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Surfaces of Finite Total Curvature

Kazuyuki Enomoto
(榎 本 一 之)

東京理科大学 基礎工学部

Introduction

Let M be a connected, oriented, noncompact, complete surface in the 3–dimensional Euclidean space E^3. We denote the second fundamental form of M by B, the mean curavature by H and Gaussian curavature by K.

In this paper, we study the geometry of a surface M which satisfies $\int_M |B|\, dM < \infty$, where dM is the area element of M. For surfaces which satisfy $\int_M |B|^2\, dM < \infty$, rather than $\int_M |B|\, dM < \infty$, extensive studies have been made. Since $|B|^2 = 4H^2 - 2K \geq 2|K|$, the condition $\int_M |B|^2\, dM < \infty$ implies $\int_M |K|\, dM < \infty$. A theorem by Huber ([H]) says that an abstract noncompact complete 2–dimensional Riemannian manifold satisfying $\int_M |K|\, dM < \infty$ is conformally equivalent to a compact Riemann surface $\bar{M}$ punctured at a finite number of points $p_1, \cdots, p_k$. If M is a minimal surface with $\int_M |B|^2\, dM < \infty$ (or equivalently, $\left|\int_M K\, dM\right| < \infty$), Osserman showed that the Gauss map $G : M \to S^2$ is continuously extended to a map from $\bar{M}$ to S^2 ([O]). White ([W]) extended Osserman's theorem to surfaces with $\int_M |B|^2\, dM < \infty$ and $K \leq 0$. The continuous extendability of the Gauss map cannot be derived without the condition $K \leq 0$, but Müller–Šverák ([M–S]) showed that if M satisfies $\int_M |B|^2\, dM < \infty$, then M is properly immersed.

Let M be a connected, oriented, noncompact, complete, properly immersed surface which satisfies a pointwise condition $|B(x)| \leq C|x|^{-1-\varepsilon}$ for x in M, where $|x|$ is the Euclidean distance from the origin and C and ε are positive constants. In [E], the author showed that for M satisfying this condition we have $\int_M |B|^2\, dM < \infty$ and the Gauss map is continuously extended to a map from $\bar{M}$. Basic tools to study those surfaces are given by Kasue and Sugahara in [K] and [K–S]. In this paper we will study a surface M which satisfies a stronger condition $|B(x)| \leq C|x|^{-2-\varepsilon}$ and show that we have $\int_M |B|\, dM < \infty$ (Theorem 1). We will also show that such a surface has a property that each end has an asymptotic plane (Theorem 2).

In general, of course, the condition $\int_M |B|\,dM < \infty$ does not give any pointwise estimate for $|B|$. Moreover, since $\int_M |B|\,dM < \infty$ does not necessarily imply $\int_M |K|\,dM < \infty$, Huber's theorem cannot be applied for these surfaces and the geometry or the topology of M near infinity can be very complicated in general. But if M is a surface of revolution, we can say something about the geometry of the ends. In Theorem 3, we will show that if M is a surface of revolution and satisfies $\int_M |B|\,dM < \infty$, then each end of M has an asymptotic plane. We note that even when M is a surface of revolution, $\int_M |B|\,dM < \infty$ does not imply that $|B|$ is uniformly bounded.

1. Surfaces whose second fundamantal forms decay uniformly

Let M be a connected, oriented, noncompact, complete, properly immersed surface in E^3. For x in M, $|x|$ will denote the Euclidean distance to x from the origin.

Theorem 1. If there exist positive constants C and ε such that $|B(x)| \leq C|x|^{-2-\varepsilon}$ for all x in M, then we have $\int_M |B|\,dM < \infty$.

Proof. Since M is properly immersed and the second fundamental form satisfies $|B(x)| \leq C|x|^{-2-\varepsilon}$, $M(t) = \{x \in M : |x| \geq t\}$ is a union of a finite number of surfaces $M_1(t), \cdots, M_q(t)$, and for $\lambda = 1, \cdots, q$ and sufficiently large t , $M_\lambda(t)$ is diffeomorphic to $\partial M_\lambda(t) \times [t, \infty)$ ([K] Lemma 2). $\partial M_\lambda(t)$ is a closed curve whose length is less than $C_1 t$, where C_1 is a constant which does not depend on t ([K–S] Lemma 6). Let $r : M \to \mathbf{R}$ be a function on M which is defined by $r(x) = |x|$ for x in M. Then there exists a positive constant C_2 such that $|\nabla r| \geq C_2$ if r is sufficiently large ([K] Lemma 2). Hence, denoting the line element of $\partial M_\lambda(t)$ by ds, we have $dM \leq C_3\, dr ds$ for some positive constant C_3. Now, if R is sufficiently large, we have

$$\begin{aligned} \int_{M_\lambda(R)} |B|\,dM &\leq C_3 \int_R^\infty \left(\int_{\partial M_\lambda(r)} |B|\,ds \right) dr \\ &\leq C_3 \int_R^\infty (Cr^{-2-\varepsilon}\text{Length}(\partial M_\lambda(r)))\,dr \\ &\leq C_3 \int_R^\infty CC_1 r^{-1-\varepsilon}\,dr \\ &< \infty \,. \end{aligned}$$

This proves the theorem.

Theorem 2 If there exist positive constants C and ε such that $|B(x)| \leq C|x|^{-2-\varepsilon}$ for all x in M, then each end of M has an asymptotic plane, i.e., for each $\lambda = 1, \cdots, q$ there exists a plane P_λ such that $\sup_{x \in M_\lambda(t)} \inf_{y \in P_\lambda} |x-y|$ tends to 0 as $t \to \infty$.

Proof. By Lemma 1.4 in [E], our assumption implies that the normal component of the position vector on each end of M tends to a constant vector. To make it more precise, let x be a point in $M_\lambda(t)$ and $N = N(x)$ be a unit normal vector of M at x. Then, as $|x| \to \infty$, $N(x)$ converges to a constant unit vector E_λ and $\langle x, N(x) \rangle$ converges to a constant a_λ. Let P_λ be the plane defined by $\{y \in E^3 : \langle y, E_\lambda \rangle = a_\lambda\}$. First we note that $\inf_{y \in P_\lambda} |x - y| = \langle x, E_\lambda \rangle - a_\lambda$ for x in M_λ. By moving the origin of E^3 if necassary, we may assume that $a_\lambda \neq 0$. Since

$$\begin{aligned} \lim_{|x| \to \infty} \langle x, a_\lambda E_\lambda \rangle &= \lim_{|x| \to \infty} \langle x, \langle x, N \rangle N \rangle \\ &= \lim_{|x| \to \infty} \langle x, N \rangle^2 \\ &= a_\lambda^2, \end{aligned}$$

we see that $\langle x, E_\lambda \rangle - a_\lambda$ tends to 0 as $|x| \to \infty$. This shows that each end of M has an asymptotic plane.

2. Surfaces of revolution

In this section, M will be a surface of revolution in E^3 which is defined by $M = \{(x, y, z) = (r\cos\theta,\ r\sin\theta,\ f(r)) : r_0 \leq r < \infty,\ 0 \leq \theta < 2\pi\}$. The norm of the second fundamental form and the area element of M are given as

$$|B| = r^{-1}(1 + f'^2)^{-3/2}(r^2 f''^2 + f'^2(1 + f'^2)^2)^{1/2}$$

$$dM = r(1 + f'^2)^{1/2}\, dr d\theta.$$

Theorem 3. Let M be a surface of revolution given above. If $\int_M |B|\, dM < \infty$, then $\lim_{r \to \infty} f(r) = C$ for some constant C, i.e., each end of M has an asymptotic plane.

Proof. Since

$$\int_M |B|\, dM = \int_{r_0}^{\infty} dr \int_0^{2\pi} \frac{(r^2 f''^2 + f'^2(1 + f'^2)^2)^{1/2}}{1 + f'^2} d\theta$$

$$= 2\pi \int_{r_0}^{\infty} \frac{(r^2 f''^2 + f'^2(1+f'^2)^2)^{1/2}}{1+f'^2} dr$$
$$\geq 2\pi \int_{r_0}^{\infty} |f'|\, dr,$$

the condition $\int_M |B|\, dM < \infty$ implies that $\lim_{r\to\infty} f(r) = C$ for some constant C.

The following example shows that $|B|$ may not be bounded on a surface of revolution with $\int_M |B|\, dM < \infty$.

Example. Let $f(r)$ be a function which has the following property;

$$f''(r) = \begin{cases} n & \text{for} \quad n+\frac{1}{n^4} \leq r \leq n+\frac{2}{n^4} \\ 0 & \text{for} \quad n+\frac{3}{n^4} \leq r \leq n+1 \\ < n & \text{for} \quad n+\frac{2}{n^4} < r < n+\frac{3}{n^4} \\ \geq 0 & \text{for all } r \end{cases}$$

Then for $n \leq r < n+1$, we have

$$\sum_{k=1}^{n-1} \frac{1}{k^3} < f'(r) - f'(0) < \sum_{k=1}^{n} \frac{2}{k^3}.$$

Hence, by a suitable choice for $f'(0)$, we can have $f'(r) = O(r^{-2})$. If M is a surface of revolution obtained from $f(r)$, we have

$$\begin{aligned} \int_M |B|\, dM &= 2\pi \int_0^{\infty} \frac{(r^2 f''^2 + f'^2(1+f'^2)^2)^{1/2}}{1+f'^2} dr \\ &\leq 2\pi \int_0^{\infty} r|f''|\, dr + 2\pi \int_0^{\infty} |f'|\, dr \\ &< \infty. \end{aligned}$$

But $|B|$ is not bounded, since $|B| \approx n$ if $n+\frac{1}{n^4} \leq r \leq n+\frac{2}{n^4}$.

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Faculty of Industrial Science
and Technology
Science Univerity of Tokyo
Oshamambe, Hokkaido
049–35 Japan
e–mail: enomoto@it.osha.sut.ac.jp