数理解析研究所講究録
993 巻 1997 年 183-191

Grating soliton の理論的解析

愛媛大理 飯塚剛*

January 28, 1997

Abstract

Optical solitons in fiber grating in which the dielectric constant varies periodically along it, are investigated. It is proved that the envelope of the Bloch wave obeys the nonlinear Schrödinger equation. As an application, gap solitons in a super-lattice are dicussed. The concept and method presented in this paper are useful not only for explaining recently observed Bragg grating soliton but for analyzing a large class of nonlinear periodic systems.

1 Introduction

Recently, soliton propagation in fiber grating, in which the dielectric constant changes periodically along it, was observed[1] for the first time. One of the remarkable points in this experiment is that the dispersion is due to the grating structure rather than the background medium. The large grating dispersion shortens the length for soliton formationn by several orders, which implies advantages in technological applications. The band structure of the dispersion relation allows the solitons to propagate at much less speed than the light speed in the conventional uniform fiber[1].

Nonlinear wave propagations in periodic optical fiber have been studied mainly by theoretical approaches. [2]~[8] One of the most important results of these works is that the stable pulse exists due to the balance between the nonlinearity associated with the Kerr effect and the dispersion associated with the periodic variation in the dielectric constant along the fiber. The gap soliton[2] is one of such a pulse whose spectrum lies in the photonic band gap. There reported two kinds of theoretical approaches (except fot numerical simulations).

The first one[3]~[6] is based on a set of nonlinear equations describing a coupling between forward and backward traveling waves. The soliton-like solution for the nonlinear coupled equations which are considered to be a generalization of the massive Thirring model, has been obtained[6]. In deriving the coupled equations, we impose the following two assumptions. One is that the frequency of the traveling waves satisfies the Bragg condition[3]. The other is that the refractive index $n(x)$ have the following form[3]

$$n(x) = n_0 + n_1 \cos \alpha x, \qquad (n_1 \ll n_0) \tag{1.1}$$

where n_0, n_1 and α are constants and x represents a coordinate along the fiber. Fiber gratings have been fabricated by holographic setup[9] or phase mask scanning[10]. By modifying the writing intensity, gratings with complex coupling coefficient profile can be written to suit specific applications[10]. Therefore, the restriction on the grating (1.1) is undesirable for general analysis.

The second one[7, 8] is based on the nonlinear Schrödinger (NLS) equation. This approach is more general[8] compared to the first one because we do not impose the Bragg condition and can

*e-mail: iizuka@sci.ehime-u.ac.jp

choose any periodic function $n(x)$. A key point of this analysis is the modulation of Bloch waves of the underlying periodic structure. The envelope function of the Bloch wave obeys the NLS equation which is usually obtained as the modulation of the monochromatic wave in 'uniform' media. However, in the analysis[7, 8] the normalization length L is introduced. The Bloch function is assumed to be periodic with a period L. This restriction seems too strong because the wave number of the Bloch wave (carrier) has to be special value and the spectrum of the wave to be discrete. In realistic situations the carriers is not necessarily periodic, and the spectrum is continuum.

A general method for Bloch wave modulations in nonlinear media has been presented[11, 12] appled to some nonlinear models. Comparing with the above mentioned theory[7, 8], it has the following two advantages. First, in our method the Bloch waves needs not to be periodic. Thus the wave number of the Bloch wave can be set any real number. Second, we can directly include the higher correction terms. Due to the periodic structure, the reductive perturbation demands some additional terms which disappear for the homogeneous system. To calculate the correction terms seems difficult in the former theory[7, 8] (e_2, e_3 in the papers). In our theory[11, 12] however, explicit form of the higher correction terms are calculated directly form the Bloch wave.

The main purpose of the present article is to analyze the solitons in fiber grating by using the recently developed theory[11, 12]. In the next section we start from a fundamental nonlinear equation for electric field in the fiber grating. The modulation of the Bloch wave will be shown to obey the nonlinear Schrödinger equation. Section 3 is devoted to analysis of a typical fiber grating, where the dielectric constant changes alternatively in a stepwise manner[2] along the fiber. Explicit form of the gap soliton is shown as an example of the grating soliton. Discussions are given in the last section.

2 The Bloch wave modulation

Let us consider electromagnetic wave propagations along the optical fiber grating. We denote the electric field by $E(x,t)$, nonlinear polarization by $N(x,t)$ and the dielectric function by $\epsilon(x)$. The fundamental equation is given by[7]

$$c^2\frac{\partial^2 E(x,t)}{\partial x^2} - \epsilon(x)\frac{\partial^2 E(x,t)}{\partial t^2} = 4\pi\frac{\partial^2 N(x,t)}{\partial t^2}, \tag{2.1}$$

where c is the light speed in vacuum. We have neglected the underlying material dispersion. The nonlinear polarization term $N(x,t)$ is originated from the optical Kerr effect,

$$N(x,t) = K(x)\left(E(x,t)\right)^3. \tag{2.2}$$

The fiber grating has a periodic structure. There, the linear dielectric function $\epsilon(x)$ and the nonlinear dielectric function $K(x)$ are periodic;

$$\epsilon(x+d) = \epsilon(x), \tag{2.3a}$$
$$K(x+d) = K(x). \tag{2.3b}$$

We first examine the linear theory. Let us neglect the nonlinear term in (2.1) and consider a solution of the form $E(x,t) = Y(x)\exp(-\mathrm{i}\omega t)$.Then, $Y(x)$ obeys a linear equation;

$$c^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}Y(x) + \omega^2\epsilon(x)Y(x) = 0. \tag{2.4}$$

Owing to the Bloch theorem, there exists a Bloch solution $Y(x)$ which satisfies

$$Y(x) = Z(x,k)\mathrm{e}^{ikx}, \tag{2.5a}$$
$$Z(x+d,k) = Z(x,k), \tag{2.5b}$$

where the wave number k and the angular frequency ω are connected by a dispersion relation $\omega = \omega(k)$. It is well konwn that the dispersion relation exhibits a band structure. From now on, we call $E(x,t) = Y(x)\exp(-\mathrm{i}\omega t) = Z(x,k)\exp\mathrm{i}(kx-\omega t)$ the 'Bloch wave', which plays a similar role as the monochromatic wave $\exp\mathrm{i}(kx-\omega t)$ in homogeneous systems.

In analyzing the nonlinear wave propagations, we shall employ the following two assumptions for $E(x,t)$. First, the amplitude is small but finite. The order of it will be denoted by a smallness parameter ε. Second, up to $O(\varepsilon)$, it is regarded as a slow modulation of the Bloch wave . The characteristic length of the modulation is much longer than that of the Bloch wave ($\sim 1/k$) which is of the same order as the period of the grating structure d. Considering the above assumptions, we introduce the following perturbation expansions for $E(x,t)$;

$$\begin{aligned}E(x,t) =&\varepsilon \mathrm{e}^{-\mathrm{i}\omega t}Y(x)\Psi + c.c\\ &+\varepsilon^2 \mathrm{e}^{i(kx-\omega t)}\left(\frac{\partial}{\mathrm{i}\partial k}\right)Z(x,k)\frac{\partial\Psi}{\partial\xi} + c.c + \varepsilon^3 \mathrm{e}^{i(kx-\omega t)}\frac{1}{2}\left(\frac{\partial}{\mathrm{i}\partial k}\right)^2 Z(x,k)\frac{\partial^2\Psi}{\partial\xi^2} + c.c.\\ &+\varepsilon^3 \mathrm{e}^{-\mathrm{i}\omega t}R(x)|\Psi|^2\Psi + c.c.,\end{aligned} \tag{2.6}$$

where $\Psi = \Psi(\xi,\tau)$ is the envelope function of the Bloch wave , $c.c.$ means the complex cojugate and new independent variables ξ and τ have been defined by

$$\xi = \varepsilon\left(x - \frac{\mathrm{d}\omega}{\mathrm{d}k}t\right), \tag{2.7a}$$

$$\tau = \varepsilon^2 t. \tag{2.7b}$$

The $O(\varepsilon)$ terms in (2.6) represent modulation of the Bloch wave $Y(x)\exp(-\mathrm{i}\omega t)$. The terms in the second line are one of the most essential points in our analysis. The derivation is shown in appendix A in which the modulations of the Bloch wave in 'linear' periodic systems are analyzed. We shall refer to them as the linear correction terms. It should be noted that the explicit forms of the linear correction terms are directly calculated from $Z(x,k)$ and $\Psi(\xi,\tau)$. In the previous analysis [7, 8], however the corresponding terms are calculated through an infinite number of integrations and their summation (see for example eq.(2) in reference 7). The third line in (2.6) is the nonlinear correction term the explicit form of which will be determined later(see eq.(2.11)).

Let us substitute the perturbation expansion(2.6) into the original nonlinear equation (2.1) and compare the coefficients of $\varepsilon^n \mathrm{e}^{-\mathrm{i}\omega l t}, (n,l = 1,2,\cdots)$. At $(n,l) = (1,\pm1),(2,\pm1)$ we confirm identities and at $(n,l) = (3,1)$ we have a set of equations,

$$P(x)\left(\mathrm{i}\frac{\partial\Psi}{\partial\tau} + \frac{1}{2}\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}\frac{\partial^2\Psi}{\partial\xi^2}\right) + Q(x)|\Psi|^2\Psi = 0, \tag{2.8a}$$

$$P(x) = 2\omega\epsilon(x)Y(x), \tag{2.8b}$$

$$Q(x) = 12\pi\omega^2 K(x)|Y(x)|^2 Y(x) + \left(c^2\frac{\partial}{\partial x^2} + \omega^2\epsilon(x)\right)R(x). \tag{2.8c}$$

Regarding ξ and x in eq.(2.8a) are independent variables, we see that $P(x)$ and $Q(x)$ are proportional.

$$Q(x) = rP(x) \qquad (r:\ \text{unknown constant}) \tag{2.9}$$

With the definitions of $P(x)$ and $Q(x)$, (2.9) is rewritten into a linear equation for $R(x)$:

$$\left(c^2\frac{\partial}{\partial x^2} + \omega^2\epsilon(x)\right)R(x) = 2\omega\left(r\epsilon(x) - 6\pi\omega K(x)|Y(x)|^2\right)Y(x), \tag{2.10}$$

whose general solution is given by

$$
\begin{aligned}
R(x) =& A_1 Y(x) + A_2 Y^*(x) \\
& -2\omega Y(x) \int_0^x \left(r\epsilon(x') - 6\pi\omega K(x')|Y(x)|^2 \right) |Y(x')|^2 \mathrm{d}x' \\
& -2\omega Y^*(x) \int_0^x \left(r\epsilon(x') - 6\pi\omega K(x')|Y(x')|^2 \right) Y(x')^2 \mathrm{d}x',
\end{aligned}
\tag{2.11}
$$

where A_1 and A_2 are arbitrary constants. The term in the second line grows as x. This situation is undesirable for real physical systems. The condition for eliminating the growing term is

$$
r = \frac{6\pi\omega \int_0^d K(x)|Y(x)|^4 \mathrm{d}x}{\int_0^d \epsilon(x)|Y(x)|^2 \mathrm{d}x},
\tag{2.12}
$$

which fixes the undetermined constant r. Finally, (2.8a) and (2.9) give a closed equation for the envelope $\Psi(\xi, \tau)$

$$
\mathrm{i}\frac{\partial \Psi}{\partial \tau} + \frac{1}{2}\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}\frac{\partial^2 \Psi}{\partial \xi^2} + r|\Psi|^2\Psi = 0.
\tag{2.13}
$$

This is nothing but the well-known nonlinear Schrödinger (NLS) equation. Similar result has been reported in the previous papers[7, 8]. However, the nonlinear coefficient of the NLS equation is different; the corresponding quantity is obtained by replacing the structural period d in (2.12) by the normalization constant L[7, 8].

3 Solitons in fiber grating

Assuming that $r\mathrm{d}^2\omega/\mathrm{d}k^2$ is positive, one soliton solution (bright soliton) of the NLS equation (2.13) is given by

$$
\Psi(\xi, \tau) = \sqrt{\frac{1}{r}\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}} A \operatorname{sech}\left(A\xi - BA\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}\tau \right) \exp i \left\{ B\xi + \frac{1}{2}\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}(A^2 - B^2)\tau \right\}
\tag{3.1}
$$

where A, B are real constants. Since Ψ, Ψ_ξ, $\Psi_{\xi\xi}$ and their complex conjugates are localized moving waves, $E(x, t)$ is also moving pulse which is regarded as the envelope soliton of the Bloch wave in fiber grating. We refer to this type of pulse as 'grating soliton'. In the ref.[1], the pulse was called 'Bragg grating soliton', however in our analysis the Bragg condition is not necessary. Therefore we have dropped 'Bragg' in naming the envelope soliton.

As a special case we set $\mathrm{d}\omega/\mathrm{d}k = 0$(band edge) and $B = 0$. There, $E(x, t)$ is zero-speed soliton, that is, the soliton at rest. Neglecting higher order terms we have

$$
E(x, t) = \varepsilon\sqrt{\frac{1}{r}\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}} AY(x) \operatorname{sech}(\varepsilon Ax) \exp\left\{ -\mathrm{i}(\omega - \frac{1}{2}\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}A^2\varepsilon^2)t \right\} + c.c + O(\varepsilon^2).
\tag{3.2}
$$

From the time dependence of E, we can see the frequency shift due to the nonlinearity.

$$
\omega \to \omega' = \omega - \frac{1}{2}\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}A^2\varepsilon^2
\tag{3.3}
$$

For real materials, r given by (2.12) is positive and then by assumption $\mathrm{d}^2\omega/\mathrm{d}k^2 > 0$. Therefore, ω lies in the lower edge of a band and the modified frequency ω' lies in the gap. The above type of localized wave is known as the gap soliton[2], which is a purely nonlinear phenomenon.

If $r\mathrm{d}^2\omega/\mathrm{d}k^2$ is negative, NLS equation(2.13) has a dark soliton solution;

$$\Psi(\xi,\tau)=\sqrt{-\frac{1}{r}\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}}\,A\,\frac{1+\exp(C\xi-D\tau+\mathrm{i}\phi)}{1+\exp(C\xi-D\tau)}\exp\mathrm{i}\left\{B\xi-\frac{1}{2}\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}(2A^2+B^2)\tau\right\}, \tag{3.4}$$

where A, B, C are real and $4A^2-C^2\geq 0$. Constants D and $\exp(\mathrm{i}\phi)$ are given by

$$D=\frac{1}{2}\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}C(2B+\sqrt{4A^2-C^2}), \tag{3.5a}$$

$$\mathrm{e}^{\mathrm{i}\phi}=-\frac{C+\mathrm{i}\sqrt{4A^2-C^2}}{C-\mathrm{i}\sqrt{4A^2-C^2}}. \tag{3.5b}$$

By the same reason, $E(x,t)$ is a moving dark pulse, which can be referred to as 'grating dark soliton'. For simplicity, we set $\mathrm{d}\omega/\mathrm{d}k=0$(band edge), $B=0$, and $4A^2-C^2=0$ from which we have

$$E(x,t)=\varepsilon\sqrt{-\frac{1}{r}\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}}AY(x)\ \tanh(\varepsilon Cx)\exp\left\{-\mathrm{i}(\omega+\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}A^2\varepsilon^2)t\right\}+c.c+O(\varepsilon^2). \tag{3.6}$$

Similarly to the bright soliton case, we can see the frequency shift,

$$\omega\to\omega'=\omega+\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}A^2\varepsilon^2. \tag{3.7}$$

Remarking (3.3) and (3.7) we can conclude that the frequency shift of the zero-speed soliton (both bright and dark) is always negative.

To examine the grating solitons in detail, we consider a typical fiber grating, where the dielectric constant varies in a steepwise manner along the fiber[2]. The fiber consists of two kinds of dielectric media jointed one after the other. The dielectric functions $\epsilon(x), K(x)$ are assumed

$$\epsilon(x)=\begin{cases}(cb)^2\\(ca)^2\end{cases}\quad K(x)=\begin{cases}K_1 & (n-\frac{1}{2})d<x<nd\ \text{[case I]}\\K_2 & nd<x<(n+\frac{1}{2})d\ \text{[case II]}\end{cases}, \tag{3.8}$$

where n is integer. We refer to this type of grating as super-lattice[2]. The linearized equation (2.4) for (3.8) is equivalent to the one-dimensional Kronig-Penny model in quantum mechanics, The Bloch type solution have the following form.

$$Y(x)=\alpha_n\mathrm{e}^{\mathrm{i}b\omega(x-nd)}+\beta_n\mathrm{e}^{-\mathrm{i}b\omega(x-nd)},\quad \text{[caseI]} \tag{3.9a}$$

$$Y(x)=\gamma_n\mathrm{e}^{\mathrm{i}a\omega(x-nd)}+\delta_n\mathrm{e}^{-\mathrm{i}a\omega(x-nd)}.\quad \text{[caseII]} \tag{3.9b}$$

Continuity conditions of $Y(x)$ and $\mathrm{d}Y(x)/\mathrm{d}x$ at the connection points give

$$\begin{pmatrix}\gamma_n\\\delta_n\end{pmatrix}=\frac{1}{2a}\begin{pmatrix}a+b & a-b\\a-b & a+b\end{pmatrix}\begin{pmatrix}\alpha_n\\\beta_n\end{pmatrix}, \tag{3.10a}$$

$$\begin{pmatrix}\alpha_{n+1}\\\beta_{n+1}\end{pmatrix}=T\begin{pmatrix}\alpha_n\\\beta_n\end{pmatrix}, \tag{3.10b}$$

where the monodromy matrix T is

$$4abT = \begin{pmatrix} (a+b)^2 e^{i(a+b)\omega d/2} & (a^2-b^2)e^{i(a+b)\omega d/2} \\ (a^2-b^2)e^{-i(a+b)\omega d/2} & (a+b)^2 e^{-i(a+b)\omega d/2} \end{pmatrix} - \begin{pmatrix} (a-b)^2 e^{-i(a-b)\omega d/2} & (a^2-b^2)e^{-i(a-b)\omega d/2} \\ (a^2-b^2)e^{i(a-b)\omega d/2} & (a-b)^2 e^{i(a-b)\omega d/2} \end{pmatrix} \tag{3.11}$$

The eigenvalue of the monodromy matrix, λ, is expressed as

$$\lambda = \frac{1}{2}(tr\ T \pm \sqrt{(tr\ T)^2 - 4}). \tag{3.12}$$

If $|tr\ T| < 2$, $|\lambda| = 1$ and we may set $\lambda = e^{ikd}$. Real number k is nothing but the wave number. Moreover if ${}^t(\alpha_0, \beta_0)$ is an eigen vector of T, $Y(x)$ given by (3.9) becomes the Bloch solutions:

$$Y(x+d) = Y(x)e^{ikd} \tag{3.13}$$

Substituting $\lambda = e^{ikd}$ into (3.12) and comparing real parts we have the disperdion relation $\omega = \omega(k)$.

$$\cos kd = \frac{1}{2} tr\ T = \left\{ \frac{(a+b)^2}{4ab} \cos \frac{a+b}{2}\omega d - \frac{(a-b)^2}{4ab} \cos \frac{a-b}{2}\omega d \right\}. \tag{3.14}$$

the profile of which for $a = 3$,$b = 1$,$d = 1$ is depicted in Fig.1. Since $(\alpha_n, \beta_n) = \exp(iknd)(\alpha_0, \beta_0)$, we

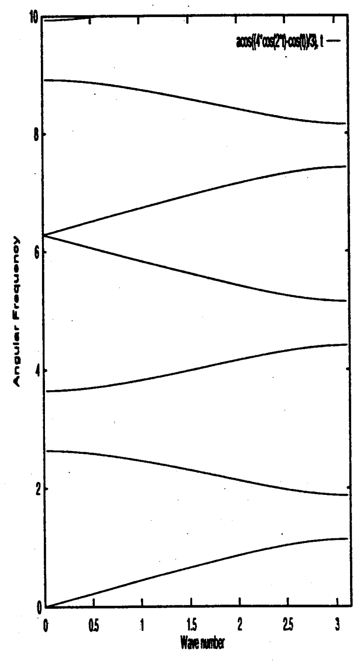

Figure 1: dispersion relation for the super lattice

obtain an explicit form of the Bloch function. In this case the modified ferquency ω' lies in the gap. Therefore, the localized wave shown in fig.4 is the gap soltion as a special case of the grating soliton.

4 Discussions

We have analyzed nonlinear electromagnetic waves in optical fiber grating, where the material dispersion is neglected. Bloch wave modulations are shown to obey the nonlinear Schrödinger (NLS) equation which allows the bright or dark soliton propagations. We have proved that the frequency shift due to the nonlinearity is always negative for both zero-speed bright and dark pulses. In the case of the super-lattice the linear problem is the Kronig-Penny model. We have shown the explicit form of the gap soliton as a special case of the grating soliton. Application to the periodic strucuture obtained from the phase scannning technique[1, 10] remains future problem.

One of the novel points in our analysis is the linear correction terms represented by the second line in (2.6). Similar terms have already been introduced by one of the authors, [11] however there are some differences. While the starting point in deriving the correction terms is a superpositon of the Bloch waves with respect to the wave number k (see eq.(A.4)in appendix A), a superpositon with respect to the frequency ω was used in the previous paper[11]. Consequently, ξ is given by $t - (\mathrm{d}k/\mathrm{d}\omega)x$[11] which is not applicable to the band edge case ($\mathrm{d}\omega/\mathrm{d}k = 0$). In the previous method[11] the gauge function $\chi(x)$ has been introduced in variable τ. On the other hand, the nonlinear correction term plays essentially the same role as $\chi(x)$ in the present method.

In our analysis the material dispersion has been dropped. Nevertheless, we can easily include this effect and the result is essentially same.

The nonlinear coefficient of the NLS equation r given in (2.12) is positive. Therefore, in general, the zero-speed bright soliton becomes the gap soliton. Contrarily, if r is negative, the zero-speed dark soliton becomes the gap dark soliton. Though r is always positive in the present system, other systems may be expected to allow negative r. For example in diatomic lattice governed by quadratic nonlinearities, r can set to be either positive or negative. By another approach bright and dark soliton have been discussed in that system[13]. Also, numerical simulations have been performed extensively in the similar system[14]. It should be noted that Bloch wave modulation is the periodic anharmonic lattice[12] is reduced to the NLS equation using the same method as the present paper.

The idea of Bloch wave modulation is applicable to a large class of nonlinear periodic systems. The acoustic wave propagation in tunnel with Helmholtz resonator ally is one of interesting system.[16] Applications to other system, such as water waves and electric circuits seem attractive problems.

References

[1] Benjamin J. Eggleton et al. Phys. Rev. Lett. **76**(1996)1627.

[2] Wei Chen and D.L.Mills, Phys. Rev. Lett. **58**(1987)160.

[3] H.G.Winful and G.D.Cooperman, Appl.Phys.Lett.**40**(1982)298.

[4] H.G.Winful, Appl.Phys.Lett.**46**(1985)527.

[5] D.L.Mills and S.E.Trullinger, Phys.Rev.B**36**(1987)947.

[6] A.B.Aceves and S.Wabnitz, Phys.Lett.A**141**(1989)37.

[7] J.E. Sipe and Herbert G. Winful, Opt. Lett. **13**(1988)132.

[8] C.Martijin de Sterke and J.E.Sipe, Phys. Rev. A **38**(1988)5149.

[9] G.Meltz, W.W.Morey and W.H.Glen, Opt.Lett. **14** (1989) 1590.

[10] J.Martin and F.Quellete, Opt.Lett. **30** (1994) 811.

[11] T. Iizuka, J.Phys.Soc.Jpn. **63**(1994)4343.

[12] T. Iizuka, J.Phys.Soc.Jpn. **64**(1995)3215.

[13] Y.S.Kivshar, O.A.Chubykalo, O.V.Usatenko and D.V.Grinyoff, Int.J.Mod.Phys.B, **22** (1995) 2963.

[14] A. Campa, A. Giansanti, A. Tenenbaum, D. Levi and O. Ragnisco, Phys. Rev. B **48** (1993) 10168.

[15] N. Yajima and J. Satsuma, Prog. Theor. Phys. **62** (1979) 370.

[16] N. Sugimoto, J. Fluid Mech **244**(1992)55. N.Sugimoto, Jpn.Soc. Mechanical Engineers (1993) 284.

Appendix A: Linear correction terms

In this appendix we derive the linear correction terms, that is the second line in eq. (2.6). For this purpose, we pay attention to the modulation of the Bloch wave in linear periodic system. Owing to the linearity of the system, the general solution is given by a superposition of the Bloch waves $Y(x)\mathrm{e}^{-\mathrm{i}\omega t} = Z(x,k)\mathrm{e}^{\mathrm{i}(kx-\omega t)}$:

$$u(x,t) = \int_{-\infty}^{\infty} \rho(k) Z(x,k) \mathrm{e}^{\mathrm{i}(kx-\omega t)} \mathrm{d}k \tag{A.1}$$

We assume that the weight function $\rho(k)$ distirbutes mainly in the vinicity of a certain wave number k_0 in a band. We denote the width of the distrubution by $\varepsilon\Delta$ ($\varepsilon \ll 1$), and introduce K and $\tilde{\rho}$ as

$$k = k_0 + \varepsilon K, \tag{A.2}$$
$$\rho(k) = \varepsilon^{-1}\tilde{\rho}(K). \tag{A.3}$$

Substituting them into (A.1), we have

$$\begin{aligned} u(x,t) =& \mathrm{e}^{\mathrm{i}(k_0 x-\omega_0 t)} Z(x,k_0) \\ &\cdot \int_{-\Delta}^{\Delta} \tilde{\rho}(K) \left\{ 1 + \varepsilon K \frac{\partial}{\partial k} + \frac{1}{2}\left(\varepsilon K \frac{\partial}{\partial k}\right)^2 \right\} \exp\mathrm{i}\left\{ K\varepsilon(x - \frac{\mathrm{d}\omega_0}{\mathrm{d}k}t) - \frac{K^2}{2}\frac{\mathrm{d}^2\omega_0}{\mathrm{d}k^2}\varepsilon^2 t \right\} \mathrm{d}K. \end{aligned} \tag{A.4}$$

where $\omega_0 = \omega(k_0)$, $\mathrm{d}\omega_0/\mathrm{d}k = \mathrm{d}\omega(k_0)/\mathrm{d}k$, and $\mathrm{d}^2\omega_0/\mathrm{d}k^2 = \mathrm{d}^2\omega(k_0)/\mathrm{d}k^2$. In the followings, we replace k_0,ω_0 by k,ω, respectively. Introducing new independent varibles ξ, τ and a dependent variable $\Psi(\xi,\tau)$;

$$\xi = \varepsilon\left(x - \frac{\mathrm{d}\omega}{\mathrm{d}k}t\right), \tag{A.5}$$
$$\tau = \varepsilon^2 t, \tag{A.6}$$
$$\Psi(\xi,\tau) = \int_{-\Delta}^{\Delta} \tilde{\rho}(K) \exp\mathrm{i}\left(K\xi - \frac{K^2}{2}\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}\tau\right) \mathrm{d}K. \tag{A.7}$$

we have

$$u(x,t) = \mathrm{e}^{\mathrm{i}(kx-\omega t)} \left\{ Z(x,k) + \varepsilon \frac{\partial Z(x,k)}{\mathrm{i}\partial k} \frac{\partial}{\partial \xi} + \frac{\varepsilon^2}{2} \left(\frac{\partial}{\mathrm{i}\partial k} \right)^2 Z(x,k) \frac{\partial^2}{\partial \xi^2} \right\} \Psi \tag{A.8}$$

The first term represents the modulation of the Bloch wave due to the dispersion effect. The second and third terms are linear correction term due to the periodic inhomogeneity of the system. Except for the factor ε and conplex conjugations, the first and sencond lines in (2.6) are same as (A.8). The envelope function (A.7) obeys the free Schrödinger equation,

$$\mathrm{i}\frac{\partial \Psi}{\partial \tau} + \frac{1}{2}\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}\frac{\partial^2 \Psi}{\partial \xi^2} = 0. \tag{A.9}$$

The reductive perturbation method introduced in §2 is a nonlinear analogue of the method presented in this appendix.