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A Divide and Conquer Approach to the Minimum k-Way Cut Problem

上土井 陽子 (Yoko Kamidoi)† 若林 真一 (Shin'ichi Wakabayashi)‡

† Faculty of Information Science, Hiroshima City University
‡ Faculty of Engineering, Hiroshima University

Abstract

This paper presents algorithms for computing a minimum 3-way cut and a minimum 4-way cut of an undirected weighted graph G. Let $G=(V,E)$ be an undirected graph with n vertices, m edges and positive edge weights. Goldschmidt *et al.* presented an algorithm for the minimum k-way cut problem with fixed k, that requires $O(n^4)$ and $O(n^9)$ maximum flow computations, respectively, to compute a minimum 3-way cut and a minimum 4-way cut of G. In this paper, we first show some properties on minimum 3-way cuts and minimum 4-way cuts, which indicate a recursive structure of the minimum k-way cut problem when $k=3$ and 4. Then, based on those properties, we give divide-and-conquer algorithms for computing a minimum 3-way cut and a minimum 4-way cut of G, which require $O(n^3)$ and $O(n^4)$ maximum flow computations, respectively. This means that the proposed algorithms are the fastest ones ever known.

1. Introduction

Computing a minimum cut of a graph is one of the important problems in graph theory [3]. Let $G=(V,E)$ be an undirected graph. Given $k(\geq 2)$ disjoint nonempty subsets, $S_1, S_2, \ldots$, and S_k, of V, an edge set $C \subseteq E$ is an $(S_1, S_2, \ldots, S_k)$ *-terminal cut* of G if $G'=(V, E-C)$ has no paths from any $s \in S_i$ to any $t \in S_j$ if $i \neq j$. An edge set $C \subseteq E$ is a *k-way cut* of G if there are k disjoint vertex subsets, $Y_1, Y_2, \ldots, Y_{(k-1)}$ and Y_k, such that C is a $(Y_1, Y_2, \ldots, Y_k)$ -terminal cut of G. The cost of a cut C is defined as the total of the edge costs in C. A k-way cut C is called *minimum* if it has the smallest cut cost among any k-way cuts of G. This paper discusses the problem of finding a minimum three-way cut and a minimum four-way cut of an undirected graph G.

Dahlhaus *et al.* [2] showed that the k-terminal cut problem is NP-hard for arbitrary k and even for $k=3$. They also proposed a minimum k-terminal cut algorithm for a planar undirected graph. Gomory and Hu [4] showed that $O(n)$ executions of a minimum two-terminal algorithm is enough to compute a minimum two-way cut of an undirected graph. Goldschmidt and Hochbaum [5] showed a polynomial time algorithm for computing a minimum k-way cut for fixed k. This result showed that the k-way cut problem is easier than the k-terminal cut problem of an undirected graph. In their algorithm, the minimum two-terminal cut algorithm is repeatedly applied. The algorithm for the minimum k-way cut problem with fixed k has $O(n^{k^2-3k/2+2})$ computation time for even k and $O(n^{k^2-3k/2+5/2})$ computation time for odd k. Saran and Vazirani [10] proposed two approximation algorithms for the minimum k-way cut problem. One algorithm requires $n-1$ maximum flow computations for finding a set of twice-optimal k-way cuts, one for each value of k between 2 and n. Hao and Orlin [6] showed that the minimum 2-way cut problem can be solved in the running time for solving a single maximum flow problem. Recently, Kapoor [8] gave an algorithm for finding a minimum three-way cut, which requires $O(n^3)$ maximum flow computations. Kapoor also gave an approximation technique for the multi-way cut problem, and showed an algorithm for the minimum k-way cut problem, which requires $O(kn(m+nlogn))$ steps and gave an approximation of $2(1-1/k)$. Hochbaum and Shimoys [7] gave an $O(n^2)$ algorithm for finding a minimum 3-way cut of an unweighted planar graph.

All algorithms shown above are ordinary deterministic algorithms, and thus they can always find optimal solutions. On the other hand, Karger and Stein [9] proposed a randomized Monte Carlo algorithm which finds a minimum 2-way cut with high probability in $O(n^2log^3n)$ time. They also gave a randomized Monte Carlo algorithm for the minimum k-way cut problem, which solves the problem in $O(n^{2(k-1)}log^3n)$ time. Note that, those randomized Monte Carlo algorithms may fail to find an optimal solution, that is due to the nature of randomized Monte Carlo algorithms.

In this paper, first, we will show several properties on minimum 3-way cuts and minimum 4-way cuts, which indicate a recursive structure of the minimum k-way cut problem when $k=3$ and 4. Then, based on those properties, we will present a divide-and-conquer strategy for the minimum 3-way and 4-way cut problems, and propose two polynomial time algorithms, each of which computes a minimum 3-way cut and a minimum 4-way cut of G, respectively. These algorithms require $O(n^3)$ and $O(n^4)$ maximum flow computations, respectively. This means that the proposed algorithms are the fastest deterministic algorithms ever known. For the minimum 3-way cut problem, the number of maximum flow computations required in the algorithm is the same as one of the algorithm proposed by Kapoor [8]. For the minimum 4-way cut problem, the number of maximum flow computations required in the algorithm is very much smaller than the one proposed by Goldschmidt and Hochbaum [5], which requires $O(n^9)$ maximum flow computations.

2. Preliminaries

In the following, we give some definitions and terminologies.

Given an undirected graph $G = (V, E)$ and k mutually disjoint nonempty subsets of V, we call the problem of finding a minimum $(T_1, T_2, \ldots, T_k)$-terminal cut of G the *minimum k-terminal cut problem.* Given an undirected graph $G = (V, E)$ and an integer $k(\geq 2)$, we call the problem of finding a minimum k-way cut of G the *minimum k-way cut problem.* From definitions, any minimal $(T_1, T_2, \ldots, T_k)$-terminal cut C can be represented as a k-way cut $(V_1; V_2; \ldots; V_k)$ where $T_i \subseteq V_i$, $1 \leq i \leq k$, and $V_1 \cup V_2 \cup \ldots \cup V_k = V$.

Definition 1 *Let $G = (V, E)$ be an undirected graph. Given a nonempty vertex subset X, let $G(X) = (X, E_X)$ be an induced subgraph of G by X with the edge cost function c_X such that for any edge $e \in E_X$, $c_X(e) = c(e)$.* □

Let X be a subset of vertices of $G = (V, E)$. $\overline{X}$ is the complement of X, i.e., $\overline{X} = V - X$.

Definition 2 *For an undirected graph $G = (V, E)$, let $C = (X; \overline{X})$ and $D = (Y; \overline{Y})$ be two-way cuts of G. C is said to be intersected with D if the following four equations hold.*

$$\overline{X} \cap \overline{Y} \neq \emptyset,\ \overline{X} \cap Y \neq \emptyset,\ X \cap \overline{Y} \neq \emptyset,\ X \cap Y \neq \emptyset. \quad \square$$

Theorem 1 *Let $G = (V, E)$ be an undirected graph, and $k(\geq 2)$ be an integer. For any vertex $x \in V$, there are $(k-1)$ distinct vertices $u_1, u_2, \ldots, u_{(k-1)}$, such that a minimum $(x, u_1, u_2, \ldots, u_{(k-1)})$-terminal cut is a minimum k-way cut of G.* □

From Theorem 1, if there exists a minimum k-terminal cut algorithm for G, we can solve the minimum k-way cut problem in polynomial time by applying it in $O(n^{k-1})$ times. For example, if $k = 2$, the minimum k-terminal problem becomes the famous minimum (s, t)-terminal cut problem, which can be solved in polynomial time based on the Ford-Fulkerson's min-cut max-flow theorem [1]. Thus, the minimum 2-way cut problem can be solved by applying the min-cut max-flow algorithm in $O(n)$ times. Dahlhaus *et al.* showed, however, that for even a fixed constant $k(\geq 3)$, the minimum k-terminal cut problem for a general graph is NP-hard [2]. So, it is hopeless to devise a minimum k-way cut algorithm based on Theorem 1. For the general minimum k-way cut problem, we should adopt another approach. In this paper, we present a divide-and-conquer approach to the minimum k-way cut problem when $k = 3$ and $k = 4$, and propose polynomial time algorithms.

3. Properties

In this section, we show several properties on minimum 3-way cuts and minimum 4-way cuts of G. In the next section, those properties will be used to derive a divide-and-conquer strategy to solve the minimum 3-way and 4-way cut problems. For any k-way cut $C = (S_1; S_2; \ldots; S_k)$, let denote the cost of C, $c(C) = \sum_{e \in C} c(e)$, by $c(S_1; S_2; \ldots; S_k)$.

3.1. Properties on Three-Way Cuts

Given an undirected graph $G = (V, E)$, let c_{2min} and c_{3min} be the costs of a minimum 2-way cut and a minimum 3-way cut of G, respectively. Then, the following lemma holds.

Lemma 1 *Let $G = (V, E)$ be an undirected graph. For any minimum 3-way cut $(R; S; T)$ of G, the following holds.*

$$c_{2min} \leq \min\{c(R; \overline{R}), c(S; \overline{S}), c(T; \overline{T})\} \leq \frac{2}{3} c_{3min}.$$

[Proof] From the definition, for any 3-way cut of G, denoted $(R; S; T)$, the following holds.

$$\begin{aligned} (R; S; T) &= (R; S) \cup (S; T) \cup (R; T), \\ c(R; S; T) &= c(R; S) + c(S; T) + c(R; T), \end{aligned}$$

where $(R; S)$, $(S; T)$, and $(R; T)$ are 2-way cuts on $G(R \cup S)$, $G(S \cup T)$, and $G(R \cup T)$, respectively. From the second equation, we see that there is a 2-way cut $(Y; Z)$, $Y, Z \in \{R, S, T\}, Y \neq Z$ such that

$$c(Y; Z) \geq \frac{1}{3} c(R; S; T) = \frac{1}{3} c_{3min}.$$

Without loss of generality, we assume that $(Y; Z) = (S; T)$. Then, we have

$$c(R; S) + c(R; T) \leq \frac{2}{3} c_{3min}.$$

On the other hand, since $S \cup T = \overline{R}$, we have

$$(R; S) \cup (R; T) = (R; \overline{R}).$$

Hence, $c(R; \overline{R}) \leq \frac{2}{3} c_{3min}$.

Since $(R; \overline{R})$, $(S; \overline{S})$, and $(T; \overline{T})$ are 2-way cuts of G, it is clear that $c(X; \overline{X}) = c_{2min} \leq c(P; \overline{P})$, $P \in \{R, S, T\}$. Thus, the lemma holds. □

Assume that there is a minimum 3-way cut $(R; S; T)$ of G such that $c(R; \overline{R}) = \min\{c(R; \overline{R}), c(S; \overline{S}), c(T; \overline{T})\}$. Let $(X; \overline{X})$ be a 2-way cut of G. Then, depending on the relation between $(R; \overline{R})$ and $(X; \overline{X})$, the following Lemmas 2 and 3 hold.

Lemma 2 *Given a graph $G = (V, E)$ and a 2-way cut $(X; \overline{X})$ of G such that $c(X; \overline{X}) \leq \frac{2}{3} c_{3min}$, if there is a minimum 3-way cut $(R; S; T)$ of G such that $c(R; \overline{R}) \leq \frac{2}{3} c_{3min}$ and $(R; \overline{R})$ is intersected with $(X; \overline{X})$, then at least one of $(X; \overline{X} \cap R; \overline{X} \cap \overline{R})$ or $(\overline{X}; X \cap R; X \cap \overline{R})$ is a minimum 3-way cut of G.*

[Proof] Since $(R; \overline{R})$ is intersected with $(X; \overline{X})$, the following hold.

$$\begin{aligned} X \cap R &\neq \emptyset, \\ X \cap \overline{R} &\neq \emptyset, \\ \overline{X} \cap R &\neq \emptyset, \\ \overline{X} \cap \overline{R} &\neq \emptyset. \end{aligned}$$

Since $R = (R \cap X) \cup (R \cap \overline{X})$ and $\overline{R} = (\overline{R} \cap X) \cup (\overline{R} \cap \overline{X})$, we have

$$\begin{aligned} (R;\overline{R}) &= (((R \cap X) \cup (R \cap \overline{X})); ((\overline{R} \cap X) \cup (\overline{R} \cap \overline{X}))) \\ &= ((R \cap X); (\overline{R} \cap X)) \cup ((R \cap X); (\overline{R} \cap \overline{X})) \cup \\ &\quad ((R \cap \overline{X}); (\overline{R} \cap X)) \cup ((R \cap \overline{X}); (\overline{R} \cap \overline{X})), \\ c(R;\overline{R}) &= c((R \cap X); (\overline{R} \cap X)) + c((R \cap X); (\overline{R} \cap \overline{X})) + \\ &\quad c((R \cap \overline{X}); (\overline{R} \cap X)) + c((R \cap \overline{X}); (\overline{R} \cap \overline{X})). \end{aligned}$$

From the assumption, $c(R;\overline{R}) \leq \frac{2}{3}c(R;S;T) = \frac{2}{3}c_{3min}$. Thus, we have

$$c((R \cap X); (\overline{R} \cap X)) + c((R \cap \overline{X}); (\overline{R} \cap \overline{X})) \leq \frac{2}{3}c_{3min},$$

which implies

$$\min\{c((R \cap X); (\overline{R} \cap X)), c((R \cap \overline{X}); (\overline{R} \cap \overline{X}))\} \leq \frac{1}{3}c_{3min}.$$

On the other hand, from the assumption, we have $c(X;\overline{X}) \leq \frac{2}{3}c_{3min}$. If $c((R \cap X); (\overline{R} \cap X)) \leq c((R \cap \overline{X}); (\overline{R} \cap \overline{X}))$, then let us consider a 3-way cut $(\overline{X}; (R \cap X); (\overline{R} \cap X))$. Then, we have

$$\begin{aligned} (\overline{X}; (R \cap X); (\overline{R} \cap X)) &= (X;\overline{X}) \cup ((R \cap X); (\overline{R} \cap X)), \\ c(\overline{X}; (R \cap X); (\overline{R} \cap X)) &= c(X;\overline{X}) \cup c((R \cap X); (\overline{R} \cap X)) \\ &\leq \frac{2}{3}c_{3min} + \frac{1}{3}c_{3min} \\ &= c_{3min}. \end{aligned}$$

Thus, $(\overline{X}; (R \cap X); (\overline{R} \cap X))$ is a minimum 3-way cut of G.

If $c((R \cap \overline{X}); (\overline{R} \cap \overline{X})) \leq c((R \cap X); (\overline{R} \cap X))$, then we have a similar discussion to show that a 3-way cut $(X; (R \cap \overline{X}); (\overline{R} \cap \overline{X}))$ is a minimum 3-way cut of G. □

Lemma 3 *Given a graph $G = (V, E)$ and a 2-way cut $(X;\overline{X})$ of G, if there is a minimum 3-way cut of G, denoted $(R;S;T)$, such that $c(X;\overline{X}) \leq c(R;\overline{R})$, $R \subseteq X$, $\overline{X} \cap S \neq \emptyset$, and $\overline{X} \cap T \neq \emptyset$, then $(X;Y;\overline{Y})$ is a minimum 3-way cut of G, where $(Y;\overline{Y})$ is a minimum 2-way cut of $G(\overline{X})$.*

[Proof] Consider a 2-way cut $((\overline{X} \cap S); (\overline{X} \cap T))$ on $G(\overline{X})$. Then, we have

$$\begin{aligned} ((\overline{X} \cap S); (\overline{X} \cap T)) &\subseteq (S;T), \\ c((\overline{X} \cap S); (\overline{X} \cap T)) &\leq c(S;T). \end{aligned}$$

On the other hand, since $(Y;\overline{Y})$ is a minimum 2-way cut of $G(\overline{X})$, we have

$$c(Y;\overline{Y}) \leq c((\overline{X} \cap S); (\overline{X} \cap T)) \leq c(S;T).$$

From the assumption,

$$c(X;\overline{X}) \leq c(R;\overline{R}).$$

Therefore,

$$\begin{aligned} c(X;Y;\overline{Y}) &= c(X;\overline{X}) + c(Y;\overline{Y}) \\ &\leq c(R;\overline{R}) + c(S;T) \\ &= c(R;S;T). \end{aligned}$$

Thus, the lemma holds. □

3.2. Properties on Four-Way Cuts

Let c_{4min} be the cost of a minimum 4-way cut of G. Then, the following lemma holds.

Lemma 4 *Let $G = (V, E)$ be an undirected graph. For any minimum 4-way cut $(R;S;T;U)$ of G, the following holds.*

$$c_{2min} \leq \min\{c(R;\overline{R}), c(S;\overline{S}), c(T;\overline{T}), c(U;\overline{U})\} \leq \frac{1}{2}c_{4min}.$$

[Proof] Since $c(R;\overline{R})$, $c(S;\overline{S})$, $c(T;\overline{T})$, and $c(U;\overline{U})$ are 2-way cuts of G, it is obvious that the following holds.

$$c_{2min} \leq \min\{c(R;\overline{R}), c(S;\overline{S}), c(T;\overline{T}), c(U;\overline{U})\}.$$

Now, we consider the second part of inequations. We prove the result by contradiction. Consider a minimum 4-way cut $(R;S;T;U)$ which satisfies the following inequation.

$$\min\{c(R;\overline{R}), c(S;\overline{S}), c(T;\overline{T}), c(U;\overline{U})\} > \frac{1}{2}c_{4min}.$$

From the definition, we have

$$\begin{aligned} & (R;S;T;U) \\ = \; & (R;S) \cup (R;T) \cup (R;U) \cup (S;T) \cup (S;U) \cup (T;U), \\ & c(R;S;T;U) \\ = \; & c(R;S) + c(R;T) + c(R;U) + c(S;T) + c(S;U) + c(T;U). \end{aligned}$$

On the other hand, we also have

$$\begin{aligned} (R;\overline{R}) &= (R;S) \cup (R;T) \cup (R;U), \\ (S;\overline{S}) &= (R;S) \cup (S;T) \cup (S;U), \\ (T;\overline{T}) &= (R;T) \cup (S;T) \cup (T;U), \\ (U;\overline{U}) &= (R;U) \cup (S;U) \cup (T;U), \\ c(R;\overline{R}) &= c(R;S) + c(R;T) + c(R;U), \\ c(S;\overline{S}) &= c(R;S) + c(S;T) + c(S;U), \\ c(T;\overline{T}) &= c(R;T) + c(S;T) + c(T;U), \\ c(U;\overline{U}) &= c(R;U) + c(S;U) + c(T;U). \end{aligned}$$

Thus, we have

$$
\begin{aligned}
& (R;S;T;U) \\
= \ & (R;S) \cup (R;T) \cup (R;U) \cup (S;T) \cup (S;U) \cup (T;U) \\
= \ & (R;\overline{R}) \cup (S;\overline{S}) \cup (T;\overline{T}) \cup (U;\overline{U}). \\
& c(R;S;T;U) \\
= \ & c(R;S) + c(R;T) + c(R;U) + c(S;T) + c(S;U) + c(T;U) \\
= \ & \frac{1}{2}\{c(R;\overline{R}) + c(S;\overline{S}) + c(T;\overline{T}) + c(U;\overline{U})\}.
\end{aligned}
$$

From the assumption, we have

$$
\begin{aligned}
c(R;S;T;U) &> \frac{1}{2}\left(\frac{1}{2}c_{4min} + \frac{1}{2}c_{4min} + \frac{1}{2}c_{4min} + \frac{1}{2}c_{4min}\right) \\
&= c_{4min}.
\end{aligned}
$$

This is a contradiction. Thus, the lemma holds. □

Assume that there is a minimum 4-way cut $(R;S;T;U)$ of G such that $c(R;\overline{R}) = \min\{c(R;\overline{R}), c(S;\overline{S}), c(T;\overline{T}), c(U;\overline{U})\}$. Let $(X;\overline{X})$ be a 2-way cut of G. Then, depending on the relation between $(R;\overline{R})$ and $(X;\overline{X})$, the following Lemmas 5, 6, 7 and 8 hold.

Lemma 5 *Given a graph $G = (V, E)$ and a 2-way cut $(X;\overline{X})$ of G such that $c(X;\overline{X}) \leq \frac{1}{2}c_{4min}$, if there is a minimum 4-way cut $(R;S;T;U)$ of G such that $c(R;\overline{R}) \leq \frac{1}{2}c_{4min}$ and $(R;\overline{R})$ is intersected with $(X;\overline{X})$, then $(X \cap R; X \cap \overline{R}; \overline{X} \cap R; \overline{X} \cap \overline{R})$ is a minimum 4-way cut of G.*

[Proof] Since $(X;\overline{X})$ is intersected with $(R;\overline{R})$, the following equation hold.

$$
\begin{aligned}
X \cap R &\neq \emptyset, \\
X \cap \overline{R} &\neq \emptyset, \\
\overline{X} \cap R &\neq \emptyset, \\
\overline{X} \cap \overline{R} &\neq \emptyset.
\end{aligned}
$$

Furthermore, we have

$$(X \cap R) \cup (X \cap \overline{R}) \cup (\overline{X} \cap R) \cup (\overline{X} \cap \overline{R}) = V.$$

Thus, $(X \cap R; X \cap \overline{R}; \overline{X} \cap R; \overline{X} \cap \overline{R})$ is indeed a 4-way cut of G. Next, we show the minimality of $(X \cap R; X \cap \overline{R}; \overline{X} \cap R; \overline{X} \cap \overline{R})$. We have

$$
\begin{aligned}
& (X \cap R; X \cap \overline{R}; \overline{X} \cap R; \overline{X} \cap \overline{R}) \\
= \ & (X \cap R; X \cap \overline{R}) \cup (X \cap R; \overline{X} \cap R) \cup \\
& (X \cap R; \overline{X} \cap \overline{R}) \cup (X \cap \overline{R}; \overline{X} \cap R) \cup \\
& (X \cap \overline{R}; \overline{X} \cap \overline{R}) \cup (\overline{X} \cap R; \overline{X} \cap \overline{R}) \\
= \ & [(X \cap R; \overline{X} \cap R) \cup (X \cap R; \overline{X} \cap \overline{R}) \cup \\
& (X \cap \overline{R}; \overline{X} \cap R) \cup (X \cap \overline{R}; \overline{X} \cap \overline{R})] \cup \\
& [(X \cap R; X \cap \overline{R}) \cup (X \cap R; \overline{X} \cap \overline{R}) \cup \\
& (X \cap \overline{R}; \overline{X} \cap R) \cup (\overline{X} \cap R; \overline{X} \cap \overline{R})] \\
= \ & (X;\overline{X}) \cup (R;\overline{R}). \\
& c(X \cap R; X \cap \overline{R}; \overline{X} \cap R; \overline{X} \cap \overline{R}) \\
= \ & c(X \cap R; X \cap \overline{R}) + c(X \cap R; \overline{X} \cap R) \cup \\
& c(X \cap R; \overline{X} \cap \overline{R}) + c(X \cap \overline{R}; \overline{X} \cap R) \cup \\
& c(X \cap \overline{R}; \overline{X} \cap \overline{R}) + c(\overline{X} \cap R; \overline{X} \cap \overline{R}) \\
\leq \ & [c(X \cap R; \overline{X} \cap R) + c(X \cap R; \overline{X} \cap \overline{R}) + \\
& c(X \cap \overline{R}; \overline{X} \cap R) + c(X \cap \overline{R}; \overline{X} \cap \overline{R})] + \\
& [c(X \cap R; X \cap \overline{R}) + c(X \cap R; \overline{X} \cap \overline{R}) + \\
& c(X \cap \overline{R}; \overline{X} \cap R) + c(\overline{X} \cap R; \overline{X} \cap \overline{R})] \\
= \ & c(X;\overline{X}) + c(R;\overline{R}) \\
\leq \ & \frac{1}{2}c_{4min} + \frac{1}{2}c_{4min} \\
= \ & c_{4min}.
\end{aligned}
$$

Thus, the lemma holds. □

Lemma 6 *Given an undirected graph $G = (V, E)$ and a 2-way cut, $(X;\overline{X})$, of G, if there is a minimum 4-way cut, $(R;S;T;U)$, of G such that $R \subseteq X$, $c(X;\overline{X}) \leq c(R;\overline{R})$, $\overline{X} \cap S \neq \emptyset$, $\overline{X} \cap T \neq \emptyset$, and $\overline{X} \cap U \neq \emptyset$, then there is a minimum 4-way cut, denoted $(X;Y;Z;W)$, such that $(Y;Z;W)$ is a minimum 3-way cut of $G(\overline{X})$.*

[Proof] Since $R \subseteq X$, we have $\overline{X} \subseteq S \cup T \cup U$. Then, $(\overline{X} \cap S; \overline{X} \cap T; \overline{X} \cap U)$ is a 3-way cut of $G(\overline{X})$. Since $(Y;Z;W)$ is a minimum 3-way cut of $G(\overline{X})$, we have

$$c(Y;Z;W) \leq c(\overline{X} \cap U; \overline{X} \cap T; \overline{U} \cap U) \leq c(S;T;U).$$

Consider a 4-way cut $(X;Y;Z;W)$ of G. Then, we have the following equations.

$$
\begin{aligned}
(X;Y;Z;W) &= (X;\overline{X}) \cup (Y;Z;W) \\
c(X;Y;Z;W) &= c(X;\overline{X}) + c(Y;Z;W) \\
&\leq c(R;\overline{R}) + c(S;T;U) \\
&= c(R;S;T;U)
\end{aligned}
$$

Thus, the lemma holds. □

Lemma 7 *Given an undirected graph $G = (V, E)$, let $(X;\overline{X})$ be a 2-way cut of G. If there is a minimum 4-way cut, denoted $(R;S;T;U)$, of G such that $X = R \cup S$, then $(R';S';T';U')$ is also a minimum 4-way cut of G, where $(R';S')$ and $(T';U')$ are minimum 2-way cuts of $G(X)$ and $G(\overline{X})$, respectively.*

[Proof] From the assumption, we have $X = R \cup S$ and $\overline{X} = T \cup U$. For $G(X)$, we have

$$c(R';S') \leq c(R;S).$$

For $G(\overline{X})$, we have

$$c(T';U') \leq c(T;U).$$

Then,

$$
\begin{aligned}
c(R';S';T';U') &= c(X;\overline{X}) + c(R';S') + c(T';U') \\
&\leq c(R \cup S; T \cup U) + c(R;S) + c(T;U) \\
&= c(R;S;T;U).
\end{aligned}
$$

Thus, the lemma holds. □

Lemma 8 *Given an undirected graph $G = (V, E)$, let $(X; \overline{X})$ be a 2-way cut of G. If there is a minimum 4-way cut, denoted $(R; S; T; U)$, of G such that $(X; \overline{X})$ is intersected with $(R; \overline{R})$ and $(S; \overline{S})$, $X \subset R \cup S$, and $c(X; \overline{X}) \leq \min\{c(R; \overline{R}), c(S; \overline{S}), c(T; \overline{T}), c(U; \overline{U})\}$, then $(R'; S'; T'; U')$ is also a minimum 4-way cut of G, where $(R'; S')$ and $(T'; U')$ are minimum 2-way cuts of $G(X)$ and $G(\overline{X})$, respectively.*

[Proof] Without loss of generality, we assume that $c(R; \overline{R}) \leq c(S; \overline{S})$. Since $(X; \overline{X})$ is intersected with $(R; \overline{R})$, $(R \cap X; R \cap \overline{X}; \overline{R} \cap X; \overline{R} \cap \overline{X})$ is a 4-way cut of G.

$$\begin{aligned}
& c(R \cap X; R \cap \overline{X}; \overline{R} \cap X; \overline{R} \cap \overline{X}) \\
\leq\ & c(R; \overline{R}) + c(X; \overline{X}) \\
\leq\ & \frac{1}{2}\{c(R; \overline{R}) + c(S; \overline{S})\} + \\
& \min\{c(R; \overline{R}), c(S; \overline{S}), c(T; \overline{T}), c(U; \overline{U})\} \\
\leq\ & \frac{1}{2}\{c(R; \overline{R}) + c(S; \overline{S}) + c(T; \overline{T}) + c(U; \overline{U})\} \\
=\ & \frac{1}{2}\{2 \times (c(R; S) + c(R; T) + c(R; U) + \\
& c(S; T) + c(S; U) + c(T; U))\} \\
=\ & c(R; S; T; U) \\
=\ & c_{4min}.
\end{aligned}$$

Thus, $(R \cap X; R \cap \overline{X}; \overline{R} \cap X; \overline{R} \cap \overline{X})$ is a minimum 4-way cut of G. Since $(R \cap X; \overline{R} \cap X)$ and $(R \cap \overline{X}; \overline{R} \cap \overline{X})$ are 2-way cuts of $G(X)$ and $G(\overline{X})$, respectively, we have

$$\begin{aligned}
c(R'; S') &\leq c(R \cap X; \overline{R} \cap X), \\
c(T'; U') &\leq c(R \cap \overline{X}; \overline{R} \cap \overline{X}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
& c(R \cap X; R \cap \overline{X}; \overline{R} \cap X; \overline{R} \cap \overline{X}) \\
=\ & c(X; \overline{X}) + c(R \cap X; \overline{R} \cap X) + c(R \cap \overline{X}; \overline{R} \cap \overline{X}) \\
\geq\ & c(X; \overline{X}) + c(R'; S') + c(T'; U') \\
=\ & c(R'; S'; T'; U')
\end{aligned}$$

Thus, the lemma holds. □

4. A Divide and Conquer Approach

In this section, first, we show a recursive structure of minimum 3-way cuts and minimum 4-way cuts of an undirected graph G. Then, we present two main theorems, which will be a base to construct algorithms for computing a minimum 3-way cut and a minimum 4-way cut of G.

Lemma 9 *Given an undirected graph $G = (V, E)$, let $(X; \overline{X})$ be a 2-way cut. Let $(Y; \overline{Y})$ and $(Z; \overline{Z})$ be minimum 2-way cuts of $G(\overline{X})$ and $G(X)$, respectively. If there is a minimum 3-way cut $(R; S; T)$ of G such that $c(X; \overline{X}) \leq \min\{c(R; \overline{R}), c(S; \overline{S}), c(T; \overline{T})\}$, then at least one of the following four properties holds.*

(i) $(X; Y; \overline{Y})$ is a minimum 3-way cut of G.
(ii) $(\overline{X}; Z; \overline{Z})$ is a minimum 3-way cut of G.
(iii) There is a minimum 3-way cut, denoted $(R'; S'; T')$, such that $X \subset R'$.
(iv) There is a minimum 3-way cut, denoted $(R''; S''; T'')$, such that $\overline{X} \subset R''$.

[Proof] Without loss of generality, we assume that $c(R; \overline{R}) = \min\{c(R; \overline{R}), c(S; \overline{S}), c(T; \overline{T})\}$. Consider the relation between $(X; \overline{X})$ and $(R; \overline{R})$. Then, there are four cases. That is, (1) $(X; \overline{X})$ is intersected with $(R; \overline{R})$, (2) $X \subset R$, (3) $R \subseteq X$, and (4) $X \cap R = \emptyset$.

First, consider the case (1). From Lemma 2, at least $(X; (\overline{X} \cap R); (\overline{X} \cap \overline{R}))$ or $(\overline{X}; (X \cap R); (X \cap \overline{R}))$ is a minimum 3-way cut of G. Consider the case that $(X; (\overline{X} \cap R); (\overline{X} \cap \overline{R}))$ is a minimum 3-way cut of G. In this case, $((\overline{X} \cap R); (\overline{X} \cap \overline{R}))$ is a 2-way cut of $G(\overline{X})$. Then, since $(Y; \overline{Y})$ is a minimum 2-way cut of $G(\overline{X})$, we have $c(Y; \overline{Y}) \leq c((\overline{X} \cap R); (\overline{X} \cap \overline{R}))$. Therefore, we have

$$c(X; Y; \overline{Y}) \leq c(X; (\overline{X} \cap R); (\overline{X} \cap \overline{R})) = c_{3min}.$$

Thus, $(X; Y; \overline{Y})$ is a minimum 3-way cut of G. For the case that $(\overline{X}; (X \cap R); (X \cap \overline{R}))$ is a minimum 3-way cut of G, we have a similar discussion to show that $(\overline{X}; Z; \overline{Z})$ is a minimum 3-way cut of G. Consequently, for the case (1), at least one of the properties (i) or (ii) is satisfied.

Next, consider the case (2). In this case, it is clear that the property (iii) is satisfied.

Next, consider the case (3). This case is further classified into the following cases. That is, (3-1) $\overline{X} \cap S \neq \emptyset$, and $\overline{X} \cap T \neq \emptyset$, (3-2) there is a P such that $X = R \cup P$, $P \in \{S, T\}$, and (3-3) there is a Q such that $\overline{X} \subset Q$, $Q \in \{S, T\}$.

Consider the case (3-1). From Lemma 3, we see that the property (i) holds. Consider the case (3-2). In this case, it is clear that the property (ii) holds. Consider the case (3-3). In this case, we see that the property (iv) holds.

Finally, consider the case (4). In this case, we have $R \subseteq \overline{X}$. Let $X' = \overline{X}$. Then, this is the same case as the case (3). Thus, the lemma holds. □

Lemma 10 *Given an undirected graph $G = (V, E)$, let $(X; \overline{X})$ be a 2-way cut. Let $(Y; \overline{Y})$ and $(Z; \overline{Z})$ be minimum 2-way cuts of $G(\overline{X})$ and $G(X)$, respectively. Let $(R; S; T)$ and $(R'; S'; T')$ be minimum 3-way cuts of $G(\overline{X})$ and $G(X)$, respectively. If there is a minimum 4-way cut $(A; B; C; D)$ of G such that $c(X; \overline{X}) \leq \min\{c(A; \overline{A}), c(B; \overline{B}), c(C; \overline{C}), c(D; \overline{D})\}$, then at least one of the following five properties holds.*

(i) $(X; R; S; T)$ is a minimum 4-way cut of G.
(ii) $(\overline{X}; R'; S'; T')$ is a minimum 4-way cut of G.
(iii) $(Y; \overline{Y}; Z; \overline{Z})$ is a minimum 4-way cut of G.
(iv) There is a minimum 4-way cut, denoted $(A'; B'; C'; D')$, such that $X \subset A'$.
(v) There is a minimum 4-way cut, denoted $(A''; B''; C''; D'')$, such that $\overline{X} \subset A''$.

[Proof] Without loss of generality, we assume that $c(A; \overline{A}) = \min\{c(A; \overline{A}), c(B; \overline{B}), c(C; \overline{C}), c(D; \overline{D})\}$. Consider the relation between $(X; \overline{X})$ and $(A; \overline{A})$. Then, there are four cases. That is, (1) $(X;\overline{X})$ is intersected with $(A;\overline{A})$, (2) $X \subset A$, (3) $A \subseteq X$, and (4) $X \cap A = \emptyset$.

First, consider the case (1). From Lemma 5, $(X \cap A; X \cap \overline{A}; \overline{X} \cap A; \overline{X} \cap \overline{A})$ is a minimum 4-way cut of G. Since $(X \cap A; X \cap \overline{A})$ and $(\overline{X} \cap A; \overline{X} \cap \overline{A})$ are 2-way cuts of $G(X)$ and $G(\overline{X})$, respectively, we have

$$\begin{aligned} c(Z;\overline{Z}) &\leq c(X \cap A; X \cap \overline{A}), \\ c(Y;\overline{Y}) &\leq c(\overline{X} \cap A; \overline{X} \cap \overline{A}). \end{aligned}$$

Therefore, we have

$$\begin{aligned} & c(Y;\overline{Y};Z;\overline{Z}) \\ =\ & c(X;\overline{X}) + c(Y;\overline{Y}) + c(Z;\overline{Z}) \\ \leq\ & c(X;\overline{X}) + c(\overline{X} \cap A; \overline{X} \cap \overline{A}) + c(X \cap A; X \cap \overline{A}) \\ =\ & c(X \cap A; X \cap \overline{A}; \overline{X} \cap A; \overline{X} \cap \overline{A}) \\ =\ & c_{4min}. \end{aligned}$$

Thus, the property (iii) holds.

Next, consider the case (2). For this case, the property (iv) holds.

Next, consider the case (3). This case is further classified into the following four cases. That is, (3-1) $\overline{X} \cap B \neq \emptyset$, $\overline{X} \cap C \neq \emptyset$, and $\overline{X} \cap D \neq \emptyset$, (3-2) there are $L, M, N \in \{B, C, D\}, L \neq M, M \neq N, L \neq N$, such that $X \subseteq A \cup L \cup M$, $\overline{X} \subseteq M \cup N$, (3-3) there are $L, M, N \in \{B, C, D\}, L \neq M, M \neq N, L \neq N$, such that $X \subset A \cup L \cup M \cup N$, $\overline{X} \subset M \cup N$, $(X;\overline{X})$ is intersected with $(M;\overline{M})$ and $(N;\overline{N})$, and (3-4) there is $L \in \{B, C, D\}$ such that $\overline{X} \subset L$.

Consider the case (3-1). From Lemma 6, the property (i) holds. Consider the case (3-2). If $X = A \cup L$ and $\overline{X} = M \cup N$, then from Lemma 7, the property (iii) holds. Consider otherwise. Then, we have $X \cap A \neq \emptyset$, $X \cap L \neq \emptyset$, and $X \cap M \neq \emptyset$. From the assumption, we have $c(X;\overline{X}) \leq c(N;\overline{N})$. Let $X' = \overline{X}$. Then, we see from Lemma 6 that the property (ii) holds. Next, consider the case (3-3). Let $X' = \overline{X}$. Then, from Lemma 8, we see that the property (iii) holds. Consider the case (3-4). For this case, it is obvious that the property (iv) holds.

Finally, consider the case (4). Since $X \cap A = \emptyset$, we have $A \subseteq \overline{X}$. Let $X' = \overline{X}$. Then, this is the same as the case (3). Thus, the lemma holds. □

Lemmas 9 and 10 tell us that a minimum 3-way cut and a minimum 4-way cut can be computed recursively.

Definition 3 *Let u and v be distinct vertices of a graph $G = (V, E)$. We can construct a new graph G' by fusing the two vertices, namely by replacing them by a single new vertex x such that every edge that was incident with u or v in G is now incident with x in G'. Given a subset X of V, let Shrink(G, X) be a graph obtained by fusing all the vertices in X, and removing all the self-loop edges from the resulting graph.* □

From Lemmas 9 and 10, and the definition of *Shrink*(G, X), we can show the following main theorems.

Theorem 2 *Let $G = (V, E)$ be a graph, and $(X;\overline{X})$ be a 2-way cut of G. Let $(Y;\overline{Y})$ and $(Z;\overline{Z})$ be minimum 2-way cuts of $G(\overline{X})$ and $G(X)$, respectively. Let $(R';S';T')$ be a minimum 3-way cut of Shrink(G, X), and $(R'';S'';T'')$ be a minimum 3-way cut of Shrink$(G, \overline{X})$. If there is a minimum 3-way cut $(R;S;T)$ of G such that $c(X;\overline{X}) \leq \min\{c(R;\overline{R}), c(S;\overline{S}), c(T;\overline{T})\}$, then at least one of the following 3-way cuts of G, $(X;Y;\overline{Y})$, $(\overline{X};Z;\overline{Z})$, $(R';S';T')$, and $(R'';S'';T'')$, is a minimum 3-way cut of G.* □

Theorem 3 *Let $G = (V, E)$ be a graph, and $(X;\overline{X})$ be a 2-way cut of G. Let $(Y;\overline{Y})$ and $(Z;\overline{Z})$ be minimum 2-way cuts of $G(\overline{X})$ and $G(X)$, respectively. Let $(R;S;T)$ and $(R';S';T')$ be minimum 3-way cuts of $G(\overline{X})$ and $G(X)$, respectively. Let $(A';B';C';D')$ and $(A'';B'';C'';D'')$ be minimum 4-way cuts of Shrink(G, X) and Shrink$(G, \overline{X})$, respectively. If there is a minimum 4-way cut $(A;B;C;D)$ of G such that $c(X;\overline{X}) \leq \min\{c(A;\overline{A}), c(B;\overline{B}), c(C;\overline{C}), c(D;\overline{D})\}$, then at least one of the following 4-way cuts of G, $(X; R; S; T)$, $(\overline{X}; R'; S'; T')$, $(Y;\overline{Y};Z;\overline{X})$, $(A';B';C';D')$ and $(A'';B'';C'';D'')$, is a minimum 4-way cut of G.* □

5. Algorithms

Based on Theorems 2 and 3, we can present simple divide-and-conquer algorithms for computing a minimum three-way cut and a minimum four-way cut of an undirected graph. From Theorems 2 and 3, we find a recursive structure of the minimum 3-way and 4-way cut problems. For example, consider the minimum 3-way cut problem. Then, given a graph $G = (V, E)$, we can find a minimum 3-way cut of G by computing some combinations of minimum 2-way cuts, or by computing minimum 3-way cuts of *Shrink*(G, X) and *Shrink*$(G, \overline{X})$ for some 2-way cut $(X;\overline{X})$ of G. If both *Shrink*(G, X) and *Shrink*$(G, \overline{X})$ are smaller than G in the number of vertices, then we see that the minimum 3-way cut problem can be solved in a divide-and-conquer manner. For some minimum 3-way and 4-way cuts, denoted $(R;S;T)$ and $(R';S';T';U')$, let $c_{3-2min} = \min\{c(R;\overline{R}), c(S;\overline{S}), c(T;\overline{T})\}$, and $c_{4-2min} = \min\{c(R';\overline{R'}), c(S';\overline{S'}), c(T';\overline{T'}), c(U';\overline{U'})\}$. Then, the problem we should consider is thus the following: How do we find a 2-way cut $(X;\overline{X})$ of G such that (i) $c(X;\overline{X}) \leq c_{3-2min}$ or $c(X;\overline{X}) \leq c_{4-2min}$, and (ii) $|X| \geq 2$ and $|\overline{X}| \geq 2$?

In the following, we will show a method for finding a two-way cut satisfying the above condition.

Lemma 11 *Given an undirected graph $G = (V, E)$, let x_1, x_2, x_3, x_4 be four distinct vertices in V such that a $(\{x_1, x_2\}, \{x_3, x_4\})$-terminal cut of G is minimum in its cost among all those $(\{u, v\}, \{w, x\})$-terminal cuts of G for any four distinct vertices, u, v, w, x, in V. Let denote this $(\{x_1, x_2\}, \{x_3, x_4\})$-terminal cut by $(X;\overline{X})$. Then, if there is a minimum 3-way cut $(R;S;T)$ of G satisfying $|R|, |S|, |T| \geq 2$, then $c(X;\overline{X}) \leq c_{3-2min}$, where $c_{3-2min} = \min\{c(R;\overline{R}), c(S;\overline{S}), c(T;\overline{T})\}$. If there is a minimum 4-way cut $(R';S';T';U')$ of G satisfying $|R'|, |S'|, |T'|, |U'| \geq 2$, then $c(X;\overline{X}) \leq c_{4-2min}$, where $c_{4-2min} = \min\{c(R';\overline{R'}), c(S';\overline{S'}), c(T';\overline{T'}), c(U';\overline{U'})\}$.*

[Proof] Consider the case of finding a minimum 3-way cut. For the case of finding a minimum 4-way cut, we can prove the lemma by giving the similar discussion shown below. Without loss of generality, $c(R;\overline{R}) = c_{3-2min}$. From the assumption of $(X;\overline{X})$, there are four distinct vertices x_1, x_2, x_3, x_4 such that $x_1, x_2 \in X$ and $x_3, x_4 \in \overline{X}$, and $(X;\overline{X})$ is a minimum $(\{x_1, x_2\}, \{x_3, x_4\})$-terminal cut of G. Since $|R| \geq 2$ and $|\overline{R}| \geq 2$, we can choose two distinct vertices, say u and v, from R and two distinct vertices, say w and x, from $\overline{R}$. Let $(Y;\overline{Y})$ be a minimum $(\{u, v\}, \{w, x\})$-terminal cut of G. Then, from the assumption, it is always true that $c(X;\overline{X}) \leq c(Y;\overline{Y})$ holds for any $u, v \in R, u \neq v$ and $w, x \in \overline{R}, w \neq x$. Thus, the lemma holds. □

Based on Lemma 11, given a graph $G = (V, E)$, we will present a procedure to find a 2-way cut, $(X;\overline{X})$, of G, which satisfies (i) $|X| \geq 2$ and $|\overline{X}| \geq 2$, and (ii) $c(X;\overline{X}) \leq c_{3-2min}$ and $c(X;\overline{X}) \leq c_{4-2min}$ for any 3-way, and 4-way cuts of G. A straightforward way to find $(X;\overline{X})$ would be as follows. We enumerate all the combinations of four distinct vertices of G, say u, v, w, and x, and for each set of vertices, we find a minimum $(\{u, v\}, \{w, x\})$-terminal cut of G. Among all the combinations of four vertices, we select one set of vertices, say $\{u', v', w', x'\}$ such that the cost of a minimum $(\{u', v'\}, \{w', x'\})$-terminal cut of G is minimum among all the other combinations of four vertices. Then, let $(X;\overline{X})$ be the minimum $(\{u', v'\}, \{w', x'\})$-terminal cut of G. Note that, for given distinct four vertices of G, finding a minimum $(\{u, v\}, \{w, x\})$-terminal cut of G is easy. First, we add two new vertices s and t to G, and then add new edges (s, u), (s, v), (t, w), and (t, x). We define the costs of new edges as ∞. Then, we find a minimum (s, t)-terminal cut of G by applying a minimum 2-terminal cut algorithm.

The procedure shown above, however, would require $O(n^4)$ min-cut max-flow computations. In the following, we will show an efficient method to compute $(X;\overline{X})$, which requires $O(n^2)$ min-cut max-flow computations. First, we pay attention to the following fact.

Fact 1 *Given an undirected graph $G = (V, E)$, let $(X;\overline{X})$ be a two-way cut of G. Let $S = \{u, v, w, x\}$ be four distinct vertices in V. Let n_X and $n_{\overline{X}}$ be the numbers of vertices in S, which are contained in X and $\overline{X}$, respectively. Then, one of the following conditions holds. (i) $n_X = n_{\overline{X}} = 2$. (ii) $\max\{n_X, n_{\overline{X}}\} = 3$ and $\min\{n_X, n_{\overline{X}}\} = 1$. (iii) $\max\{n_X, n_{\overline{X}}\} = 4$ and $\min\{n_X, n_{\overline{X}}\} = 0$.* □

This Fact gives the base of our algorithm for computing $(X;\overline{X})$. Assume that a fixed set of four distinct vertices, say $S_0 = \{u_0, v_0, w_0, x_0\}$, is given in advance. For any distinct four vertices of G, say $\{u, v, w, x\}$, consider a minimum $(\{u, v\}, \{w, x\})$-terminal cut of G, denoted $(Y;\overline{Y})$. Then, from Fact 1, one of the following conditions holds.

Case (1) S_0 is partitioned into two subsets, say T and U, each of which consists of two elements, respectively, so that $(Y;\overline{Y})$ is a minimum (T, U)-terminal cut of G.

Case (2) S_0 is partitioned into two subsets, say T' and U', each of which consists of three and one elements, respectively, so that $(Y;\overline{Y})$ is a minimum $(T', U' \cup \{y\})$-terminal cut of G, where y is a vertex in G.

Case (3) S_0 is not partitioned so that $(Y;\overline{Y})$ is a minimum $(S_0, \{y, z\})$-terminal cut of G, where y and z are vertices in G.

From those results mentioned above, we can present a procedure to find $(X;\overline{X})$, which satisfies the conditions given previously. We call this procedure the procedure *Divide*(G). Due to space limitation, description of the procedure *Divide*(G) is omitted. For this procedure, we can show the following theorem.

Theorem 4 *The function Divide(G) finds a smallest cost cut in all minimum $(\{u, v\}, \{w, x\})$-terminal cuts for any distinct four vertices in G, by applying at most $O(n^2)$ min-cut max-flow computations.*

[Proof] Correctness of the function was derived from Fact 1, as we discussed previously. Since there are doubly nested loops on vertices in G, it is clear that the function, which executes min-cut max-flow computation once, was invoked in $O(n^2)$ times in total. □

5.1. The Four-way Cut Algorithm

Due to space limitation, in this paper, we only present an algorithm for computing a minimum 4-way cut of a given graph G. The proposed algorithm is based on Theorem 3. Note that, there is a special case, in which for given G, there is no minimum 4-way cut, $(R; S; T; U)$ such that $|R| \geq 2$, $|S| \geq 2$, $|T| \geq 2$, and $|U| \geq 2$. In such a case, we can not compute a minimum 4-way cut by applying the function *Divide*, and we should treat this case separately.

The following are functions, which are used in the proposed algorithm.

(i) MIN-ONE-TERM-4WAY(G) computes a smallest cost four-way cut $(\{x\}; Y; Z; W)$ in all four-way cuts constructed by a minimum three-way cut in $G(V - \{x\})$ and a two-way cut $(\{x\}; V - \{x\})$, where $x \in V$. [$O(n^3)$ maximum flow computations]

(ii) CONST-4WAY-CUT(X) constructs a four-way cut $\mathcal{C}$ of G by using the combination of $(X;\overline{X})$ and a minimum three-way cut in $G(\overline{X})$. [$O(n^2)$ maximum flow computations]

(iii) OTHER-4WAY-CUT(X) constructs a four-way cut $\mathcal{C}$ of G by using the combination of $(X;\overline{X})$, a minimum two-way cut in $G(\overline{X})$ and a minimum two-way cut in $G(X)$. [2 maximum flow computations]

(iv) ENUMERATE-ALL-4CUTS(G) enumerates all 4-way cuts of G, and returns the one with the smallest cost.

Algorithm MIN-QUADRI-PARTITION(G)
input an undirected graph $G = (V, E)$.

begin
$\mathcal{C}_0$ = MIN-ONE-TERM-4WAY(G);
$\mathcal{C}_1$ = MIN-4WAY-CUT(G);
return MIN($\mathcal{C}_0$, $\mathcal{C}_1$)
end.

Recursive Procedure MIN-4WAY-CUT(G)
input an undirected graph $G = (V, E)$.

begin
if $|V| < 6$ **then** return ENUMERATE-ALL-4CUTS(G);
else begin
$(X; \overline{X}) \leftarrow$ *Divide*(G);
$G_X \leftarrow$ *Shrink*(G, X);
$G_{\overline{X}} \leftarrow$ *Shrink*($G, \overline{X}$);
$\mathcal{C}_0 \leftarrow$ CONST-4WAY-CUT(X);
$\mathcal{C}_1 \leftarrow$ CONST-4WAY-CUT($\overline{X}$);
$\mathcal{C}_2 \leftarrow$ OTHER-4WAY-CUT(X);
$\mathcal{C}_3 \leftarrow$ MIN-4WAY-CUT(G_X);
$\mathcal{C}_4 \leftarrow$ MIN-4WAY-CUT($G_{\overline{X}}$);
return MIN($\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$)
end
end

5.2. Computation Time

Correctness of the proposed algorithms can be easily shown from Theorems 2 and 3. For the time complexity of the algorithms, we can show the following theorem.

Theorem 5 *For an undirected graph $G = (V, E)$, the algorithm MIN-TRI-PARTITION(G) and the algorithm MIN-QUADRI-PARTITION(G) compute a minimum 3-way cut and a minimum 4-way cut by applying $O(n^3)$ and $O(n^4)$ maximum flow computations, respectively.*

[Proof] In the following, we consider computation time of the algorithm MIN-TRI-PARTITION(G). Computation time of the algorithm MIN-QUADRI-PARTITION(G) can be discussed similarly.

Given a graph $G = (V, E)$, let K be the total number of invocations of the procedure MIN-3WAY-CUT in the algorithm. Them, from the description of the algorithm, it is easy to show that the algorithm invokes the min-cut max-flow procedure in $O(Kn^2)$ times. Thus, in the following, we derive an upper bound of K.

First, we define a rooted tree called *computation tree* $T = (N, A)$ as follows. Each vertex, v, of T has a weight, denoted $w(v)$. Each vertex in T corresponds to an invocation of MIN-3WAY-CUT in the algorithm. The root of T corresponds to the first invocation of MIN-3WAY-CUT, whose actual parameter is G itself. Assume that G' is an input graph of MIN-3WAY-CUT, and applying *Divide* to G', two new graphs, G_X = *Shrink*(G', X) and $G_{\overline{X}}$ = *Shrink*($G', \overline{X}$) are produced. Then, in T, there are three vertices, u, v, and w, which correspond to G', G_X, and $G_{\overline{X}}$, respectively, and there are edges (u, v) and (u, w). The weights of u, v, and w are the number of vertices in G', G_X, and $G_{\overline{X}}$. For simplicity, we assume that in the algorithm, if a given graph has more than three vertices, then MIN-3WAY-CUT will be applied to continue the recursive calls of MIN-3WAY-CUT, although, in the actual algorithm, if a given graph has less than six vertices, the recursive calls will terminate. Then, the weight of a vertex has the following properties. (i) Let r be the root of T. Then, $w(r) = |V| = n$. (ii) For each internal vertex v, let u and w be its left and right sons, respectively. Then, $w(v) \geq 4$, $w(u) \geq 3$, $w(w) \geq 3$, and $w(v) + 2 = w(u) + w(w)$. (iii) For each leaf v, $w(v) = 3$.

Now, it is clear that T is a full binary tree, i.e., a binary tree whose any internal vertex has left and right sons. Let $I(T)$ and $L(T)$ be the numbers of internal vertices and leafs, respectively. Then, we can easily show that $L(T) = I(T) + 1$. Let SUM be the total of weights of all leafs. Then, from the properties of the weights of vertices, we can show that $SUM = w(r) + I(T) \times 2$. On the other hand, it is obvious that $SUM = L(T) \times 3$. Since $w(r) = n$, we have $n + I(T) \times 2 = L(T) \times 3$. Substituting the equation $L(T) = I(T) + 1$, we finally get $n + I(T) \times 2 = (I(T) + 1) \times 3$. Thus, we have $n - 3 = I(T)$ and $L(T) = I(T) + 1 = n - 2$. Consequently, the total number of invocation of MIN-3WAY-CUT is $I(T) + L(T) = 2n - 5$. This shows that the algorithm MIN-TRI-PARTITION invokes the min-cut max-flow procedure in $O(n^3)$ times. □

Note that there have been a number of min-cut max-flow algorithms [1]. Time complexity of finding a minimum (s, t)-terminal cut of a general undirected weighted graph G is bounded by $O(n^3)$.

6. Conclusion

We have presented divide-and-conquer algorithms for computing a minimum three-way cut and a minimum four-way cut of an undirected weighted graph. As future work, we will consider an extension of the proposed algorithms for the minimum k-way cut problem for general $k \geq 5$.

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