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On the convergence of the Feynman path integral defined through broken line paths

(Feynman による量子化の正当性について一経路積分の収束)

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Abstract. We study the convergence of the Feynman path integral defined through broken line paths in non-relativistic quantum mechanics. This path integral is very familiar with us and well known to be useful. But its rigorous proof of its convergence was given little except for special cases. In the preceding paper, using the idea in the theory of difference methods and the theory of pseudo-differential operators, the author showed for a class of potentials that this path integral converges and gives the solution of the Schrödinger equation. In the present paper we generalize this result. We note that our result is gauge invariant.

In the present paper we consider some charged particles in an electromagnetic field. For the sake of simplicity we suppose charge $=$ one and mass $=$ $m>0$. Let $x\in R^{d}$ and $t\in[0,T]$. We denote by $E(t,x)=(E_{1},\cdots,E_{d})\in R^{d}$ and $(B_{jk}(t,x))_{1\leq j<k\leq d}\in R^{d(d-1)/2}$ the electric strength and the magnetic strength tensor respectively. Let us introduce the electromagnetic potentials $V(t,x)\in R, A(t,x)=(A_{1},\cdots,A_{d})\in R^{d}$, which satisfy

$$E_{j}=-\frac{\partial A_{j}}{\partial t}-\frac{\partial V}{\partial x_{j}}\ (j=1,\cdots,d),$$

$$d(\sum_{j=1}^{d}A_{j}dx_{j})=\sum_{1\leq j<k\leq d}B_{jk}dx_{j}\wedge dx_{k}\quad \text{on } R^{d}. \tag{0.1}$$

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It is well known that the electromagnetic potentials are not unique, but gauge dependent. That is, the gauge transformation

$$V' = V - \frac{\partial \psi}{\partial t}, \quad A'_j = A_j + \frac{\partial \psi}{\partial x_j} \quad (j = 1, 2, \cdots, d) \tag{0.2}$$

leaves E and $(B_{jk}(t,x))_{1 \leq j < k \leq d}$ unchanged.

In classical mechanics the Lagrangian function is given by

$$\mathcal{L}(t, x, \dot{x}) = \frac{m}{2}|\dot{x}|^2 + (\dot{x} \cdot A - V). \tag{0.3}$$

Let $(R^d)^{[s,t]}$ be the space of all paths $\gamma : [s,t] \to R^d$ and $S(\gamma)$ the classical action

$$S(\gamma) = \int_s^t \mathcal{L}(\theta, \gamma(\theta), \dot{\gamma}(\theta)) d\theta, \quad \dot{\gamma}(\theta) = \frac{d\gamma}{dt}(\theta) \tag{0.4}$$

for $\gamma \in (R^d)^{[s,t]}$. Then each classical path γ,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}(t, \gamma(t), \dot{\gamma}(t)) = \frac{\partial \mathcal{L}}{\partial x}(t, \gamma(t), \dot{\gamma}(t)),$$

is given by the condition that $S(\gamma)$ is an extremum in the variational problem, which is called the principle of least action (cf.[3]). So $S(\gamma)$ is fundamental in classical mechanics.

Let $L^2 = L^2(R^d)$ denote the space of all square integrable functions on R^d with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$. The Hamilton function

$$\mathcal{H}(t, x, p) = \frac{1}{2m}|p - A|^2 + V \tag{0.5}$$

is defined by the Legendre transformation $\mathcal{H} = \dot{x} \cdot p - \mathcal{L}, p = \partial \mathcal{L} / \partial \dot{x}$ of $\mathcal{L}(t, x, \dot{x})$ in $\dot{x}$. In quantum mechanics the Hamiltonian operator $H(t)$ can be obtained after replacing p_j by $\hbar i^{-1} \partial / \partial x_j$ in $\mathcal{H}(t, x, p)$. It should be noticed that $H(t)$ has the ordering ambiguity (cf.[15]). Then the time evolution $U(t,s)f$ of the probability amplitude, i.e. $\|U(t,s)f\| = 1$, is described by the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}u(t) = H(t)u(t), \quad u(s) = f. \tag{0.6}$$

On the other hand Feynman in [4] and [5] proposed another method of the quantization. He claimed that the value of the probability amplitude $U(t,s)f$ at x is written as the sum, in some sense, of $N^{-1}(\exp i\hbar^{-1}S(\gamma))f(\gamma(s))$ over all paths $\gamma \in (R^d)^{[s,t]}$ such that $\gamma(t) = x$. N is a normalization factor independent of x and γ. This is called the Feynman path integral or simply the path integral.

Since then, much work has been devoted by physists and mathematicians to give the rigorous meaning of the Feynman path integral. In [1], [2], [12], [13], [14],[16] et al. they studied the case where A is a constant magnetic potential and V is a sum of $|x|^2$ and the Fourier transform of a complex measure of bounded variation on R^d. On the other hand Fujiwara and Yajima in [6], [7], [17] studied the path integral defined through piecewise classical paths. They showed its convergence for a class of potentials and that this path integral satisfies the Schrödinger equation (1.6) with $H(t)$ determined by

$$H(t) = \frac{1}{2m}\sum_{j=1}^{d}(\hbar D_{x_j} - A_j)^2 + V, \quad D_{x_j} = \frac{1}{i}\frac{\partial}{\partial x_j}. \tag{0.7}$$

In the preceding paper [10] the author studied the path integral defined through broken line paths, which is very familiar with us and well known to be useful. Let $\Delta : 0 = t_0 < t_1 < \cdots < t_n = t$ be an arbitrary subdivision of the interval $[0,t]$ and put $|\Delta| = \max_{1\leq j\leq n}(t_j - t_{j-1})$. Let $x^{(j)} \in R^d$ ($j = 0,1,\cdots,n-1$) and denote by $\gamma_\Delta = \gamma_\Delta(x^{(0)}, x^{(1)},\cdots,x^{(n-1)},x) \in (R^d)^{[0,t]}$ the broken line path joining $(t_j, x^{(j)})$ ($j = 0,1,\cdots,n, x^{(n)} = x$). Let $\mathcal{S}$ be the space of rapidly decreasing functions on R^d. We define the operator $\mathcal{C}(\Delta)$ on

$\mathcal{S}$ by the oscillator integral

$$
\begin{aligned}
&\mathcal{C}(\Delta)f \\
&= \prod_{j=1}^{n} \sqrt{\frac{m}{2\pi i\hbar(t_j - t_{j-1})}}^{\,d} \mathrm{Os} - \int_{R^d} \cdots \int_{R^d} e^{i\hbar^{-1}S(\gamma_\Delta)} f(x^{(0)})dx^{(0)}dx^{(1)} \cdots dx^{(n-1)} \\
&\equiv \prod_{j=1}^{n} \sqrt{\frac{m}{2\pi i\hbar(t_j - t_{j-1})}}^{\,d} \lim_{\epsilon \to 0} \int_{R^d} \cdots \int_{R^d} e^{i\hbar^{-1}S(\gamma_\Delta)} \chi(\epsilon x^{(1)}) \cdots \chi(\epsilon x^{(n-1)}) \\
&\qquad\qquad f(x^{(0)})dx^{(0)}dx^{(1)} \cdots dx^{(n-1)},
\end{aligned} \tag{0.8}
$$

where $\chi(x) \in \mathcal{S}$ such that $\chi(0) = 1$ and $\sqrt{i} = e^{i\pi/4}$. The path integral is defined by $\lim_{|\Delta| \to 0} \mathcal{C}(\Delta)f$. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_d)$ we write $\partial_x^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_d)^{\alpha_d}$ and $|\alpha| = \sum_{j=1}^d \alpha_j$. In [10] the author proved that this path integral can be defined and satisfies the Schrödinger equation (1.6) with $H(t)$ determined by (1.7), assuming the below. There exist constants $\delta > 0$, $v \geq 0$, and C_α such that

$$
\begin{aligned}
&|\partial_x^\alpha A_j(t,x)| \leq C_\alpha < x >^{-(1+\delta)}, \ |\alpha| \geq 2, \quad |\partial_x^\alpha \partial_t A_j(t,x)| \leq C_\alpha, \ |\alpha| \geq 1, \\
&|\partial_x^\alpha V(t,x)| \leq C_\alpha, \ |\alpha| \geq 2, \quad |\partial_x^\alpha \partial_t V(t,x)| \leq C_\alpha < x >^v, \ |\alpha| \geq 1
\end{aligned}
$$

on $(t,x) \in [0,T] \times R^d$, where $< x >= \sqrt{1 + |x|^2}$.

In the present paper we generalize this result. We obtain the following.

Theorem. *Let $\partial_x^\alpha E_j(t,x), \partial_x^\alpha B_{jk}(t,x)$, and $\partial_t \partial_x^\alpha B_{jk}(t,x)$ be continuous on $[0,T] \times R^d$ for all α and suppose*

$$
\begin{aligned}
&|\partial_x^\alpha B_{jk}(t,x)| \leq C_\alpha < x >^{-(1+\delta)}, \ |\alpha| \geq 1, \\
&|\partial_x^\alpha E_j(t,x)| \leq C_\alpha, \ |\alpha| \geq 1, \quad (t,x) \in [0,T] \times R^d
\end{aligned} \tag{0.9}
$$

for some $\delta > 0$ and C_α. Let V and A_j $(j = 1, 2, \cdots, d)$ be arbitrary continuously differentiable potentials on $[0,T] \times R^d$. Then $\mathcal{C}(\Delta)$ can be extended to a bounded operator on L^2 and for $f \in L^2$ $\mathcal{C}(\Delta)f$ converges in L^2 uniformly in

$t \in [0,T]$ as $|\Delta| \to 0$. So the path integral through broken line paths can be defined. In addition, this path integral satisfies the Schrödinger equation with $H(t)$ determined by (1.7).

The outline of the proof of Theorem is as follows. We use the idea in difference methods and the theory of pseudo-differential operators as in [10]. We denote by B^a $(a \geq 0)$ the weighted Sobolev space $\{f \in L^2;\ \|f\|_{B^a} \equiv \| <\cdot>^a f\| + \| <\cdot>^a \hat{f}\| < \infty\}$, where $\hat{f}$ is the Fourier transform $\int e^{-ix\cdot\xi} f(x)dx$. Let us define $\gamma^{t,s}_{x,y} \in (R^d)^{[s,t]}$ by

$$\gamma^{t,s}_{x,y} = y + \frac{\theta - s}{t-s}(x-y) \qquad (s \leq \theta \leq t) \tag{0.10}$$

and set for $f \in \mathcal{S}$

$$\mathcal{C}(t,s)f = \begin{cases} \sqrt{m/(2\pi i\hbar(t-s))}^{\,d} \int \exp(i\hbar^{-1} S(\gamma^{t,s}_{x,y}))f(y)dy, & s < t, \\ f, & s = t. \end{cases} \tag{0.11}$$

Then we can easily write for $f \in \mathcal{S}$

$$\mathcal{C}(\Delta)f = \lim_{\epsilon \to 0} \mathcal{C}(t,t_{n-1})\chi(\epsilon\cdot)\mathcal{C}(t_{n-1},t_{n-2})\chi(\epsilon\cdot)\cdots\mathcal{C}(t_2,t_1)\chi(\epsilon\cdot)\mathcal{C}(t_1,0)f. \tag{0.12}$$

First we can show that there exist potentials V and A satisfying

$$\begin{gathered} V = 0, \quad |\partial^\alpha_x A_j(t,x)| + |\partial_t \partial^\alpha_x A_j(t,x)| \leq C_\alpha, \\ |\alpha| \geq 1, \quad (t,x) \in [0,T] \times R^d \end{gathered} \tag{0.13}$$

for some C_α. We fix these V and A for the moment. Then we can prove :

(i) $\mathcal{C}(t,s)$ is extended to a bounded operator on L^2 and satisfies

$$\|\mathcal{C}(t,s)f\| \leq e^{K(t-s)}\|f\|, \quad 0 \leq t-s \leq \rho^*, f \in L^2 \tag{0.14}$$

for some constants $\rho^* > 0$ and $K \geq 0$.

(ii) $\mathcal{C}(\Delta)$ is extended to a bounded operator on L^2 and satisfies

$$\mathcal{C}(\Delta)f = \mathcal{C}(t,t_{n-1})\mathcal{C}(t_{n-1},t_{n-2})\cdots\mathcal{C}(t_2,t_1)\mathcal{C}(t_1,0)f, \quad f \in L^2. \quad (0.15)$$

(iii) There exists an $a \geq 0$ such that

$$\lim_{\rho\to+0}\sup_{0\leq s<T}\|i\hbar\frac{\mathcal{C}(s+\rho,s)f-f}{\rho} - H(s)f\| = 0$$

$$\text{uniformly in } f \in \{g; \|g\|_{B^a} \leq C < \infty\} \quad (0.16)$$

for any C.

The properties (i) and (ii) above can be proved by the analogous arguments in [10]. But to prove (iii) we need a different method from that in [10]. We can prove (iii) by showing the boundedness theorem on B^a of the operators $\mathcal{C}(t,s)$, $\partial_t\mathcal{C}(t,s)$, and etc. From (i)-(iii) Theorem can be proved by the same arguments as in [10]. For the general potentials V and A, applying the arguments about the gauge transformation, we can prove Theorem.

This result will be published in [11].

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