality, assume that $G_2[X]$ consists of at least two components. There are vertices $x^* \in X$ and $y^* \in V - X$ such that $r_\lambda(x^*, y^*) = r_\lambda(X) \ge c_{G_2'}(X) + 1$. Without loss of generality, assume that $x^* \in X \cap T_1$. Note that $c_{G_2}(X \cap T_2) \ge r_\lambda(X \cap T_2) \ge 2$ and $c_{G_2}(X \cap T_1) \ge r_\lambda(X \cap T_1) \ge r_\lambda(x^*, y^*) \ge c_{G_2'}(X) + 1$ hold. This implies $c_{G_2}(X) = c_{G_2}(X \cap T_1) + c_{G_2}(X \cap T_2) \ge (c_{G_2'}(X)+1)+2$, contradicting $c_{G_2'}(X) = c_{G_2}(X) - 2$.

(ii) It is sufficient to show that $G_2'[T_1 \cup T_2]$ is connected. Since the removal of the admissible edge e_1 does not increase the number of v-components, T_1 remains a v-component in $G_2 - e_1$. If T_2 remains a v-component in $G_2 - e_2$, then $G[T_1]$ and $G[T_2]$ are joined by the edges (u_1, u_2) and (w_1, w_2) obtained by switching e_1 and e_2 in G_2'. If T_2 consists of two components T_2^1 and T_2^2 in $G_2 - e_2$, then $u_2 \ne v \ne w_2$ holds and u_2 and w_2 are separated by T_2^1. Assume $u_2 \in T_2^1$ and $w_2 \in T_2^1$ without loss of generality. Now T_2^1 (resp., T_2^2) and T_1 are joined by the edges (u_1, u_2) (resp., (w_1, w_2)). This implies that $G_2'[T_1 \cup T_2]$ is a component since T_1 remains a v-component in $G_2 - e_1$. Therefore if v remains a cut vertex in G_2', then $T_1 \cup T_2$ is a v-component (otherwise, clearly, $p(G_2 - v) = 1$).

(iii) Assume that there are vertices $x, y \in V$ such that $\kappa_{G_2}(x, y) = 2$ but $\kappa_{G_2'}(x, y) = 1$. Let $v' \in V$ denote a cut vertex in G_2' that separates x and y. Clearly, $v' \ne v$ (because $v = v'$ would imply $\kappa_{G_2}(x, y) = 1$). Let $W_1, W_2, \cdots, W_q$ $(q \ge 2)$ be the v'-components of G_2', where $x \in W_1$ and $y \in W_2$. Since a cut vertex v' does not separate x and y in G_2, $e_1 \in E_{G_2}(W_1, W_2)$ or $e_2 \in E_{G_2}(W_1, W_2)$ holds. Also note that no edge other than e_1 and e_2 cannot belong to $E_{G_2}(W_1, W_2)$. We can easily see that $G_2[W_1 \cup W_2 \cup \{v'\}]$ contains u_1, w_1, u_2, and w_2. Then note that $u_i, w_i \in W_j$ cannot hold for any i, j with $1 \le i \le j \le 2$. Otherwise (assume $u_1, w_1 \in W_1$ without loss of generality) then $e_2 \in E_{G_2}(W_1, W_2)$ holds (assume $u_2 \in W_1$ and $w_2 \in W_2$ without loss of generality). Now $(w_1, w_2) \in E_{G_2'}(W_1, W_2)$ holds and $G_2'[W_1]$ and $G_2'[W_2]$ are both connected from definition of W_1 and W_2, contradicting that cut vertex v' separates x and y in G_2'. Therefore, for each $i = 1, 2$, we have now $e_i = (u_i, w_i) \in E_{G_2}(W_1, W_2)$ or $u_i = v'$ or $w_i = v'$.

We first consider the case of $e_1 \in E_{G_2}(W_1, W_2)$. Then $v' \in T_1$ holds since $G_2[T_1] - e_1$ is connected by Claim 6.1. Hence $e_2 \in E_{G_2}(W_1, W_2)$ holds since $v' \in T_1$ implies $u_2 \ne v' \ne w_2$. Let $v \notin W_2$ and $u_1, u_2 \in W_1$ without loss of generality. Now $\Gamma_{G_2'}(T_2 \cap W_2) \cap (T_2 - W_2) = \emptyset$ holds since v' is a cut vertex of G_2' and $v' \notin T_2$ hold. Note that $E_{G_2'}(T_2 \cap W_2, V -$

$(T_2 \cap W_2)) = \{(w_1, w_2)\}$ since T_2 is a v-component of G_2 and $u_2 \in W_1$ holds. This implies $\Gamma_{G_2}(T_2 \cap W_2) = \{u_2\}$ holds and hence e_2 is a bridge of G_2 from $E_{G_2}(W_1, W_2) = \{e_1, e_2\}$, which contradicts $\lambda(G_2) \geq 2$.

We then consider the case of $e_1 \notin E_{G_2}(W_1, W_2)$ holds, i.e., $v' = u_1 \in T_1$ or $v' = w_1 \in T_1$ holds. This implies that $e_2 \in E_{G_2}(W_1, W_2)$ holds and $v' \notin T_2$. Therefore, this clearly leads to a contradiction, in a similar way to above case of $e_1 \in E_{G_2}(W_1, W_2)$. □

From the above claim, Property 3.3 is proved.

7 Correctness of Step IV

Let $G_3 = (V, E \cup F_3)$ be obtained from G_2 after Step III. Now clearly $|F_3| = \lceil \alpha(G)/2 \rceil$ This G_3 has exactly one cut vertex v.

The correctness of Step IV clearly follows if we prove Property 3.6. The proof is now given below via two claims.

Claim 7.1 *G_3 has no edge in F_3 incident to the cut vertex v.* □

Claim 7.2 *$p(G - v) = p(G_3 - v) + |F_3|$ holds. That is, deleting any edge $e \in F_3$ increases the number of v-components in G_3.*

Proof. If $p(G-v) < p(G_3-v)+|F_3|$ holds, then there is at least one edge $e \in F_3$ with $p((G_3 - e) - v) = p(G_3 - v)$. Then e is admissible with respect to v since Claim 7.1 implies that any edge in F_3 is not incident to v, contradicting construction of G_3. □

This claim implies that since G_3 has no edge in F_3 incident to the cut vertex v, a graph $H = (W, F_3)$ is a forest, where a vertex set W of H is obtained by removing the cut vertex v and contracting each component of $G-v$ to one vertex. Now Claim 7.2 implies Property 3.6 since $|F_3| = \lceil \alpha(G)/2 \rceil$ holds from construction.

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