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ASYMPTOTIC DISTRIBUTION OF EIGENVALUES FOR PAULI OPERATORS WITH NONCONSTANT MAGNETIC FIELDS

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Introduction

The aim here is to study the asymptotic distribution of discrete eigenvalues near the bottom of essential spectrum for two and three dimensional Pauli operators perturbed by electric potentials falling off at infinity. The special emphasis is placed on the case that the Pauli operators have nonconstant magnetic fields.

The Pauli operator describes the motion of a particle with spin in a magnetic field and it acts on the space $L^2(R^3) \otimes C^2$. The unperturbed Pauli operator is given by

$$H_P = (-i\nabla - A)^2 - \sigma \cdot B$$

under a suitable normalization of units, where $A : R^3 \to R^3$ is a magnetic potential, $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ with components

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is the vector of 2×2 Pauli matrices and $B = \nabla \times A$ is a magnetic field. We write $(x, z) = (x_1, x_2, z)$ for the coordinates over the three dimensional space $R^3 = R_x^2 \times R_z$. Throughout the entire discussion, we suppose that the magnetic field B has a constant direction. For notational brevity, the field is assumed to be directed along the positive z axis, so that B takes the form

$$B(x) = (0, 0, b(x)).$$

Since the magnetic field B is a closed two form, it is easily seen that B is independent of the z variable. We identify $B(x)$ with the function $b(x)$. Let $A(x) = (a_1(x), a_2(x), 0)$ with real function $a_j \in C^1(R_x^2)$ be a magnetic potential associated with $b(x)$. Then $b(x) = \partial_1 a_2 - \partial_2 a_1$, $\partial_j = \partial/\partial x_j$, and the Pauli operator takes the simple form

$$H_P = \begin{pmatrix} H_+ - \partial_z^2 & 0 \\ 0 & H_- - \partial_z^2 \end{pmatrix},$$

where

$$H_\pm = H_0 \mp b, \quad H_0 = \Pi_1^2 + \Pi_2^2, \quad \Pi_j = -i\partial_j - a_j. \tag{0.1}$$

The magnetic field b is represented as the commutator $b = i[\Pi_2, \Pi_1]$ and hence $H_\pm$ can be rewritten as

$$H_\pm = (\Pi_1 \pm i\Pi_2)^*(\Pi_1 \pm i\Pi_2). \tag{0.2}$$

This implies that $H_\pm \geq 0$ is nonnegative. If, in particular, $b(x) > c > 0$ is positive, then $H_- \geq c$ becomes strictly positive. On the other hand, it is known ([1, 6]) that H_+ has zero as an eigenvalue with infinite multiplicities. We further know (see [3] for example) that the non-zero spectra of operators H_+ and H_- coincide with each other. Thus H_+ has zero as the bottom of its essential spectrum and the bottom is an isolated eigenvalue with inifinite multiplicities.

We first discuss the two dimensional case. We consider the Pauli operator

$$H(V) = H_+ - V = \Pi_1^2 + \Pi_2^2 - b - V \tag{0.3}$$

perturbed by an electric potential $V(x)$. As stated above, the unperturbed operator H_+ has zero as an isolated eigenvalue with infinite multiplicities. If the perturbation $V(x)$ falling off at infinity is added to this operator, then the infinite multipicities are resolved and the above operator $H(V)$ has discrete (positive or negative) eigenvalues accumulating at the origin. We are concerned with how the inifinite multiplicities of zero eigenvalue are resolved. The aim is to study the asymptotic distribution near the origin of such discrete eigenvalues.

We shall formulate the obtained result more precisely. We assume that the magnetic field $b(x) \in C^1(R_x^2)$ fulfills the following assumption : There exists β, $0 < \beta \leq 1$, such that

$$(b) \qquad 1/C \leq b(x) \leq C, \qquad |\nabla b(x)| \leq C\langle x\rangle^{-\beta}$$

for some $C > 1$, where $\langle x\rangle = (1+|x|^2)^{1/2}$. If $V(x)$ is a real bounded function, then the operator $H(V)$ formally defined by (0.3) admits a unique self–adjoint realization in the space $L^2 = L^2(R_x^2)$ with natural domain $\{u \in L^2 : H(V)u \in L^2\}$. We denote by the same notation $H(V)$ this self–adjoint realization. We now mention the first main theorem.

Theorem 1. *Let assumption* (b) *be fulfilled. Assume that a real function* $V(x) \in C^1(R_x^2)$ *satisfies*

$$|V(x)| \leq C\langle x\rangle^{-m}, \qquad |\nabla V(x)| \leq C\langle x\rangle^{-m-1}, \quad C > 0,$$

for some $m > 0$ *and that*

$$\limsup_{\lambda\to 0} \lambda^{2/m} \int_{(1-\delta)\lambda < |V(x)| < (1+\delta)\lambda} dx = o(1), \quad \delta \to 0.$$

Then one has :

(i) Let $N(H(V) < -\lambda)$, $\lambda > 0$, denote the number of negative eigenvalues less than $-\lambda$ of operator $H(V)$. Then

$$N(H(V) < -\lambda) = (2\pi)^{-1} \int_{V(x)>\lambda} b(x)\, dx + o(\lambda^{-2/m}), \quad \lambda \to 0.$$

(ii) Let $0 < c < b_0/3$, $b_0 = \inf b(x)$, be fixed and let $N(\lambda < H(V) < c)$, $0 < \lambda < c$, be the number of positive eigenvalues lying in the interval (λ, c) of operator $H(V)$. Then

$$N(\lambda < H(V) < c) = (2\pi)^{-1} \int_{V(x)<-\lambda} b(x)\, dx + o(\lambda^{-2/m}), \quad \lambda \to 0.$$

Next we proceed to the three dimensional case. Let $b(x) \in C^1(R_x^2)$ be again the magnetic field satisfying the assumption (b). We consider the three dimensional perturbed Pauli operator

$$H = H(V) = \Pi_1^2 + \Pi_2^2 - \partial_z^2 - b - V,$$

which acts on the space $L^2(R^3) = L^2(R_x^2 \times R_z)$, where $V = V(x,z)$ is a real function decaying at infinity. The essential spectrum of the unperturbed three dimensional Pauli operator without potential V begins at the origin and occupies the whole positive axis. On the other hand, the perturbed operator H has an infinite number of negative eigenvalues accumulating the origin. The second main theorem is formulated as follows.

Theorem 2. *Let $H = H(V)$ be as above. Suppose that the magnetic field $b(x)$ fulfills the assumption (b). If a real function $V(x,z) \in C^1(R^3)$ satisfies*

$$\langle x,z\rangle^{-m}/C \le V(x,z) \le C\,\langle x,z\rangle^{-m}, \quad |\nabla V(x,z)| \le C\,\langle x,z\rangle^{-m-1}, \quad C > 1,$$

for some $m > 0$, where $\langle x,z\rangle = (1 + |x|^2 + |z|^2)^{1/2}$, then one has :

(i) If $0 < m < 2$, then

$$N(H < -\lambda) = 2(2\pi)^{-2} \int_{V(x,z)>\lambda} b(x)(V(x,z) - \lambda)^{1/2}\, dxdz\, (1 + o(1)), \quad \lambda \to 0.$$

(ii) Assume that $m > 2$. Let $w(x)$ be defined as

$$w(x) = \int_{-\infty}^{\infty} V(x,z)\, dz.$$

If $w(x)$ fulfills the additional assumption

$$\limsup_{\lambda \to 0} \lambda^{2/(m-1)} \int_{(1+\delta)\lambda > w(x) > (1-\delta)\lambda} dx = o(1), \quad \delta \to 0,$$

then

$$N(H < -\lambda) = (2\pi)^{-1} \int_{w(x) > 2\lambda^{1/2}} b(x)\, dx\, (1 + o(1)), \quad \lambda \to 0.$$

Remark. The above theorem can be extended to a certain class of potentials with indefinite sign or weak local singularities. Such a class of potentials includes the negative Coulomb potential as one of typical examples.

The problem on the asymptotic distribution of eigenvalues for Pauli operators perturbed by electric potentials has been already studied by [7, 9] when $b(x) = b > 0$ is a uniform magnetic field. Both the works make an essential use of the uniformity of magnetic fields and the methods developed there do not seem to apply directly to the case of nonconstant magnetic fields. Roughly speaking, the difficulty arises from the fact that magnetic potentials which actually appear in Pauli operators undergo nonlocal changes even under local changes of magnetic fields. This makes it difficult to control nonconstant magnetic fields by a local approximation of uniform magnetic fields. To prove the two main theorems, some new devices are required in many states of the proof. We also note that the present method may extend to the case of periodic magnetic fields for which the second assumption in (b) is in general violated. We will discuss the matter in detail elsewhere ([12]).

Recently several works have been done on the spectral problems of Pauli operators with nonconstant magnetic fields. For example, the Lieb–Thirring inequality for negative eigenvalues has been discussed in [5, 8] and the asymptotic behavior of ground state densities in the strong field limit has been studied in [4]. The present work is motivated by these works.

Sketch of proof of Theorem 1

Theorem 2 follows from Theorem 1. We here give only s sketch for the proof of the first theorem. The detailed proof of both the theorems can be found in [11] (see [10] also). For brevity, we assume that $V(x) > 0$ is strictrly positive, and we consider only the number $N(H(V) < -\lambda)$ of negative eigenvalues less that $-\lambda$ of operator $H(V)$. The proof is based on the min–max principle and on the perturbation theory for singular numbers of compact operators.

We start by fixing several notations. For given self–adjoint operator T, we use the notations $N(T < \lambda)$ and $N(T > \lambda)$ to denote the number of eigenvalues less and more than λ of T, respectively. Let $H_\pm$ be as in (0.1). As previously stated, H_+ has the remarkable spectral property that H_+ has zero, bottom of its spectrum, as an isolated eigenvalue with infinite multiplicities and also the non–zero spectra of operators H_+ and H_- coincide with each other. We denote by $P : L^2(R^2_x) \to L^2(R^2_x)$

the eigenprojection associated with the zero eigenspace $\operatorname{Ker} H_+$ of H_+ and we write Q for $Id - P$, Id being the identity operator. By assumption (b), $H_- \geq b_0 > 0$, $b_0 = \inf b(x)$, is strictly positive and hence we have

$$QH_+Q \geq b_0\, Q \tag{1.1}$$

in the form sense.

(1) Let $0 < c < b_0/2$ be fixed. Then we have the form inequalities

$$PVQ + QVP \lesseqgtr \pm cQ \pm PV^2P/c$$

and hence it follows that

$$N(H(V) < -\lambda) \lesseqgtr N(P(V \pm V^2/c)P > \lambda) + N(Q(H_+ - V \mp c)Q < -\lambda).$$

By (1.1), the quantities $N(Q(H_+ - V \mp c)Q < -\lambda)$ are seen to be bounded uniformly in $\lambda > 0$ small eough. On the other hand, $V(x)^2 = O(|x|^{-2m})$ falls off at infinity faster than $V(x)$ and hence this is treated as a negligible term by a perturbation method. Thus we have

$$N(H(V) < -\lambda) \sim N(PVP > \lambda), \quad \lambda \to 0.$$

The problem is now reduced to the study on the asymptotic distribution of compact operator PVP. If we denote by $\{e_j\}_{j=1}^{\infty}$ an orthonormal system of $\operatorname{Ker} H_+$, then this operator is realized as the infinite matrix with component (Ve_j, e_k), $(\ ,\ )$ being the L^2 scalar product in $L^2(R_x^2)$. Let $\varphi(x)$ be a solution to

$$\Delta\varphi = b, \tag{1.2}$$

so that the magnetic potential $(a_1(x), a_2(x))$ associated with the field $b(x)$ is chosen to be divergenceless

$$a_1(x) = -\partial_2\varphi(x), \qquad a_2(x) = \partial_1\varphi(x).$$

Hence a simple calculation yields the relation

$$\Pi_1 + i\Pi_2 = -ie^{-\varphi}(\partial_1 + i\partial_2)e^{\varphi}.$$

This, together with (0.2), implies that

$$u_l(x) = (x_1 + ix_2)^l \exp(-\varphi(x)) = r^l \exp(il\theta)\exp(-\varphi(x)), \quad l \geq 0 \ \text{(integer)},$$

spans the zero eigenspace $\operatorname{Ker} H_+$, where (r, θ) stands for thr polar coordinates over the plane R_x^2. If, in particular, $b = b(r)$ is spherically symmetric, then so is $\varphi = \varphi(r)$ and hence $\{u_l\}$ forms an orthogonal system of eigenfunctions spanning $\operatorname{Ker} H_+$. If, in addition, $V = V(r)$ is also spherically symmetric, then the operator PVP under consideration is realized as the diagonal matrix with $\lambda_l = p_l/q_l$ as eigenvalues, where

$$p_l = 2\pi \int_0^\infty V(r) r^{2l+1} \exp(-2\varphi(r))\, dr, \quad q_l = 2\pi \int_0^\infty r^{2l+1} \exp(-2\varphi(r))\, dr.$$

Thus the theorem is obtained by studying the asymptotic behavior as $l \to \infty$ of λ_l with aid of the stationary phase method, provided that magnetic fields and electric potentials are both spherically symmetric.

(2) If $V(x) = O(|x|^{-m})$ falls off very slowly at infinity, then the theorem is established through a local approximation of uniform magnetic fields.

Proposition 1.1. *Let β, $0 < \beta \le 1$, be as in assumption (b). If $m < 2\beta/3$, then*

$$N(H(V) < -\lambda) = (2\pi)^{-1} \int_{V(x)>\lambda} b(x)\,dx + o(\lambda^{-2/m}), \quad \lambda \to 0.$$

The proof of this proposition uses the min–max principle and it is based on the following lemma due to [2].

Lemma 1.2. *Let Q_R be a cube with side R. Let H_B be the Schrödinger operator with constant magnetic field $B > 0$. If we consider H_B under zero Dirichlet boundary conditions over the domain Q_R and denote by $N_D(H_B < \lambda; Q_R)$, $\lambda > 0$, the number of eigenvalues less that λ, then there exists $c > 0$ independent of λ, R and Λ, $0 < \Lambda < R/2$, such that :*

(1) $N_D(H_B < \lambda; Q_R) \le (2\pi)^{-1} B\,|Q_R|\, f(\lambda/B)$

(2) $N_D(H_B < \lambda; Q_R) \ge (2\pi)^{-1}(1 - \Lambda/R)^2 B\,|Q_R|\, f((\lambda - c\Lambda^{-2})/B)$,

where $|Q_R| = R^2$ is the measure of cube Q_R and

$$f(\lambda) = \#\{n \in N_* = N \cup \{0\} : 2n + 1 \le \lambda\}.$$

(3) In order to prove the theorem for a wider class of potentials decaying not necessarily slowly at infinity, we use the following simple commutator relation :

$$PVP = PV^{1/2}(P+Q)V^{1/2}P = (PV^{1/2}P)^2 + P[V^{1/2}, Q]V^{1/2}P.$$

Roughly speaking, the second operator on the right side takes the form

$$P[V^{1/2}, Q]V^{1/2}P = P\langle x\rangle^{-m-1}BP$$

for some bounded operator B. This enables us to deal with it as a negligible operator. We make repeated use of this procedure to conclude that

$$N(PVP > \lambda) \sim N(PV^{1/2^k}P > \lambda^{1/2^k}) + N(PUP > \lambda), \quad \lambda \to 0,$$

for some $U(r) \ge 0$ with compact support. If $k \gg 1$ is taken large enough, then we can apply Proposition 1.1 to the first term on the right side, which determines the leading term of the asymptotic formula in the theorem.

(4) It remains to control the error term $N(PUP > \lambda)$ with spherically symmetric nonnegative function $U(r)$ compactly supported. We can prove the following

Lemma 1.3. *Let $U(r)$ be as above. Then*

$$N(PUP > \lambda) = O(\lambda^{-\epsilon}), \quad \lambda \to 0,$$

for any $\epsilon > 0$ small enough.

The lemma above completes the proof of the theorem. We shall briefly explain how to prove this key lemma. Several new notations are required. Let $\mathcal{A}(C)$ be the class of analytic functions over the complex plane C. For given real function $\psi(x) \in C^2(R_x^2)$, we define the subspace $\mathcal{K}_\psi(R_x^2)$ of $L^2(R_x^2)$ by

$$\mathcal{K}_\psi(R_x^2) = \{u \in L^2(R_x^2) : u = he^{-\psi} \text{ with } h \in \mathcal{A}(C)\}$$

and we denote by P_ψ the orthogonal projection on $\mathcal{K}_\psi(R_x^2)$. Such a subspace is easily seen to be closed. Let $\varphi(x)$ be as in (1.2). By construction, the zero eigenspace $\operatorname{Ker} H_+$ just coincides with $\mathcal{K}_\varphi(R_x^2)$ and hence the eigenprojection P is realized as the projection P_φ on $\mathcal{K}_\varphi(R_x^2)$. The lemma below is obtained as a simple application of the min–max principle.

Lemma 1.4. *Let $\psi_j(x) \in C^2(R_x^2)$, $1 \le j \le 2$, be a real function and let $\chi(x) \ge 0$ be a bounded function with compact support. Write $\mathcal{K}_j$ and P_j for $\mathcal{K}_{\psi_j}(R_x^2)$ and P_{ψ_j}, respectively. If $\psi_1(x) \le \psi_2(x)$, then one has*

$$N(P_1\chi P_1 > \lambda) \le N(P_2\chi P_2 > \lambda/\gamma),$$

where

$$\gamma = \max_{x \in \operatorname{supp} \chi} \exp(2\theta(x)), \qquad \theta(x) = \psi_2(x) - \psi_1(x) \ge 0.$$

This lemma implies the key lemma. We can construct a real solution $\varphi(x) \in C^2(R_x^2)$ to equation (1.2) with bound

$$\varphi(x) = O(\exp(cr^2)), \qquad r = |x| \to \infty,$$

for some $c > 0$. We apply Lemma 1.4 with $\psi_1 = \varphi$ and $\psi_2 = \psi = \exp(c(r^2 + 1))$. Since $U(r)$ and $\psi(r)$ are spherically symmetric, the operator $P_\psi U P_\psi$ is realized as a diagonal matrix. The bound in Lemma 1.3 is obtained by evaluating the eigenvalues of such an infinite diagonal matrix.

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