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Minimum Self-Dual Decompositions of Positive Dual-Minor Boolean Functions*

Jan C. Bioch† Toshihide Ibaraki‡ Kazuhisa Makino‡

Abstract

In this paper we consider decompositions of a positive dual-minor Boolean function f into $f = f_1 f_2 \ldots\ldots f_k$, where all f_j are positive and self-dual. It is shown that the minimum k having such a decomposition equals the chromatic number of a graph associated with f, and the problem of deciding whether a decomposition of size k exists is co-NP-hard, for $k \geq 2$. We also consider the canonical decomposition of f and show that the complexity of finding a canonical decomposition is equivalent to deciding whether two positive Boolean functions are mutually dual. Finally, for the class of path functions including the class of positive read-once functions, we show that the sizes of minimum decompositions and minimum canonical decompositions are equal, and present a polynomial total time algorithm to generate all minimal canonical decompositions.

1 Introduction

1.1 Motivation and Results

Positive self-dual Boolean functions arise in various contexts under different names. For example, they can be interpreted as non-dominated coteries [6, 16], as strong simple games (or social choice functions) [19], as symmetric tight (i.e., Nash-solvable) games of two players [13], as self-dual antichains in clutters [4], as maximal intersecting families of sets [10], or as critical bipartite hypergraphs [2]. In all these contexts, a positive Boolean function is interpreted as a family of sets satisfying certain conditions that depend on the domains of applications. Self-dual functions also occur in logic, lattice theory, operations research (e.g maximal independent sets [5], minimal transversals of hypergraphs [10]), artificial intelligence (e.g., diagnosis [10]), computational learning (e.g., identification of positive Boolean functions [5]), database theory (e.g., the additional key problem [10]), coherent systems of reliability theory [22], and in various areas of Boolean function theory, such as threshold logic [20], regular Boolean functions [9, 21] and circuit theory.

One of the real-world applications is in coterie theory. Coteries play an important role in distributed systems, as they are used as a means to realize mutual exclusion [11, 16]. Non-dominated (ND) coteries are important in practice, since those are the coteries with maximal efficiency when implemented to realize mutual exclusion. As noted above, ND coteries correspond to positive self-dual Boolean functions [6, 16], and it is important to know how to compose a large ND coterie with some specified property (e.g., with high availability) from small ND coteries. In other words, one of the fundamental problems in this area is how to decompose a given positive self-dual function into smaller positive self-dual functions, as it explains how to represent and how to construct these functions by us-

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†Department of Computer Science, Erasmus University Rotterdam, P.O. Box 1738, 3000 DR Rotterdam, The Netherlands. (bioch@cs.few.eur.nl)

‡Department of Applied Mathematics and Physics, Graduate School of Engineering, Kyoto University, Kyoto 606, Japan. ({ibaraki,makino}@kuamp.kyoto-u.ac.jp)

ing simpler elements. Such a representation of a large ND coterie by simple smaller ND coteries is important in applying it to real distributed systems as it gives a simple and efficient means to check whether given vectors (generated in the distributed systems) belong to the true set or not.

It is shown in [7, 16, 19] that any positive self-dual function can be decomposed into a set of basic majority functions (the basic majority function is the only self-dual function containing three variables). Other types of decompositions are also found in [16, 22]. A key step in the procedure of decomposition into basic majority functions is how to decompose a given positive *dual-minor* function f into a conjunction of positive *self-dual* functions:

$$f = f_1 f_2 \cdots f_k.$$

In [7] we introduced canonical decompositions, where each f_j has certain special structure (see Subsection 1.3), and gave an algorithm to compute all minimal canonical decompositions. Although the size of a canonical decomposition is already rather small, it may not be minimum, and the question of how to compute minimum decompositions was left open.

In Section 2 of this paper, we show that the size of a minimum decomposition of a positive dual-minor function f equals the chromatic number of a graph associated with f. The complexity of k-decomposability is shown to be coNP-hard if $k \geq 2$, under the assumption that f is given by the set of minimal true vectors $\min T(f)$. On the other hand, if the minimal vectors of its dual f^d are given as input, the problem is solvable in polynomial time if $k \leq 2$ and NP-complete if $k \geq 4$. In this case, the question is open if $k = 3$. In Subsection 3.1, it is shown that the complexity of finding a minimal canonical decomposition is equivalent to the problem whether two positive functions are mutually dual or not. The latter problem is related to many other interesting problems as discussed in e.g., [5, 10]. However, the complexity of the mutual duality problem is still a major open problem, although Fredman and Khachiyan [12] showed that this problem is quasi-polynomial, and therefore it is unlikely to be NP-hard. It is then discussed in Subsection 3.2 when prime implicants of f can induce all minimal canonical decompositions. Finally, in Section 4, we discuss the decomposability problem for the class of path functions, which includes the class of positive read-once functions, and show that all the above problems are solvable in polynomial time over this class. The path functions are important in such applications as coteries since they have simple representations which can be efficiently handled algorithmically. It is also shown that, for path functions, the sizes of minimum decompositions and minimum canonical decompositions are equal.

1.2 Definitions and basic properties

Positive Boolean functions

A *Boolean function*, or in short a *function* is a mapping $f : \{0,1\}^n \mapsto \{0,1\}$. Let $v \in \{0,1\}^n$ be a *Boolean vector*, or in short a *vector*, for which we introduce a notation $ON(v) = \{i \mid v_i = 1\}$. If $f(v) = 1$ (resp. 0), then v is called a *true* (resp. *false*) vector of f. The set of all true vectors (resp. false vectors) is denoted by $T(f)$ (resp. $F(f)$). For a function f, the minimal elements in $T(f)$ (resp. maximal elements in $F(f)$) are called the minimal true vectors (resp. maximal false vectors) of f, and the set of all minimal true vectors (resp. maximal false vectors) is denoted by $\min T(f)$ (resp. $\max F(f)$). A function f is called *positive* (or *monotone*) if $v \leq w$ always implies $f(v) \leq f(w)$. It is known that a positive function f is uniquely determined by $\min T(f)$, and that f has the unique *minimal disjunctive form* (MDF) consisting of all the prime implicants of f, such that all the literals of each prime-implicant are uncomplemented. There is one-to-one correspondence between $\min T(f)$ and the set of all prime implicants of f, where a vector v corresponds to the term t_v defined by $t_v = \bigwedge_{i \in ON(v)} x_i$. For example, the vector (101) corresponds to $x_1 x_3$, and a positive function $f = x_1 x_2 \vee x_2 x_3 \vee x_3 x_1$ (which is also denoted by a simplified form $f = 12 \vee 23 \vee 31$ in this paper) has $\min T(f) = \{(110), (011), (101)\}$. Finally, the constant functions $f \equiv 0$ and $f \equiv 1$ are denoted respectively by $\perp$ and $\top$.

Dual-comparable functions

The *dual* of a function f, denoted f^d, is defined by

$$f^d(x) = \bar{f}(\bar{x}),$$

where $\bar{f}$ and $\bar{x}$ denote the complement of f and x, respectively. As is well-known, the MDF expression defining f^d is obtained from that of f by exchanging $\vee$ and $\wedge$ (where $\wedge$ is also denoted by $\cdot$ or omitted if there is no confusion), as well as the constants 0 and 1. Call a vector w a *transversal* of f if $ON(w)$ satisfies $ON(w) \cap ON(v) \neq \emptyset$ for all $v \in \min T(f)$. It is known that, for a positive function f, $w \in T(f^d)$ (resp. $w \in \min T(f^d)$) holds if and only if w is a transversal (resp. minimal transversal) of f. Denote $f \leq g$ if these functions satisfy $f(x) \leq g(x)$ for all $x \in \{0,1\}^n$. It is easy to see that $(f \vee g)^d = f^d g^d$, $(fg)^d = f^d \vee g^d$, $f \leq g$ implies $f^d \geq g^d$, and so on. A function f is called *dual-minor* if $f \leq f^d$, *dual-major* if $f \geq f^d$, *dual-comparable* if $f \leq f^d$ or $f \geq f^d$, and *self-dual* if $f^d = f$.

For example, $f = 123$ is dual-minor since $f^d = 1 \vee 2 \vee 3$ satisfies $f \leq f^d$. Similarly, the dual of $f = 12 \vee 23 \vee 31$ is

$$f^d = (1 \vee 2)(2 \vee 3)(3 \vee 1) = 12 \vee 23 \vee 31.$$

This function is self-dual, and is called the *basic majority function*; it is known to be the only positive self-dual function containing three variables. There is no positive self-dual function of two variables. However, the function $f = x$ is a positive self-dual function of one variable. The functions f and g are called *mutually dual* if $f^d = g$.

The following lemmas give characterizations of dual-comparable functions (see [6, 7, 16, 20] for the proofs).

Lemma 1 *Let f be a function. Then:*

(i) *f is dual-minor if and only if $x \in T(f) \Rightarrow \bar{x} \notin T(f)$.*

(ii) *f is dual-major if and only if $x \notin T(f) \Rightarrow \bar{x} \in T(f)$.*

(iii) *f is self-dual if and only if $x \in T(f) \Leftrightarrow \bar{x} \notin T(f)$.* □

Lemma 2 *Let f be a positive function. Then f is dual-minor if and only if every pair of $v, w \in \min T(f)$ satisfies $ON(v) \cap ON(w) \neq \emptyset$.* □

The *contra-dual* f^* of f is defined by

$$f^*(x) = f(\bar{x}).$$

For example, the contra-dual of $f = 12 \vee 23 \vee 31$ is $f^* = \bar{1}\bar{2} \vee \bar{2}\bar{3} \vee \bar{3}\bar{1}$, where $\bar{i}$ stands for literal $\bar{x}_i$. By definition, $T(f^*) = \{\bar{x} \mid x \in T(f)\}$, and hence $|T(f)| = |T(f^*)|$. It is known that the four operations, identity, d, $*$ and complementation, are idempotent, commute and satisfy the relation $\alpha\beta = \gamma$, where α, β, γ are three different operations: $(\bar{f})^d = \overline{(f^d)} = f^*$, $(\bar{f})^* = \overline{(f^*)} = f^d$, $(f^d)^* = (f^*)^d = \bar{f}$ and so on. It is also trivial to see that: $(fg)^* = f^* g^*$, $(f \vee g)^* = f^* \vee g^*$, $f \leq g \Rightarrow f^* \leq g^*$, and so on.

1.3 Decompositions of positive dual-minor functions

Let f be a positive dual-minor function. Then f is called *k-decomposable* if f can be represented by

$$f = f_1 f_2 \dots f_k, \tag{1}$$

where f_j, $j = 1, 2 \dots, k$, are all positive and self-dual. An equivalent representation is

$$f^d = f_1 \vee f_2 \dots \vee f_k.$$

A decomposition (1) is called *minimal* if none of its components f_j can be deleted from the expression. In [7] we have studied a special class of decompositions called *canonical decompositions.* For this, let f and g be functions, and define the *extension of f with respect to g* by

$$f \uparrow g = f \vee f^d g, \tag{2}$$

which may be considered as a generalization of Shannon's decomposition (e.g., [7, 16, 20]). This extension has an important property that, if g is self-dual and f is dual-minor, then $f \uparrow g$ is self-dual. In particular, since every variable x_i itself is a positive self-dual function, $f \uparrow x_i$ is self-dual. A decomposition of $f = f_1 f_2 \cdots f_k$ is called a *canonical decomposition*, if each component f_j is such an extension of f by a single variable. Now, let t be a positive term (i.e. a conjunction of uncomplemented literals) $t = \bigwedge_{i \in P} x_i$. Then

$$f \uparrow t = \bigwedge_{i \in P} f \uparrow x_i$$

holds. We say that t *induces* a canonical decomposition if $f = f \uparrow t$. Conversely, it is easy to see that any canonical decomposition is induced by some term. The next lemma proved in [7] is fundamental in characterizing (minimal) canonical decompositions.

Lemma 3 [7] *Let f be a positive dual-minor function and let $t = \bigwedge_{i \in P} x_i$ be a positive term.*

(i) *t induces a canonical decomposition of f if and only if $t \leq f \vee f^*$.*

(ii) *t induces a minimal canonical decomposition of f if and only if $t \leq f \vee f^*$ and $\bigwedge_{i \in P \setminus \{j\}} x_i \not\leq f \vee f^*$ for all $j \in P$.* □

2 Minimum Decompositions

2.1 Minimum decomposition and chromatic number

In this subsection we show that the minimum decomposition size of a positive dual-minor function f equals the chromatic number of a graph G_f associated with f (for definitions and terminologies of graphs, see e.g., [3]).

Definition 1 *Let f be a positive dual-minor function, and let V_f denote the set $\min T(f^d) \setminus \min T(f)$. The graph $G_f = (V_f, E_f)$ associated with f is then defined by $(v, w) \in E_f \iff ON(v) \cap ON(w) = \emptyset$. Furthermore, let $\Delta(f)$ and $\delta(f)$ denote the size of a minimum decomposition and the size of a minimum canonical decomposition of f, respectively, and let $\chi(f)$ denote the chromatic number of G_f (i.e., the minimum number of colors needed to color all vertices in V_f so that no adjacent vertices receive the same color).*

Theorem 1 *Let f be a positive dual-minor function. Then $\Delta(f) = \chi(f)$.* □

2.2 Complexity of minimum decomposability

By Theorem 1, in order to compute $\Delta(f)$ for a positive dual-minor function f, we first construct a graph G_f by dualizing f, and then compute $\chi(f)$. However, this algorithm is not of polynomial time, since the number of prime implicants in the dual may be exponentially more than that of the original function. Furthermore, it is unlikely to have a polynomial time algorithm whatever, because we show in this section that the problem of minimum decomposition is co-NP-hard.

We first discuss the complexity of k-decomposability, assuming that $\min T(f)$ is given, and subsequently the same question for the case in which $\min T(f^d)$ is given.

Problem k-DECOMPOSABILITY
Input: A positive dual-minor function f, i.e., $\min T(f)$.
Question: Is f k-decomposable ?

Theorem 2 *For $k \geq 2$, problem k-DECOMPOSABILITY is co-NP-hard.* □

It is noted here that whether or not k-DECOMPOSABILITY belongs to co-NP is not obvious. For example, the argument in Subsection 2.1 cannot be directly used because set V_f of G_f may contain exponentially many vertices in $|\min T(f)|$. Furthermore, this theorem does not say anything about the case $k = 1$. The complexity of 1-DECOMPOSABILITY, i.e., problem of deciding if $f = f^d$ for a positive dual-minor function f, is a major open problem [5, 10], but is unlikely to be NP-hard [12]. This problem is also polynomially equivalent to the mutual duality problem, which will be discussed in Section 3.

In the second part of this section, we will discuss the complexity of k-DECOMPOSABILITY if $\min T(f^d)$ is given instead of $\min T(f)$. This discussion is relevant because the problem of computing $\min T(f^d)$ from $\min T(f)$ is not trivial (e.g., [5, 10, 12]).

Theorem 3 *Let f be a positive dual-minor function. For $k \leq 2$, given $\min T(f^d)$, deciding if f is k-decomposable is polynomially solvable.* □

Theorem 4 *Let f be a positive dual-minor function. For $k \geq 4$, given $\min T(f^d)$, deciding if f is k-decomposable is NP-complete.* □

We remark that the above problem is still open for $k = 3$.

3 Minimal Canonical Decompositions

In this section, we concentrate on the canonical decompositions defined in section 1.3 and, based on Lemma 3, clarify their structures.

3.1 Equivalence with the mutual duality problem

We first show that the complexity of checking if a term t induces a canonical decomposition or a minimal canonical decomposition is equivalent to the mutual duality problem, whose complexity status is not known, but which is known to be equivalent to many other problems including the problem of dualizing a positive function [5, 10].

Theorem 5 *The following three problems are polynomially equivalent.*

(i) (*Mutual duality*) *Given positive functions f and g, i.e., $\min T(f)$ and $\min T(g)$, decide if $f = g^d$.*

(ii) (*Canonical decomposition*) *Given a positive dual-minor function f (i.e., $\min T(f)$) and a positive term t, decide if t induces a canonical decomposition of f.*

(iii) (*Minimal canonical decomposition*) *Given a positive dual-minor function f (i.e., $\min T(f)$) and a positive term t, decide if t induces a minimal canonical decomposition of f.* □

Given a positive term $t = \bigwedge_{j \in P} x_j$ and a positive dual-minor function f, where t induces a canonical decomposition of f, it may be sometimes asked to find a term $t^* \geq t$ that induces a minimal canonical decomposition. To solve this, we first check if $\bigwedge_{j \in P \setminus \{i\}} x_j$ induces a canonical decomposition of f for each $i \in P$; if so, let $P := P \setminus \{i\}$; otherwise, output a term $t^* = \bigwedge_{j \in P} x_j$ for the current P. By repeating this procedure until a term is output, we can obtain a desired term. Justification of this algorithm is similar to that of the algorithm for finding a minimal true vector of a positive function [5, 23]. Furthermore, this algorithm is of polynomial time if one of the problems in Theorem 5 can be solved in polynomial time.

If we start from $t = \bigwedge_{j=1}^{n} x_j$, this algorithm can be used to generate one minimal canonical decomposition of f.

3.2 All minimal canonical decompositions from prime implicants

As shown in [7], any prime implicant t of a positive dual-minor function f induces a canonical decomposition. However, in general, there may be other terms (which may not be even implicants of f) which induce minimal canonical decompositions. We consider in this subsection the condition with which all minimal canonical decompositions are obtainable from prime implicants of f.

Lemma 4 *Let f be a positive dual-minor function. Then the set of all prime implicants of f precisely induces all minimal canonical decompositions of f if and only if*

$$\min T(f) \cap \min T(f^d) = \emptyset$$

holds. □

If $\min T(f) \cap \min T(f^d) = \emptyset$ holds, the correspondence between the set of all prime implicants and the set of minimal canonical decompositions is nominally one to one and onto, as obvious from the above proof. Here, "nominally" means that canonical decompositions $f_1 f_2 \cdots f_k$ and $f'_1 f'_2 \cdots f'_k$ are considered to be different if the inducing terms t and t' are different. However, it is sometimes possible that different terms t and t' induce the same decomposition in the sense that $f_j \equiv f'_j$ (as functions) holds for all $j = 1, 2, \ldots, k$. For example, a positive dual-minor function $f = 12 \vee 13$ and its dual $f^d = 1 \vee 23$ have $f_2 = f \vee f^d x_2 = 12 \vee 23 \vee 31$ and $f_3 = f \vee f^d x_3 = 12 \vee 23 \vee 31$. Therefore, two minimal canonical decompositions $f_1 f_2$ and $f_1 f_3$ induced by $t = 12$ and $t' = 13$ are the same, even if they are nominally different.

Theorem 6 *Let f be a positive dual-minor function. Given $\min T(f)$, checking if the set of all prime implicants of f precisely induces all minimal canonical decompositions of f can be done in polynomial time.* □

4 Path and read-once functions

As discussed in Sections 2 and 3, minimal canonical decompositions and minimum decompositions for general positive dual-minor functions appear to be intractable. Therefore, it is natural to consider nontrivial subclasses of positive dual-minor functions, for which these problems are polynomially solvable. As such a subclass, we introduce the class of positive dual-minor path functions, and show that all minimal canonical decompositions can be induced from prime implicants of f, and that the sizes of minimum canonical decompositions and minimum decompositions are equal. A polynomial total time algorithm that generates all minimal canonical decompositions is also presented.

In this section, we assume that each edge of a graph $G = (V, E)$ has a label of a positive literal x_i, where the same label x_i appears at most once. For $e \in E$, let $L(e) = i$ if x_i is the label of e, and for $S \subseteq E$, let $L(S) = \cup_{e \in S} L(e)$. A positive function f is called a *path* function if there exists a graph $G_f = (V, E)$ (with source $s \in V$ and sink $t \in V$), such that

$$\begin{aligned} \min T(f) = \{v \mid ON(v) = L(S), \\ S \subseteq E \text{ is a minimal } s\text{-}t \text{ path in } G_f\}. \end{aligned} \tag{3}$$

For example, $f = x_1x_4 \vee x_2x_5 \vee x_1x_3x_5 \vee x_2x_3x_4$ is a path function because the graph of Figure 1 satisfies (3). It is easy to see that, given a graph

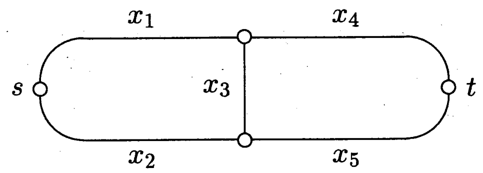

Figure 1: The graph G_f representing a path function $f = x_1x_4 \vee x_2x_5 \vee x_1x_3x_5 \vee x_2x_3x_4$.

G_f representing a path function f, $\min T(f^d)$ can be obtained by

$$\begin{aligned} \min T(f^d) = \{v \mid ON(v) = L(S), \\ S \subseteq E \text{ is a minimal } s\text{-}t \text{ cut in } G_f\}, \end{aligned} \tag{4}$$

where $S \subseteq E$ is an *s-t cut* if removing S from G_f separates s and t in the resulting graph. For the function in Figure 1, we have

$$\min T(f) = \{10010, 01001, 10101, 01110\} \tag{5}$$
$$\min T(f^d) = \{11000, 10101, 01110, 00011\}. \tag{6}$$

Path functions are well studied in reliability theory [8].

A Boolean expression is called *read-once* [1, 18] if it contains at most one occurrence of each variable, where an expression is given by using operations: conjunction, disjunction and negation. For instance, $\bar{x}_1 \vee x_2(x_3 \vee x_4\bar{x}_5)$ is a read-once expression. Read-once expressions are also called *repetition-free* [14], *irredundant* [15], *μ-formulas* [23] or *Boolean trees*. A function is called *read-once* if it has a read-once expression. It is known

[18] that a positive read-once function f is a path function whose G_f (with source $s \in V$ and sink $t \in V$) is an *s-t series-parallel graph.* In such a graph, parallel (resp. series) edges correspond to disjunctions (resp. conjunctions). For example, $x_1 \vee x_2(x_3 \vee x_4x_5)$ is represented by the series-parallel graph in Figure 2, and its $\min T(f)$ given by (3) is $\min T(f) = \{(10000), (01100), (01011)\}$.

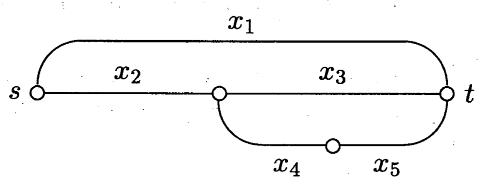

Figure 2: A read-once function $x_1 \vee x_2(x_3 \vee x_4x_5)$.

Lemma 5 *Let f be a dual-minor (but not self-dual) path function. Then the set of all prime implicants t of f precisely induces all minimal canonical decompositions of f.* □

An algorithm to generate items is called a *polynomial total time* algorithm if its running time is polynomial in the size of both input and output [17].

Theorem 7 *Let f be a dual-minor path function. Then there is a polynomial total time algorithm for computing all minimal canonical decompositions $f = f_1f_2 \cdots f_k$ (i.e., all $\min T(f_j)$ of such decompositions are output).* □

Corollary 1 *Let f be a positive dual-minor read-once function. Then there is a polynomial total time algorithm for computing all minimal canonical decompositions $f = f_1f_2 \cdots f_k$.* □

Now we turn to a *minimum* canonical decomposition.

Theorem 8 *Let f be a dual-minor path function. Then, given $\min T(f)$, a term t that induces a minimum canonical decomposition of f can be found in $O(n|\min T(f)|)$ time. Furthermore, there is a polynomial total time algorithm for computing all $\min T(f_j)$ of a minimum canonical decomposition $f = f_1f_2 \cdots f_k$.* □

Corollary 2 *Let f be a positive dual-minor read-once function. Then, given $\min T(f)$, a term t that induces a minimum canonical decomposition of f can be found in $O(n|\min\ T(f)|)$ time. Furthermore, there is a polynomial total time algorithm for computing all $\min T(f_j)$ of a minimum canonical decomposition $f = f_1f_2 \cdots f_k$.* □

Finally, we consider the relation between minimum canonical decompositions and minimum decompositions. Recall that, for a positive dual-minor function f, $\Delta(f)$ denotes the size of a minimum decomposition of f, and $\delta(f)$ denotes the size of a minimum canonical decomposition of f.

Lemma 6 *Let f be a dual-minor path function. Then $\Delta(f) = \delta(f)$.* □

Theorem 9 *Let f be a dual-minor path function. Then, given $\min T(f)$, a term t that induces a minimum decomposition of f can be found in $O(n|\min T(f)|)$ time. Furthermore there is a polynomial total time algorithm for computing all $\min T(f_j)$ of a minimum decomposition $f = f_1f_2 \cdots f_k$.* □

Corollary 3 *Let f be a positive dual-minor and read-once function. Then, given $\min T(f)$, a term t that induces a minimum decomposition of f can be found in $O(n|\min T(f)|)$ time. Furthermore there is a polynomial total time algorithm for computing all $\min T(f_j)$ of a minimum decomposition $f = f_1f_2 \cdots f_k$.* □

Conclusion

We addressed in this paper the problem of finding minimum decompositions of positive dual-minor functions, which was first studied in [7]. The complexity of k-decompositions is clarified for the cases in which $\min T(f)$ is given, and $\min T(f^d)$ is given. In the latter case the question is left open for $k = 3$. For a canonical decomposition, which was also introduced in [7], we have shown that the complexity of canonical decomposability is polynomially equivalent to the problem of mutual duality. The complexity of the latter problem is still a major open problem [5, 10], but is unlikely to be NP-hard [12]. Finally, we have shown that all these problems are solvable in polynomial time for the class of path functions.

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