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On Shintani's zeta functions of ternary zero forms (third approach)

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1 Introduction

This report is based on my talk in the conference but some conjectural part in the talk is now completely proved. The details is in Ibukiyama and Saito [4] and the details will be omitted here.

Let (G, V) be a prehomogeneous vector space defined over $\boldsymbol{Q}$. For most cases, the theory of zeta functions associated with (G, V) has been well known and the analytic continuation and the functional equations are proved (M. Sato and T. Shintani [6] etc.) But in some rare cases, a certain volume which is usually used to define the zeta function is infinite and we need some special consideration for modification of the definion of zeta functions. In most general cases, the definition of zeta functions is not clear and not known yet.

Here, we treat the case of ternary zero forms, which is the typical example of the rare case explained above. The final definitions of zeta functions and the functional equations obtained here are not new. They are the same as the results already known by Shintani and F. Sato. But our approarch here is completely new (at least apparently) and the proofs are completely different. Also, the previously known functional equations are derived from our new 'symmetric' functional equations. The poles and residues are obtained also by our method. The author believes that our approach gives new light to the theory and is applicable to some other unknown cases.

To explain more in detail, we roughly review the theory of zeta functions associated with (G, V). Let Γ be a discrete subgroup of $G(\boldsymbol{R})$, and let L be a lattice in $V(\boldsymbol{Q})$ which is stable under the action of Γ. For the sake of simplicity, we assume that the relative invariant polynomial is spanned by the unique polynomial $P(X)$, and put $S = \{x \in V; P(x) = 0\}$. Let $V(\boldsymbol{R}) - S(\boldsymbol{R}) = \coprod_{i=1}^{r} V^i$ (disjoint), where V^i is the connected components. Put $L' = \{x \in L; P(x) \neq 0\}$. For $x \in L'$, denote by G_x, or Γ_x the stabilizer of x in $G(\boldsymbol{R})$, or in Γ, respectively. We fix a G invariant measure of $G(\boldsymbol{R})$ and V, and we can define a measure of G_x in a unified way for all x as a 'quotient' of the above two measures, since $G(\boldsymbol{R})/G_x = V(\boldsymbol{R}) - S(\boldsymbol{R})$. If $\mu(x) = vol(G_x/\Gamma_x)$ is finite, then we can at least define the zeta functions $\zeta_i(s, L)$ for each connected component V^i by

$$\zeta_i(s, L) = \sum_{x \in \Gamma \backslash L^i} \frac{\mu(x)}{|P(x)|^s},$$

where $L^i = L \cap V^i$. Then, in most usual cases, the functional equation between the vector $(\zeta_i(s, L))_{1 \leq i \leq r}$ and the vector $(\zeta_i(s, L^*))_{1 \leq i \leq r}$ is shown, where L^* is the dual of L. But, in some cases, $\mu(x)$ is not finite for some $x \in L'$. If we take only those $x \in L_i$ such that $\mu(x) < \infty$ in the summation of the definition of the zeta functions, then we cannot expect a good functional equation. So, we must add some modified terms to the definition of the zeta functions to get a reasonable functional equation. The only case where such modification is known is the case of ternary zero forms. Next we explain this.

By some historical reason, we present here ternary zero forms as a special case of the prehomogenous vector space of symmetric matrices. Put $G = GL_n(\boldsymbol{Q})$ and $V = \{x \in M_n(\boldsymbol{Q}); x = {}^t x\}$. Let ρ be the representation of G on V defined by $\rho(g)x = gx\,{}^t g$. Then $(G(\boldsymbol{C}), V(\boldsymbol{C}), \rho)$ is a prehomogenous vector space defined over $\boldsymbol{Q}$. Then the relative invariant is spanned by $P(x) = \det(x)$ and $V(\boldsymbol{R}) - S(\boldsymbol{R})$ is the union of the sets V^i of real symmetric matrices with i positive and $n - i$ negative eigenvalues ($1 \leq i \leq n$). We put $\Gamma = SL_n(\boldsymbol{Z})$. We denote by L_n the set of all symmetric matrices of integral coefficients in V and by L_n^* the set of all half integral symmetric matrices, that is,

$$L_n^* = \{x = (x_{ij}) \in V(\boldsymbol{Q}); x_{ii} \in \boldsymbol{Z},\ 2x_{ij} \in \boldsymbol{Z} \text{ for all } 1 \leq i < j \leq n\}.$$

Then L_n and L_n^* are Γ invariant lattices. We define an invariant measure of G by

$$dg = (\det g)^{-n} \prod_{1 \leq i,j \leq n} dg_{ij}.$$

For $y = (y_{ij}) \in V$, put $dy = \prod_{1 \leq i \leq j \leq n} dy_{ij}$. Write L_n or L_n^* by L. For any $x \in L^i = L \cap V^i$, let T be a relatively compact open set in V^i and let Y be the domain in $G_+ = \{g \in GL_n(R); \det g > 0\}$ which is mapped to T by the mapping $g \to \rho(g)(x)$. Let Y_0 be the fundamental domain of Y with respect to the action of Γ_x. Then, the following volume

$$\mu(x) = \int_{Y_0} dg / \int_T \det(y)^{-(n+1)/2} dy$$

is finite except for the case $(n, i) = (2, 1)$ and independent of the choice of T. When $(n, i) \neq (2, 1)$, we define

$$\zeta_i(s, L) = c_n \sum_{x \in \Gamma \backslash L^i} \frac{\mu(x)}{|\det(x)|^s},$$

where

$$c_n = \frac{2 \prod_{j=1}^n \Gamma(j/2)}{\pi^{n(n+1)/4}}.$$

When $(n, i) = (2, 1)$, $\mu(x)$ is finite, if and only if $-\det(x)$ is not square, and we cannot define zeta functions $\zeta_1(s, L)$ as usual. We need some modification in this case.

Now, we review the history around this case. When $n = 2$, we can regard $\det(x) = x_{11}x_{22} - x_{12}^2$ as a ternary quadratic form of signature $(1, 2)$. Siegel first defined zeta functions associated with indefinite quadratic forms and proved functional equations. As for ternary zero forms, he defined $\zeta_2(s, L_2)$ and $\zeta_2(s, L_2^*)$ in the above notation and proved a functional equation between these two functions. But he did not give definition of zeta function for L_2^1 and just excluded the case. Later, Shintani succeeded to define $\zeta_1(s, L_2)$ and $\zeta_1(s, L_2^*)$ and proved a functional equation. His method was somewhat mysterious. Later F. Sato gave more conceptual proof, using a so called Eisenstein series attached to $O(2, 1)$. This 'Eisenstein series' is a certain function of two complex variables, but not a function on usual symmetric domain.

Now, we give here the third method. Rough idea is as follows. We use a certain real analytic Eisenstein series of half integral weight $-k/2$ with another complex parameter σ. This Eisenstein series is a function on the upper half plane (of one variable) and we use the Mellin transform of this function. By Hecke's usual argument, we can show the functional equation of the Mellin transform. We can expand this Eisenstein series by using Whittaker functions. By a suitable choice of k and σ, we can show that 'main part' of the Mellin transform coinsides with

$$\sum_{x \in \Gamma \backslash L^1, \mu(x) < \infty} \mu(x)/|P(x)|^s.$$

But in the Fourier expansion, some coefficients becomes infinite for the above choice of σ (where k is fixed), that is, the Eisenstein series, or its Mellin transforms has a simple pole as a function of σ at the point we need. So, we take the Laurent expansion of both sides of the functional equation of the Mellin transform, and compare the constant term. Then, we get the correct definition of $\zeta_1(s, L)$ and get the functional equation of zeta functions of prehomogeneous vector space. The details will be explained in the later sections.

2 Zeta functions and functional equation

In this section, we review the definition of zeta functions and functional equations first given by T. Shintani. Our aim of this note is to give the third alternative proof of the results in this section.

First, we define the main part of our zeta functions. Define four Dirichlet series

$$\begin{aligned}
\xi_+^M(s) &= \zeta(2s) \sum_{d=1}^{\infty} h(d) \log(\epsilon_d) d^{-s}, \\
\xi_+^{*,M}(s) &= \zeta(2s) \{ \sum_{d=1}^{\infty} h(4d) \log(\epsilon_{4d}) (4d)^{-s} + 2^{-2s} \sum_{d=1}^{\infty} h(4d+1) \log(\epsilon_{4d+1}) (4d+1)^{-s} \},
\end{aligned}$$

$$\xi_-(s) = 2\zeta(2s)\sum_{d=1}^{\infty} h(-d)w_{-d}^{-1}|d|^{-s},$$
$$\xi_-^*(s) = 2\zeta(2s)\{\sum_{d=1}^{\infty} h(-4d)w_{-4d}^{-1}|4d|^{-s} + 2^{-2s}\sum_{d=1}^{\infty} h(-4d+1)w_{-4d+1}^{-1}(4d-1)^{-s}\},$$

where $h(d)$ is the class number in the narrow sense of the order $\mathcal{O}_d$ of conductor d in a quadratic field and ϵ_d is the fundamental unit of norm 1 of $\mathcal{O}_d$ and w_d is the number of units of $\mathcal{O}_d$.

Now, we define our zeta functions.

$$\begin{aligned}\xi_+(s) &= \xi_+^M(s) + \zeta(2s-1)(\frac{\zeta'(2s)}{\zeta(2s)} - \frac{\zeta'(2s-1)}{\zeta(2s-1)}),\\ \xi_+^*(s) &= \xi_+^{*,M}(s) + 2^{1-2s}\zeta(2s-1)(\frac{\zeta'(2s)}{\zeta(2s)} - \frac{\zeta'(2s-1)}{\zeta(2s-1)}),\\ &\quad +2^{-2s}(\log 2)(1-2^{-2s})^{-1}\zeta(2s-1).\end{aligned}$$

The following theorem is due to Siegel, Shintani, and F. Sato.

Theorem 1 *Four Dirichlet series $\xi_+(s)$, $\xi_+^*(s)$, $\xi_-(s)$, and $\xi_-^*(s)$ are meromorphically continued to the whole complex s plane and satisfies the following functional equations.*

1.

$$\begin{aligned}\xi_-(3/2-s) &= 2^{2s-1}\pi^{-2s+1/2}\Gamma(s)\Gamma(s-1/2)\cos(\pi s)\xi_-^*(s)\\ &\quad -2^{-1}\pi^{-2s+1/2}\Gamma(s)\Gamma(s-1/2)\zeta(2s-1).\end{aligned}$$

2.

$$\begin{aligned}\xi_+(3/2-s) &= 2^{2s-1}\pi^{-2s+1/2}\Gamma(s)\Gamma(s-1/2)((\sin\pi s)\xi_+^*(s) + \pi\xi_-^*(s))\\ &\quad +2^{-1}\pi^{-2s+1/2}\Gamma(s)\Gamma(s-1/2)\zeta(2s-1)(\sin\pi s)(\frac{\Gamma'(s)}{\Gamma(s)} - \frac{\Gamma'(s-1/2)}{\Gamma(s-1/2)}).\end{aligned}$$

Four zeta functions $\xi_-(s)$, $\xi_-^*(s)$, $\xi_+(s)$ and $\xi_+^*(s)$ are holomorphic except for $s=1$ and $s=3/2$. $\xi_-(s)$, or $\xi_-^*(s)$ has simple poles at $s=1$ with residue -4^{-1}, or -8^{-1}, and at $s=3/2$ with residue $\pi/12$, or $\pi/24$, respectively. The principal part of the Laurent expansions of $\xi_+(s)$ or $\xi_+^*(s)$ at poles are given as follows. At $s=1$, we get $\xi_+(s) = -4^{-1}(s-1)^{-2} - 2^{-1}\log(2\pi)(s-1)^{-1} + \cdots$ and $\xi_+^*(s) = -8^{-1}(s-1)^{-2} - 4^{-1}\log(2\pi)(s-1)^{-1} + \cdots$. At $s=3/2$, we get $\xi_+(s) = 12^{-1}\pi^2(s-3/2) + \cdots$, and $\xi_+^*(s) = 24^{-1}\pi^2(s-3/2)^{-1} + \cdots$.

3 Real analytic Eisenstein series

We first review the definition of Eisenstein series. (cf. Shimura [8], Sturm [11], Cohen [1]). For any $\sigma, z \in C$ with $z \neq 0$, we put $z^\sigma = e^{\sigma \log(z)}$, where $\arg(\log(z))$ is taken so that $-\pi < \arg(\log(z)) \leq \pi$. We denote the upper half plane by $H = \{z \in C; \mathrm{Im}(z) > 0\}$. For sake of simplicity, we write $e(z) = e^{2\pi i z}$. For each $s \in C$, each odd (positive or negative) integer k, and $z \in H$, define Eisenstein series $E(k, \sigma, z)$ and $E^*(k, \sigma, z)$ by

$$E(k,\sigma,z) = y^{\frac{\sigma}{2}} \sum_{d=1,\ odd}^{\infty} \sum_{c=-\infty}^{\infty} \left(\frac{4c}{d}\right) \epsilon_d^{-k} (4cz+d)^{\frac{k}{2}} |4cz+d|^{-\sigma},$$

where $\left(\frac{*}{*}\right)$ is the quadratic residue symbol whose precise meaning is as was given in Shimura [7], p.442 and $\epsilon_d = 1$, or i, if $d \equiv 1$, or $3 \bmod 4$, respectively. When $-k + 2\sigma - 4 > 0$, this series converges absolutely and uniformly. We also define $E^*(k, \sigma, z)$ by

$$\begin{aligned} E^*(k,\sigma,z) &= E(-\frac{1}{4z})(-2iz)^{\frac{k}{2}} \\ &= y^{\frac{\sigma}{2}} 2^{\frac{k}{2}-\sigma} e(-\frac{k}{8}) \sum_{d=1,\ odd}^{\infty} \sum_{b=-\infty}^{\infty} \left(\frac{-b}{d}\right) \epsilon_d^{-k} (dz+b)^{\frac{k}{2}} |dz+b|^{-\sigma}. \end{aligned}$$

To get our Dirichlet series in question, it is convenient to consider the following Eisenstein series.

$$F(k,\sigma,z) = E(k,\sigma,z) + 2^{k/2-\sigma}(e(k/8) + e(-k/8))E^*(k,\sigma,z).$$

The value of this function at $\sigma = 0$, that is , the function $F(k, 0, z)$ has been already considered by Cohen [1] and Sturm [11] for odd k with $k < -3$. But it is essential here to consider this as a function of σ. We see that $F(k, \sigma, z)$ is an (non holomorphic) automorphic form of "weight" $-k/2$, and in particular invariant by translation : $F(k, \sigma, z + 1) = F(k, \sigma, z)$. Hence we have an Fourier expansion. That is, for $z = x + iy$, we get

$$F(k,\sigma,z) = \sum_{d=-\infty}^{\infty} c_d(y) e^{2\pi i d x},$$

where $c_d(y)$ can be written explicitly by Whittaker functions, using Shimura [8]. Here we omit the formula for $c_d(y)$. We put $G(y) = \sum_{d=-\infty}^{\infty} c_d(y)$ and $H(y) = \sum_{d=-\infty}^{\infty} c_{4d}(y/4)$. Then we get

$$G(1/4y) = (-1)^{(k^2-1)/8} 2^{1/2} y^{\sigma-k/2} H(y).$$

Define two Dirichlet series by

$$\begin{aligned} \Phi_{\sigma,k}(s) &= \int_0^\infty (G(y) - c_0(y)) y^{s-1} dy, \\ \Psi_{\sigma,k}(s) &= \int_0^\infty (H(y) - c_0(y/4)) y^{s-1} dy. \end{aligned}$$

Then by usual argument of Hecke and by some estimation of the Fourier coefficients in Shimura [8], we get

Theorem 2 *For a fixed odd integer k, the Dirichlet series $\Phi_{\sigma,k}(s)$ and $\Psi_{\sigma.k}(s)$ are continued meromorphically to the whole $(\sigma, s) \in \boldsymbol{C}^2$, and satisfies the following functional equation*

$$\Phi_{\sigma,k}(\sigma - k/2 - s) = (-1)^{(k^2-1)/8} 2^{2s-2\sigma+k+1/2} \Psi_{\sigma,k}(s).$$

Moreover, the function

$$\zeta(2\sigma - k - 1)(\sigma - (k+2)/2)(\sigma - (k+3)/2)s(s-1)(s - \sigma + k/2)(s - \sigma + k/2 + 1)\Phi_{\sigma,k}(s)$$

is holomorphic on the whole $(\sigma, s) \in \boldsymbol{C}^2$.

4 Dirichlet series associated with Eisenstein series

Next, we shall explain the relation between Dirichlet series in the last section and our zeta functions in question. For each α, $\beta \in \boldsymbol{R}$ and $s \in \boldsymbol{C}$, we consider the following integral

$$I(s, \alpha, \beta) = \int_0^\infty \frac{(1+u)^{\alpha-1} u^{\beta-1}}{(1+2u)^s} du.$$

This integral converges absolutely when $\Re(\beta) > 0$ and $\Re(s) > \Re(\alpha) + \Re(\beta) - 1$. This function is related with the usual hypergeometric functions, but the details are omitted here. For each quadratic field K, or $K = \boldsymbol{Q} \oplus \boldsymbol{Q}$, we denote by d_K the discriminant of K, or 1, respectively. For sake of simplicity, we put

$$Z^*(k, \sigma, s, d_K) = |d_K|^{\sigma-k/2-1-s} \frac{L(\sigma - (k+1)/2, \chi_K)\zeta(2s)\zeta(2s - 2\sigma + k + 2)}{\zeta(2\sigma - k - 1)L(2s - \sigma + (k+3)/2, \chi_K)},$$

and

$$C_{\sigma,k} = (-1)^{(k^2-1)/8} \frac{2^{k/2-\sigma+3/2}\pi^{\sigma-k/2}}{\Gamma((\sigma-k)/2)\Gamma(\sigma/2)}.$$

We write the above Dirichle series as $Z^*(k, \sigma, s, d_K) = \sum_{n=1}^\infty a(n) n^{-s}$ and put

$$Z(k, \sigma, s, d_K) = \sum_{n=1}^\infty \frac{a(4n)}{n^s}.$$

Then, we get

$$\Phi_{\sigma,k}(s) =$$
$$C_{\sigma,k}(2\pi)^{-s}\Gamma(s)$$
$$\times \Big(\sum_{(-1)^{(k+1)/2} d_K > 0} Z^*(k, \sigma, s, d_K) I(s, \frac{\sigma - k}{2}, \frac{\sigma}{2}) + \sum_{(-1)^{(k+1)/2} d_K < 0} Z^*(k, \sigma, s, d_K) I(s, \frac{\sigma}{2}, \frac{\sigma - k}{2}) \Big)$$

and

$$\Psi_{\sigma,k}(s) =$$
$$C_{\sigma,k}(2\pi)^{-s}\Gamma(s)$$
$$\times\Big(\sum_{(-1)^{(k+1)/2}d_K>0} Z(k,\sigma,s,d_K)I(s,\frac{\sigma-k}{2},\frac{\sigma}{2}) + \sum_{(-1)^{(k+1)/2}d_K<0} Z(k,\sigma,s,d_K)I(s,\frac{\sigma}{2},\frac{\sigma-k}{2})\Big),$$

where K runs over quadratic fields or $\boldsymbol{Q}\oplus\boldsymbol{Q}$. Now first we explain easier case which has been already treated in Sturm [11], that is, the functional equation between $\xi_-(s)$ and $\xi_-^*(s)$. We review his argument for the readers convenience. We put $k=-3$. By the usual class number formula, we can easily see that

$$\pi\xi_-(s) = \zeta(2)\sum_{d_K<0} Z^*(-3,0,s,d_K)$$

and

$$\pi\xi_-^*(s) = 2^{-2s}\zeta(2)\sum_{d_K<0} Z(-3,0,s,d_K).$$

On the other hand, $I(s,\sigma/2,(\sigma+3)/2)$ is holomorphic at $\sigma=0$ and $\Gamma(\sigma/2)^{-1}I(s,\sigma/2,(\sigma+3)/2)$ vanishes at $\sigma=0$. Hence in the summation over $d_K>0$, the only term which has a pole at $\sigma=0$ remains. The only such term is the term for $d_K=1$, and that term is constant times $\zeta(2s-1)$. Then by straight forward calculation, we get the functional equation (1) in Theorem 1. For details, see Sturm [11] or Ibukiyama and Saito [4]. Next, we explain more difficult part which is new. From now on, we fix $k=1$. Then, by the classical class number formula, we can show that

$$\begin{aligned}
\xi_+^M(s) &= \sum_{d_K>0,d_K\neq 1} \zeta(2)Z^*(1,2,s,d_K),\\
\pi\xi_-(s) &= \sum_{d_K<0} \zeta(2)Z^*(1,2,s,d_K)\\
\xi_+^{*,M}(s) &= 2^{-2s}\sum_{d_K>0,d_K\neq 1} \zeta(2)Z(1,2,s,d_K),\\
\pi\xi_-^*(s) &= 2^{-2s}\sum_{d_K<0} \zeta(2)Z^*(1,2,s,d_K).
\end{aligned}$$

We put

$$\begin{aligned}
g(2,s) &= \zeta(2)^{-1}(\xi_+^M(s)I(s,1,1/2)+\pi\xi_-(s)I(s,1/2,1))\\
g^*(2,s) &= 2^{2s}\zeta(2)^{-1}(\xi_+^{*,M}(s)I(s,1,1/2)+\pi\xi_-^*(s)I(s,1/2,1)).
\end{aligned}$$

To rewrite the functional equation in the last section, we put

$$\begin{aligned}
A(\sigma,s) &= 2^{(\sigma-1)/2}\zeta(2\sigma-2)\Gamma((\sigma-1)/2)^{-1} \\
&\quad \times\frac{\Gamma(\sigma-s-1/2)}{\Gamma(1-s)}g(\sigma,\sigma-s-1/2) \\
A^*(\sigma,s) &= 2^{(\sigma-1)/2}\zeta(2\sigma-2)\Gamma((\sigma-1)/2)^{-1} \\
&\quad \times(2\pi)^{\sigma-2s-1/2}2^{2s-2\sigma+3/2}\frac{\Gamma(s)}{\Gamma(1-s)}g^*(\sigma,s) \\
B(\sigma,s) &= \frac{\Gamma(\sigma-s-1/2)}{\Gamma((\sigma+1)/2-s)}\frac{\zeta(2\sigma-1-2s)\zeta(2-2s)}{\zeta(\sigma+1-2s)} \\
&\quad \times F_2(\sigma,s) \\
B^*(\sigma,s) &= 2^{-\sigma+1}\pi^{\sigma-2s-1/2}\frac{\Gamma(s)\Gamma(s-\sigma+3/2)}{\Gamma(1-s)\Gamma(s-\sigma/2+1)} \\
&\quad \times\frac{\zeta(2s)\zeta(2s-2\sigma+3)}{\zeta(2s-\sigma+2)}\times\frac{1-2^{\sigma-2}+2^{2\sigma-3}-2^{-2s+2\sigma-3}}{1-2^{-2s+\sigma-2}} \\
&\quad \times F_1(\sigma,s),
\end{aligned}$$

where each $F_i(\sigma,s)$ ($i=1,2$) is defined by using hypergeometric functions as follows.

$$\begin{aligned}
F_1(\sigma,s) &= F(1-\frac{\sigma}{2},\frac{\sigma-1}{2};s-\frac{\sigma}{2}+1;\frac{1}{2}), \\
F_2(\sigma,s) &= F(1-\frac{\sigma}{2},\frac{\sigma-1}{2},\frac{\sigma+1}{2}-s;\frac{1}{2}).
\end{aligned}$$

Then, after cancelling some common factors, the functional equation in the last section reads

$$A(\sigma,s)+B(\sigma,s)\zeta(\sigma-1)=A^*(\sigma,s)+\zeta(\sigma-1)B^*(\sigma,s).$$

Both sides are meromorphic as functions of σ and s and have poles at $\sigma=2$ (if we fix generic s). Of course the residue of both sides at $\sigma=2$ coincides, but this is not so interesting, since it gives only the functional equation between $\zeta(2s-1)$ and $\zeta(2-2s)$. Next, we compare the constant terms (of the Laurent expansions with respect to σ at 2) of both sides. This is really interesting and gives the desired results for our purpose. As is easily shown, we get

$$\begin{aligned}
A(2,s) &= \xi_+^M(3/2-s) \\
&\quad +\sqrt{2\pi}\frac{\Gamma(3/2-s)}{\Gamma(1-s)}\xi_-(3/2-s)I(3/2-s,1/2,1), \\
A^*(2,s) &= 2^{2s-1}\pi^{1/2-2s}\Gamma(s)\Gamma(s-1/2) \\
&\quad \times\{\sin(\pi s)\xi_+^{*,M}(s)+\pi^{3/2}\sqrt{2}\frac{\xi_-^*(s)}{\Gamma(s-1/2)\Gamma(1-s)}I(s,1/2,1)\}.
\end{aligned}$$

Now, we sketch the proof of Theorem 1. We know already the functional equation between $\xi_-(s)$ and $\xi_-^*(s)$. So, we eliminate $\xi_-(s)$ in the functional equation. We can also give a formula to express $\frac{d}{d\sigma}(F_1-F_2)|_{\sigma=2}$ by derivatives or translates of Γ functions and $I(s,1/2,1)$, applying some classical formula between hypergeometric functions. We also eliminate sevral terms by using the functional equation of Riemann zeta function. Throught these calculations, we can discover how to modify the zeta functions in question and we can show the second functional equation in Theorem 1. As you can see, the original functional equation above is more symmetric than the one in Theorem 1, but contains several strange integrals and not easy to see. If we want to write down everything by relatively known functions such as Γ functions or Riemann zeta functions, it is inevitable to break the symmetry. To obtain the principal terms around poles, we need more involved calculation. For example, we must know poles and residues of functions $\frac{d}{d\sigma}F_1(\sigma,s)|_{\sigma=2}$ or $I(s,1/2,1)$ and also we need several special values of these functions. These are rather complicated calculation and omitted here. The details are in [4].

5 Zeta functions of symmetric matrices

Until last section, we explained rather pathological case of the theory. So it seems not useless to review here "simpler" general case and how it "degenerates" when $n=2$. This section is based on Ibukiyama and Saito [3]. We assume $n\geq 3$ in this section. When $n\geq 4$ is even, for each $\delta=1$, or -1, we define Dirichlet series $D(s,\delta)$ by

$$D_n^*(s,\delta) \;=\; 2(2\pi)^{-n/2}(-1)^{[n/4]}\Gamma(n/2)\sum_{(-1)^{n/2}\delta d_K>0}|d_K|^{(n-1)/2-s}\frac{L(n/2,\chi_K)\zeta(2s)\zeta(2s-n+1)}{L(2s-n/2+1,\chi_K)}.$$

Let

$$D_n^*(s,\delta)=\sum_{d=1}^{\infty}H(\frac{n}{2},d,\delta)d^{-s}.$$

Also for even $n\geq 4$, we define

$$D_n(s,\delta)=\sum_{d=1}^{\infty}H(\frac{n}{2},4d,\delta)d^{-s}.$$

The Dirichlet series $D_n^*(s,1)$, or $D_n(s,1)$ is obtained from $\Phi_{0,-n-1}(s)$, or $\Psi_{0,-n-1}(s)$ in the last section. On the other hand, the Dirichlet series $\Phi_{2,-n+3}(s)$, or $\Psi_{2,-n+3}(s)$ can be written by using $D_n^*(s,\delta)$ with $\delta=\pm1$, or and $D_n(s,\delta)$ with $\delta=\pm1$, respectively.

We use the folloowing notation:

$$b_n=\frac{|\prod_{i=1}^{[(n-1)/2]}B_{2i}|}{2^{n-1}([\frac{n-1}{2}])!},\qquad B'_{n/2}=2(\frac{n}{2})!(2\pi)^{-n/2}\zeta(\frac{n}{2})(-1)^{1+n/2}.$$

Here, B_{2i} is the Bernoulli numbers. For even n, we put

$$A_n(s)=\prod_{i=1}^{n/2-1}\zeta(2s-2i),\qquad \text{and}\qquad B_n(s)=\prod_{i=1}^{n/2}\zeta(2s-(2i-1)).$$

For odd n, we put

$$\begin{aligned} Q_n(s) &= \zeta(s-\frac{n-1}{2})\prod_{i=1}^{(n-1)/2}\zeta(2s-(2i-1)),\\ R_n(s) &= \zeta(s)\prod_{i=1}^{(n-1)/2}\zeta(2s-2i). \end{aligned}$$

We can show that the zeta functions $\zeta_i(s,L_n)$ or $\zeta_i(s,L_n^*)$ defined in section 1 depends only on $\delta=(-1)^{n-i}$ and $\epsilon=(-1)^{(n-i)(n-i+1)/2}$ and not i itself. Hence we denote these zeta functions as $\zeta(s,L,\delta,\epsilon)$ for $L=L_n$ or L_n^*.

Theorem 3 *(cf. [4]) If n is odd with $n\geq 3$, we have*

$$\begin{aligned} \zeta(s,L_n^*,\delta,\epsilon) &= b_n 2^{(n-1)s}(Q_n(s)+\epsilon\delta^{(n+1)/2}(-1)^{(n^2-1)/8}R_n(s)),\\ \zeta(s,L_n,\delta,\epsilon) &= b_n(2^{(n-1)/2}Q_n(s)+\epsilon\delta^{(n+1)/2}(-1)^{(n^2-1)/8}R_n(s)). \end{aligned}$$

If n is even with $n\geq 4$, we have

$$\begin{aligned} \zeta(s,L_n^*,\delta,\epsilon) &= b_n 2^{ns}((-1)^{[n/4]}D_n^*(s,\delta)A_n(s)+\epsilon\delta_n(-1)^{n(n+2)/8}\frac{2|B'_{n/2}|}{n}B_n(s))\\ \zeta(s,L_n,\delta,\epsilon) &= b_n((-1)^{[n/4]}D_n(s,\delta)A_n(s)+\epsilon\delta_n(-1)^{n(n+2)/8}\frac{2^{(n+2)/2}|B'_{n/2}|}{n}B_n(s)), \end{aligned}$$

where $\delta_n=1$ if $(-1)^{n/2}\equiv\delta \bmod 4$, and $\delta_n=0$ otherwise.

The functional equations between these functions are simple and easily obtained from the functional equation of the Mellin transform of the Eisenstein series. For details, see [4]

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