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1次元弾性体のダイナミクスとソリトン理論

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1 Introduction

Research on the dynamics of large deformation of one-dimensional elastic media, such as rods, belts and cables are important for engineering and its applications. Such a nonlinear problem, however, is difficult to handle theoretically because of our shortage of mathematical methods.

Recently, Goldstein and Petrich [1] have discovered the connection between dynamics of curves and soliton theory. They have shown that the dynamics of a curve in a plane is governed geometrically by the modified KdV (mKdV) equation in a particular case. Their theory has some advantages: we can treat exact nonlinear motion of curves analytically by using powerful methods of soliton theory. Another is that their formalism is so general that there is a possibility to apply the theory to various one-dimensional phenomena, such as the growth of boundaries of crystals and so on.

In their analysis, however, the dynamics is determined by assuming the velocities of curves *a priori*, not by using the basic equations of motion of the curve. Moreover, the properties of the curve, such as the constitutive equation, are not taken into consideration. Therefore we cannot use their theory directly to the problem of elastic rods.

The aim of our study is to show the way of utilizing the new tool of the soliton approach to analyze the nonlinear deformation of real elastic rods, and to clarify the meaning of their assumption of velocities from the physical point of view. Moreover, we construct discrete model for an extensible string to simulate dynamics of the rod. We use exact solutions of the mKdV equation, such as the one-soliton and the breather to study the dynamics of the rod. The analysis is performed by some perturbation methods and numerical calculations.

2 Basic equations for an elastic rod

We consider basic equations of motion for an elastic rod in a plane. Let $\mathbf{r}(\sigma, t)$ denote the position vector of the rod at time t, and σ is a parameter which represents the unstretched length of the rod. There is a metric g on the rod defined by

$$g = \frac{\partial \mathbf{r}}{\partial \sigma} \cdot \frac{\partial \mathbf{r}}{\partial \sigma}, \tag{1}$$

then the arclength s is given by

$$s = \int_0^{\sigma} \sqrt{g(\sigma', t)}\, d\sigma'. \tag{2}$$

The arclength s is used as a coordinate of the rod in the following. The unit tangent vector $\mathbf{t}$ of the rod is

$$\mathbf{t} = \frac{\partial \mathbf{r}}{\partial s} = g^{-1/2} \frac{\partial \mathbf{r}}{\partial \sigma} \tag{3}$$

and the unit vector normal to $\mathbf{t}$ is represented by $\mathbf{n}$, which is related to the curvature of the rod κ and $\mathbf{t}$ by the Serret-Frenet formula in its two-dimensional version

$$\frac{\partial}{\partial s} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix}, \tag{4}$$

where $\kappa = \partial\theta/\partial s$ and θ is an angle between the x-axis and $\mathbf{t}$.

Basic equations for motion of the rod in $\{\mathbf{t}, \mathbf{n}\}$ frame are written as [2]

$$\frac{\rho A}{\sqrt{g}} \frac{d^2 \mathbf{r}}{dt^2} = \left(\frac{\partial N}{\partial s} + \kappa Q + q_t \right) \mathbf{t} + \left(-\frac{\partial Q}{\partial s} + \kappa N + q_n \right) \mathbf{n} \tag{5}$$

$$\frac{\rho I}{\sqrt{g}}\frac{d^2\theta}{dt^2} = \frac{\partial M}{\partial s} - Q. \tag{6}$$

The right-hand sides (r.h.s.) of (5) and (6) represent the resultant force and moment acting on each material segment, respectively, and M is the moment, N the axial force, Q shear force, ρ the density, A the cross sectional area, I the geometrical moment of inertia of the rod. In (5), q_t and q_n are applied forces tangential and normal to the rod, respectively, and we will neglect these external forces in the following study. The total derivative with respect to time is written by using (2) as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{ds}{dt}\frac{\partial}{\partial s} = \frac{\partial}{\partial t} + \left(\int \frac{d}{dt}\ln\sqrt{g}ds\right)\frac{\partial}{\partial s}, \tag{7}$$

and the last term in (7) represents the stretching effect of the rod.

The constitutive equation for N we use in this study is

$$N = EA(\sqrt{g} - 1), \tag{8}$$

where E is the Young's modulus of the rod. For the constitutive equation of M, we adopt the Bernoulli-Euler assumption, and considering the stretching effect of the neutral axis, we can write

$$M = EI\kappa\sqrt{g}. \tag{9}$$

Subtracting Q from (5) and (6) and substituting constitutive equations (8) and (9) into basic equations, we obtain

$$\begin{aligned}\frac{\rho A}{\sqrt{g}}\frac{d^2\mathbf{r}}{dt^2} &= \left(EA\frac{\partial\sqrt{g}}{\partial s} + EI\kappa\frac{\partial\kappa\sqrt{g}}{\partial s} - \rho I\frac{\kappa}{\sqrt{g}}\frac{d^2\theta}{dt^2}\right)\mathbf{t} \\ &+ \left(EA\kappa(\sqrt{g}-1) - EI\frac{\partial^2\kappa\sqrt{g}}{\partial s^2} + \rho I\frac{\partial}{\partial s}\frac{1}{\sqrt{g}}\frac{d^2\theta}{dt^2}\right)\mathbf{n}. \end{aligned} \tag{10}$$

Non-dimensional form of this equation is rewritten as

$$\begin{aligned}\frac{1}{\sqrt{g}}\frac{d^2\mathbf{r}}{dt^2} &= \left(\frac{\partial\sqrt{g}}{\partial s} + \gamma\kappa\frac{\partial\kappa\sqrt{g}}{\partial s} - \frac{\gamma\kappa}{\sqrt{g}}\frac{d^2\theta}{dt^2}\right)\mathbf{t} \\ &+ \left(\kappa(\sqrt{g}-1) - \gamma\frac{\partial^2\kappa\sqrt{g}}{\partial s^2} + \gamma\frac{\partial}{\partial s}\frac{1}{\sqrt{g}}\frac{d^2\theta}{dt^2}\right)\mathbf{n}. \end{aligned} \tag{11}$$

In (11), $\mathbf{r}$ and s are normalized by s_0, which is a typical width of wave occurring in the rod, κ is normalized by $1/s_0$, A and I are normalized by y_0^2 and y_0^4, respectively, where y_0 is a diameter of the rod. The velocity v_0 is normalized by the linearized longitudinal velocity $\sqrt{E/\rho}$, and t is normalized by t_0, where $s_0/t_0 = v_0$. The non-dimensional parameter $\gamma = (y_0/s_0)^2$ represents narrowness of the rod, which is considered to be quite small in this paper.

3 Analysis of curves by the soliton theory

In this section, we summarize the analysis of curves by the soliton approach [1], [3]. Let the curve dynamics in the plane be of the form

$$\frac{d\mathbf{r}}{dt} = U\mathbf{t} + W\mathbf{n}, \tag{12}$$

where $\mathbf{r}, \mathbf{t}$, etc. are defined as in the previous section. It must be noted here that s and t derivatives do not commute in general: from (3), (4) and (12), we obtain

$$[\frac{d}{dt}, \frac{\partial}{\partial s}] = -\left(\frac{\partial U}{\partial s} - \kappa W\right)\frac{\partial}{\partial s}, \tag{13}$$

where $[\cdot,\cdot]$ is the usual Poisson bracket. By using (4), (12) and (13), we can show the following equations:

$$\frac{d\mathbf{t}}{dt} = \left(\frac{\partial W}{\partial s} + \kappa U\right)\mathbf{n}, \tag{14}$$

$$\frac{d\mathbf{n}}{dt} = -\left(\frac{\partial W}{\partial s} + \kappa U\right)\mathbf{t}, \tag{15}$$

$$\frac{d}{dt}\ln\sqrt{g} = \frac{\partial U}{\partial s} - \kappa W, \tag{16}$$

$$\frac{\partial \kappa}{\partial t} = \frac{\partial^2 W}{\partial s^2} + \kappa^2 W + \frac{\partial \kappa}{\partial s}\int \kappa W ds \equiv RW, \tag{17}$$

where the *recursion operator* R is defined by

$$R = \frac{\partial^2}{\partial s^2} + \kappa^2 + \frac{\partial \kappa}{\partial s}\int ds\, \kappa. \tag{18}$$

The crucial assumption of the soliton approach is that we seek solutions with

$$W = R^n \frac{\partial \kappa}{\partial s}, \tag{19}$$

then from (17) we obtain the dynamics of the curvature

$$\frac{\partial \kappa}{\partial t} = R^{n+1} \frac{\partial \kappa}{\partial s}. \tag{20}$$

For the lowest order case $n = 0$, we obtain the mKdV equation

$$\frac{\partial \kappa}{\partial t} = \frac{\partial^3 \kappa}{\partial s^3} + \frac{3}{2}\kappa^2 \frac{\partial \kappa}{\partial s}. \tag{21}$$

In this paper, we call (20) the (n+1)th-order mKdV equation, which is completely integrable by the soliton theory for each n.

Therefore we solve the (n+1)th-order mKdV equation and substitute solutions into

$$\frac{dx}{ds} = \cos\left(\int \kappa ds\right), \quad \frac{dy}{ds} = \sin\left(\int \kappa ds\right), \tag{22}$$

to obtain shapes of certain classes of dynamically moving curves. We note here that the assumption (19) for the form of the normal velocity has no clear physical meaning. Accordingly, it can be expected to be compatible with the basic equations of motion for real materials only in special cases. In the following sections, we will consider dynamics of an elastic rod by using this assumption, justifying it in the case $n = 0$ by identifying certain mechanically consistent solutions.

4 Comparison and symmetry

4.1 Perturbation analysis

In order to utilize the soliton approach discussed in the previous section to analyze the nonlinear deformation of real elastic rods, we must compare the dynamics of (12) with that of (11) and check in which situations the assumption of the velocity is satisfied. Differentiating (12) with respect to time gives

$$\frac{d^2\mathbf{r}}{dt^2} = \left(\frac{dU}{dt} - W\frac{\partial W}{\partial s} - \kappa UW\right)\mathbf{t} + \left(\frac{dW}{dt} + U\frac{\partial W}{\partial s} + \kappa U^2\right)\mathbf{n} \tag{23}$$

The compatibility conditions for (23) and (11) are

$$\frac{dU}{dt} - W\frac{\partial W}{\partial s} - \kappa UW = \sqrt{g}\frac{\partial\sqrt{g}}{\partial s} + \gamma\kappa\sqrt{g}\frac{\partial\kappa\sqrt{g}}{\partial s} - \gamma\kappa\frac{d^2\theta}{dt^2} \tag{24}$$

$$\begin{aligned}\frac{dW}{dt} + U\frac{\partial W}{\partial s} + \kappa U^2 &= \kappa\sqrt{g}(\sqrt{g}-1) \\ &\quad -\gamma\sqrt{g}\frac{\partial^2\kappa\sqrt{g}}{\partial s^2} + \gamma\sqrt{g}\frac{\partial}{\partial s}\frac{1}{\sqrt{g}}\frac{d^2\theta}{dt^2}\end{aligned} \tag{25}$$

The equations (16), (17), (24) and (25) for $\sqrt{g}, \kappa, W$ and U are the basic equations which connect the elastic theory and the soliton theory. In the following analysis, we assume $\kappa \to 0$ as $s \to \pm\infty$, i.e., we consider the rod as infinitely long and neglect boundary effects.

Since γ and the stretch effect are considered to be small, we will solve these equations by a perturbation method. As we can see in (20), we can set orders of variables as: $\kappa \sim \varepsilon, \partial_s \sim \varepsilon, \partial_t \sim \varepsilon^{2n+3}$ and $W \sim \varepsilon^{2n+2}$, where ε is called the booking parameter which will be related to γ afterwards. Then we expand κ and W in terms of ε as

$$\kappa = \varepsilon\kappa_0 + \varepsilon^2\kappa_1 + O\left(\varepsilon^3\right), \tag{26}$$

$$W = \varepsilon^{2n+2}w_0 + \varepsilon^{2n+3}w_1 + O\left(\varepsilon^{2n+4}\right). \tag{27}$$

In the lowest order of ε, we assume (19) in order to use the soliton approach, then we put

$$w_0 = R_0^n\frac{\partial\kappa_0}{\partial s}, \quad R_0 = \partial_s^2 + \kappa_0^2 + \frac{\partial\kappa_0}{\partial s}\int ds\kappa_0. \tag{28}$$

Next we expand $\sqrt{g}$ in terms of ε. From the balance of acceleration and axial force $\partial W/\partial t \sim \kappa\sqrt{g}(\sqrt{g}-1)$ in (25), we get the order of the perturbation as $\sqrt{g} = 1 + O(\varepsilon^{4n+4})$. Thus we can expand $\sqrt{g}$ as

$$\sqrt{g} = 1 + \varepsilon^{4n+4}G + \varepsilon^{4n+5}H + O\left(\varepsilon^{4n+6}\right). \tag{29}$$

Then we obtain from (7)

$$\frac{d}{dt} = \varepsilon^{2n+3}\frac{\partial}{\partial t} + \varepsilon^{6n+7}\frac{\partial}{\partial t}\int G ds \frac{\partial}{\partial s}. \tag{30}$$

Here we can determine U from (16) by integrating once with respect to s as

$$U = \varepsilon^{2n+2}\int \kappa_0 w_0 ds + \varepsilon^{2n+3}\int(\kappa_0 w_1 + \kappa_1 w_0)ds. \tag{31}$$

Substituting (26) and (27) into (17), we obtain from the order of ε^{2n+4} the (n+1)th-order mKdV equation

$$\frac{\partial \kappa_0}{\partial t} = R_0^{n+1}\frac{\partial \kappa_0}{\partial s} = R_0 w_0 \tag{32}$$

and from ε^{2n+5}

$$-\frac{\partial \kappa_1}{\partial t} + \frac{\partial^2 w_1}{\partial s^2} + \frac{\partial}{\partial s}\left(\kappa_0\int(\kappa_0 w_1 + \kappa_1 w_0)ds + \kappa_1\int \kappa_0 w_0 ds\right) = 0. \tag{33}$$

Here we relate ε with γ. In order to take the shearing force into account, it is natural to choose

$$\gamma = p\varepsilon^{4n+3} \tag{34}$$

by considering the balance of terms in (24) and (25), where in (34), p is a real constant. Then we substitute (31) into (24) and considering (34), we obtain from the order of ε^{4n+5}

$$\frac{\partial}{\partial t}\int \kappa_0 w_0 ds - w_0\left(\kappa_0\int \kappa_0 w_0 ds + \frac{\partial w_0}{\partial s}\right) = \frac{\partial G}{\partial s}, \tag{35}$$

from the order of ε^{4n+6}

$$\begin{aligned}&\frac{\partial}{\partial t}\int(\kappa_0 w_1 + \kappa_1 w_0)ds - \frac{\partial}{\partial s}w_0 w_1 \\ &\quad -\frac{\partial}{\partial s}\left(\int \kappa_0 w_0 ds\int(\kappa_0 w_1 + \kappa_1 w_0)ds\right) = \frac{\partial H}{\partial s} + p\kappa_0\frac{\partial \kappa_0}{\partial s}.\end{aligned} \tag{36}$$

We also substitute (31) into (25) to obtain from the order of ε^{4n+5}

$$\frac{\partial w_0}{\partial t} + \int \kappa_0 w_0 ds\left(\kappa_0\int \kappa_0 w_0 ds + \frac{\partial w_0}{\partial s}\right) = \kappa_0 G, \tag{37}$$

and from the order of ε^{4n+6}

$$\frac{\partial w_1}{\partial t} + \left(2\kappa_0 \int \kappa_0 w_0 ds + \frac{\partial w_0}{\partial s}\right) \int (\kappa_0 w_1 + \kappa_1 w_0) ds$$
$$+\frac{\partial w_1}{\partial s} \int \kappa_0 w_0 ds + \kappa_1 \left(\int \kappa_0 w_0 ds\right)^2 = \kappa_1 G + \kappa_0 H - p\frac{\partial^2 \kappa_0}{\partial s^2}. \quad (38)$$

We will solve (32), (33), (35)-(38) for κ_0, κ_1, w_1, G and H. We see that mechanical effects arise at higher order of ε (i.e., ε^{4n+5}) in comparison with the order of the dynamics for the curvature (i.e., ε^{2n+4}). This means that the dynamics of the rod is governed mainly by the geometrical constraint (17) and that the rod considered in this paper is like a "string" more than a "rod". This coincides the fact that we treat the parameter γ to be small.

4.2 Lowest order case

Let us consider the special case $n = 0$ first, i.e. κ_0 is governed by the mKdV equation. In this case, we can prove that only in the case

$$\partial_t = -v\partial_s \quad (39)$$

the basic equations are compatible [4], where v represents the soliton velocity. Therefore we find that only solutions of the travelling wave type satisfy the elastic equations, under the assumption (28) for the velocity and using this kind of perturbation analysis. In this case, the rod shows shape-steady motions.

Since $\partial_t = -v\partial_s$, the solution of the mKdV equation of the travelling wave type is given by

$$\kappa = -4\frac{\partial}{\partial s} \tan^{-1}\left(\exp(\alpha s + \alpha^3 t)\right) = -2\alpha\, \mathrm{sech}\alpha(s + \alpha^2 t). \quad (40)$$

This is a loop which propagates without deformation. The shape and the velocity of the solitary wave are determined by the value α, which is related with the initial tension of the rod. It is worth while to mention here that although the shape of the famous Elastica and the loop soliton are the same, the mechanism for force balance is different. Shapes of the Elastica are formed by the balance of stresses and an external force, while the loop soliton is the balance of stresses and the centrifugal force.

4.3 Higher order case

Let us move on to consider the case $n = 1$. In this case, we obtain the second-order mKdV equation

$$\frac{\partial \kappa}{\partial t} = \frac{\partial}{\partial s}\left(\frac{\partial^4 \kappa_0}{\partial s^4} + \frac{3}{8}\kappa_0^5 + \frac{5}{2}\left(\kappa_0(\frac{\partial \kappa_0}{\partial s})^2 + \kappa_0^2\frac{\partial^2 \kappa_0}{\partial s^2}\right)\right). \tag{41}$$

One soliton solution of this equation is given by

$$\tilde{\kappa}_1 = -4\frac{\partial}{\partial s}\tan^{-1}\left(\frac{\sin(2\alpha(s-64\alpha^4 t))}{\cosh(2\alpha(s-64\alpha^4 t))}\right). \tag{42}$$

The position of the curve in this case is given by integrating (22) as

$$x = s + \frac{\cos\Theta\ \sin\Theta - \cosh\Theta\,\sinh\Theta}{\alpha\ \cosh\Theta^2 + \alpha\ \sin\Theta^2}, \tag{43}$$

$$y = \frac{\cos\Theta\ \cosh\Theta + \sin\Theta\,\sinh\Theta}{\alpha\ \cosh\Theta^2 + \alpha\ \sin\Theta^2}, \tag{44}$$

where $\Theta = 2\,\alpha\, s - 128\,\alpha^5\, t$. This is illustrated in Fig.1. In this case, the travelling wave on the rod is no longer a simple loop, but shows more complicated deformation like a "worm."

5 Breather soliton and two-soliton on the rod

Let us consider the solution of a non-travelling wave type in this section. To avoid complicated discussions, we focus on the case of $n = 0$, i.e., κ is governed by the mKdV equation. As we prove in the previous section, only solutions of the travelling wave type satisfy the basic equations of the elastic theory under the perturbation method. Thus the idea for realization of more than two-soliton on the rod is that we put different orderings on parameters in solutions. For wxample, the breather solution for the mKdV equation is given by

$$\kappa = -4\frac{\partial}{\partial s}\tan^{-1}\left(\frac{\beta}{\alpha}\frac{\sin(2\alpha s - 8\alpha(\alpha^2 - 3\beta^2)t)}{\cosh(2\beta s + 8\beta(\beta^2 - 3\alpha^2)t)}\right). \tag{45}$$

Here we put $\beta/\alpha \ll 1$ and $\beta \ll 1$, then we can prove that (45) can satisfy the basic equations up to the perturbated order. This is illustrated in Fig.2. We see that in this breather case, the envelope will propagate like a solitary wave, and the carrier wave oscillates with a high frequency.

In the case of usual two soliton solutions, we use numerical simulations to check its stability. If the difference of the two loops are small, we can see that the collision are not distructive to each loops (Fig. 3) and maintain its shape after the collision. On the other hand, in the case that the difference of the two loops are not small, the loops are largely disturbed by the collision. These are natural results from the above discussions, because if the difference of the two loops are small, then we can consider these two loops as travelling wave type "as a whole".

References

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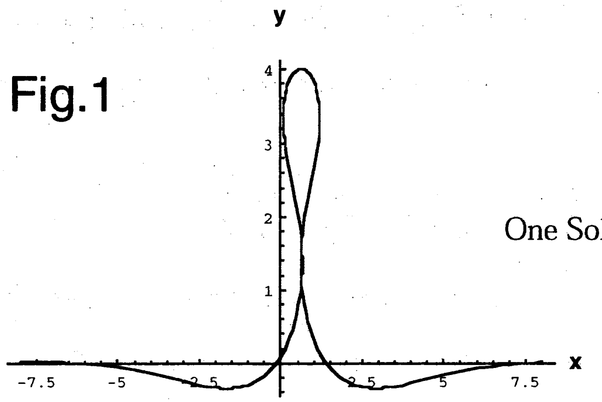
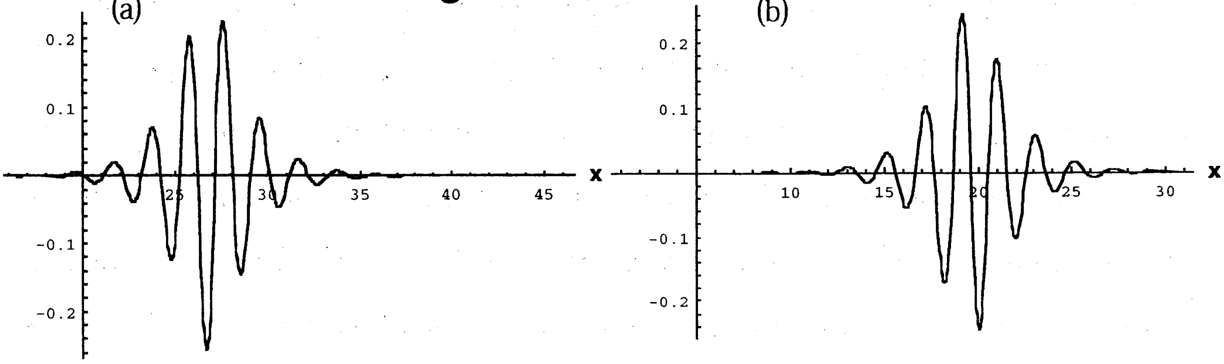
Fig.1



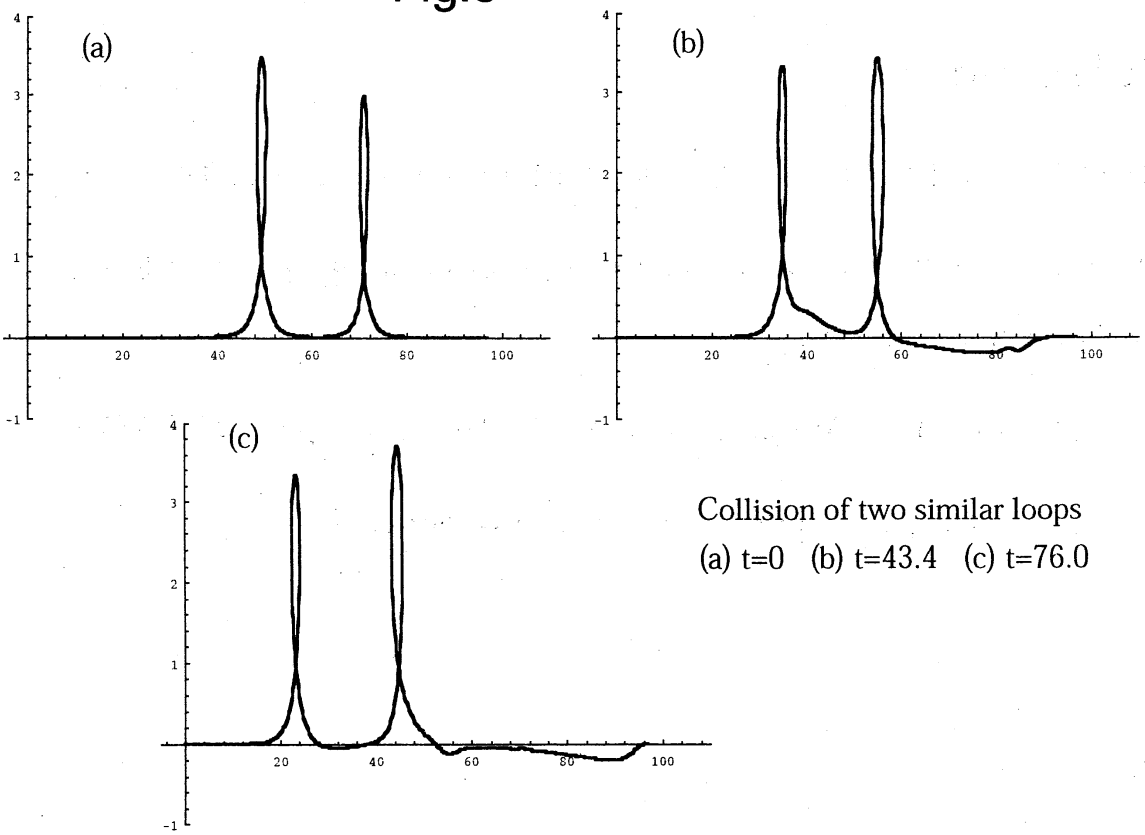
One Soliton solution of the higher mKdV equation

Fig.2



Time evolution of the breather soliton on the string (a) t=0.4 (b) t=0.7

Fig.3



Collision of two similar loops
(a) t=0 (b) t=43.4 (c) t=76.0