数理解析研究所講究録
999 巻 1997 年 21-30

ON SOME FUNCTIONAL EQUATION OF A PREHOMOGENEOUS SPACE OF SYMMETRIC MATRICES.

TAMOTSU IKEDA (KYOTO UNIVERSITY)

Introduction

We gave an expository talk on the results of the paper of J. Sweet [5],[6]. Here we will give another proof of his results.

§1. Properties of Weil constants.

Let k be a local field with char.$k \neq 2$. We fix a non-trivial additive character ψ of k. When k non-archimedean, we denote by $\mathfrak{o}$ the maximal order of k. Let ϖ and q be a prime element of $\mathfrak{o}$ and the number of elements of the residue field of $\mathfrak{o}$. We put $c_k = [k^\times : (k^\times)^2] = 4\,|2|^{-1}$. Let $\langle\ ,\ \rangle$ be the Hilbert symbol of k.

We define a Weil constant $\alpha(a)$ by the equation

$$\int_k \phi(x)\psi(ax^2)\,\mathrm{d}x = \alpha(a)|2a|^{-\frac{1}{2}} \int_k \hat{\phi}(x)\psi(-\frac{1}{4a}x^2)\,\mathrm{d}x,$$

$\phi \in \mathcal{S}(k)$. Here,

$$\hat{\phi}(x) = \int_k \phi(y)\psi(xy)\,\mathrm{d}y$$

is the Fourier transform of ϕ. Then $|\alpha(a)| = 1$, $\alpha(-a) = \overline{\alpha(a)}$, and

$$\frac{\alpha(1)\alpha(ab)}{\alpha(a)\alpha(b)} = \langle a, b\rangle.$$

For a quadratic form Q over k, we denote by d_Q and ϵ_Q the determinant and the Minkowski-Hasse invariant of Q, respectively. When Q is equivalent to a diagonal form diag$(q_1, \dots q_m)$, $\epsilon_Q = \prod_{i<j}\langle q_i, q_j\rangle$. We put $\Delta_Q = (-1)^{[\frac{m}{2}]}d_Q$. We think of d_Q and Δ_Q as elements of $k^\times/(k^\times)^2$. we put $\alpha_Q(a) = \prod_{i=1}^m \alpha(q_i a)$. It is easy to see $\alpha_Q(1) = \alpha(1)^{m-1}\alpha(d_Q)\epsilon_Q$.

Proposition 1. *For $a \in k^\times$,*

$$\sum_{x\in k^\times/(k^\times)^2} \alpha(x)\langle x, a\rangle = c_k^{\frac{1}{2}}\frac{\alpha(1)}{\alpha(a)}.$$

Proof. It is enough to show that

$$\sum_{x\in k^\times/(k^\times)^2} \alpha(x) = c_k^{\frac{1}{2}}.$$

When k is archimedean, this is clear. Let k be a non-archimedean local field. We may assume that ψ is of order 0. Let e be an integer such that $|2| = q^{-e}$. Let A be a complete representative of $\mathfrak{o}^\times/(\mathfrak{o}^\times)^2$. Then $A \cup \varpi A$ is a complete representative of $k^\times/(k^\times)^2$. Let ϕ be the characteristic function of $\mathfrak{o}$. When $x \in A \cup \varpi A$, the left hand of the equation

$$\int_k \phi(t)\psi(xt^2)\,dt = \alpha(x)\,|2x|^{-\frac{1}{2}} \int_k \hat{\phi}(t)\psi(-\frac{1}{4x}t^2)\,dt$$

is equal to 1. If x runs over A and y runs over $\mathfrak{o} - \{0\}$, then $-xy^2$ runs over $\bigcup_{r=0}^{\infty} \varpi^{2r}\mathfrak{o}^\times$ twice. If we put $t = -xy^2$, then $dt = |2y|\,dy$. Therefore

$$\sum_{x\in A} \overline{\alpha(x)} = |2|^{-\frac{1}{2}} \cdot 2|2|^{-1} \sum_{r=0}^{\infty} q^{-r} \int_{\mathfrak{o}^\times} \psi(\varpi^{2r-2e}t)\,dt.$$

Similarly,

$$\sum_{x\in \varpi A} \overline{\alpha(x)} = |2|^{-\frac{1}{2}} q^{\frac{1}{2}} \cdot 2|2|^{-1} \sum_{r=0}^{\infty} q^{-r} \int_{\mathfrak{o}^\times} \psi(\varpi^{2r-2e-1}t)\,dt.$$

Observe that $[\mathfrak{o}^\times : (\mathfrak{o}^\times)^2] = 2\,|2|^{-1}$ and

$$\int_{\mathfrak{o}^\times} \psi(\varpi^l t)dt = \begin{cases} 1 - q^{-1}, & \text{if } l \geq 0, \\ -q^{-1}, & \text{if } l = -1, \\ 0, & \text{if } l < -1. \end{cases}$$

It follows that

$$\begin{aligned} \sum_{x\in k^\times/(k^\times)^2} \alpha(x) &= \sum_{x\in A} \overline{\alpha(x)} + \sum_{x\in \varpi A} \overline{\alpha(x)} \\ &= 2q^{e/2} = 2\,|2|^{-\frac{1}{2}}. \end{aligned}$$

Now we consider a relationship of the Weil constant and the Tate functional equation. Let ω be a unitary character of $k^\times$. For $\Phi \in \mathcal{S}(k)$, the functional equation is biven by

$$\frac{\int_{k^\times} \hat{\Phi}(x)\omega^{-1}(x)|x|^{1-s}\,d^\times x}{L(1-s,\omega^{-1})} = \varepsilon(s,\omega,\psi)\frac{\int_{k^\times} \Phi(x)\omega(x)|x|^s\,d^\times x}{L(s,\omega)}, 0 < \mathrm{Re}(s) < 1$$

and both sides are meromorphically continued to whole s-plane. We put

$$\varepsilon'(s,\omega,\psi) = \varepsilon(s,\omega,\psi)\frac{L(1-s,\omega^{-1})}{L(s,\omega)}.$$

Note that even if Φ is not a Schwartz function, the functional equation holds as long as both sides converges in some vertical strip in $\{0 < \mathrm{Re}(s) < 1\}$.

We put $\chi_\theta(x) = \langle \theta, x\rangle$ for $\theta \in k^\times$.

Now we prove the following proposition.

Proposition 1. *For any character ω of $k^\times$,*

$$\sum_{\theta\in k^\times/(k^\times)^2} \frac{\alpha(1)}{\alpha(\theta)}\varepsilon'(s,\omega\chi_\theta,\psi)^{-1} = 2\,|2|^{-2s}\,\omega(\tfrac{1}{4})\varepsilon'(2s,\omega^2,\psi)^{-1}\varepsilon'(s+\tfrac{1}{2},\omega,\psi).$$

Proof. We reproduce a proof of Rallis and Schiffmann [3]. For $\varphi\in\mathcal{S}(k)$, we define a L^1-function $M_\phi\in\mathrm{L}^1(k)$ by

$$M_\varphi(x) = \begin{cases} 2|x|^{\frac{1}{2}}[\varphi(\sqrt{x})+\varphi(-\sqrt{x})] & x\in(k^\times)^2, \\ 0 & \text{otherwise.}\end{cases}$$

The for any bounded function f over k,

$$\int_k M_\varphi(y)f(y)\,dy = \int_k \varphi(y)f(y^2)\,dy.$$

In particular,

$$\begin{aligned}\int_k M_\varphi(y)\psi(xy)\,dy =& \int_k \varphi(y)\psi(xy^2)\,dy \\ =&\alpha(x)|2x|^{-\frac{1}{2}}\int_k \hat{\varphi}(y)\psi(-\frac{1}{4x}y^2)\,dy \\ =&\alpha(x)|2x|^{-\frac{1}{2}}\int_k M_{\hat{\varphi}}(y)\psi(-\frac{1}{4x}y)\,dy.\end{aligned}$$

I.e.,

$$\overline{\alpha(x)}|2x|^{\frac{1}{2}}\widehat{M_\varphi}(x) = M_{\hat{\varphi}}(-\frac{1}{4x}).$$

Since the right hand side is a Fourier transform of a L^1-function, $\widehat{M_\varphi}(x)|x|^{\frac{1}{2}}$ is bounded when $|x|\to\infty$. It follows that

$$\int_{k^\times} \widehat{M_\varphi}(x)\omega^{-1}(x)|x|^{\frac{1}{2}-s}\,d^\times x$$

is absolutely convergent for $\mathrm{Re}(s)<1$. On the other hand

$$\int_{k^\times} M_\varphi(x)\omega(x)|x|^{s+\frac{1}{2}}\,d^\times x = \int_{k^\times}\varphi(x)\omega^2(x)|x|^{2s}\,d^\times x$$

is absolutely convergent for $\Re(s)>\frac{1}{2}$. Therfore we have a functional equation

$$\begin{aligned}\int_{k^\times}\widehat{M_\varphi}(x)\omega^{-1}(x)|x|^{\frac{1}{2}-s}\,d^\times x =&\varepsilon'(s+\frac{1}{2},\omega,\psi)\int_{k^\times} M_\varphi(x)\omega(x)|x|^{s+\frac{1}{2}}\,d^\times x \\ =&\varepsilon'(s+\frac{1}{2},\omega,\psi)\int_{k^\times}\varphi(x)\omega^2(x)|x|^{2s}\,d^\times x \\ =&\varepsilon'(s+\frac{1}{2},\omega,\psi)\varepsilon'(2s,\omega^2,\psi)^{-1} \\ &\times\int_{k^\times}\varphi(x)\omega^2(x)|x|^{2s}\,d^\times x\end{aligned}$$

for $\frac{1}{2} < \Re(s) < 1$. Note that

$$\begin{aligned}
&\int_{k^\times} \widehat{M_\varphi}(x)\omega^{-1}(x)|x|^{\frac{1}{2}-s}\, d^\times x \\
=&|2|^{-\frac{1}{2}} \int_{k^\times} \alpha(x)\, \widehat{M_{\hat\varphi}}(-\frac{1}{4x})\omega^{-1}(x)|x|^{-s}\, d^\times x \\
=&|2|^{-\frac{1}{2}}\omega(-4) \int_{k^\times} \overline{\alpha(x)}\, \widehat{M_{\hat\varphi}}(x)\omega(x)|\tfrac{1}{4x}|^{-s}\, d^\times x \\
=&|2|^{2s-\frac{1}{2}}\omega(-4) \int_{k^\times} \overline{\alpha(x)}\, \widehat{M_{\hat\varphi}}(x)\omega(x)|x|^{s}\, d^\times x \\
=&|2|^{2s-\frac{1}{2}}\omega(-4) \sum_{\beta\in k^\times/(k^\times)^2} \overline{\alpha(\beta)} \int_{\beta\cdot(k^\times)^2} \widehat{M_{\hat\varphi}}(x)\omega(x)|x|^{s}\, d^\times x
\end{aligned}$$

Here,

$$\begin{aligned}
&\sum_{\beta\in k^\times/(k^\times)^2} \overline{\alpha(\beta)} \int_{\beta\cdot(k^\times)^2} \widehat{M_{\hat\varphi}}(x)\omega(x)|x|^{s}\, d^\times x \\
=&c_k^{-1} \sum_{\beta,\theta\in k^\times/(k^\times)^2} \overline{\alpha(\beta)}\chi_\beta(\theta) \int_{k^\times} \widehat{M_{\hat\varphi}}(x)\omega\chi_\theta(x)|x|^{s}\, d^\times x
\end{aligned}$$

Here, by Proposition 1, $\sum_{\beta,\in k^\times/(k^\times)^2} \overline{\alpha(\beta)}\chi_\beta(\theta) = c_k^{\frac{1}{2}}\frac{\alpha(1)}{\alpha(\theta)}\chi_\theta(-1)$. Therefore the equation above is

$$\begin{aligned}
&2^{-1}|2|^{\frac{1}{2}} \sum_{\theta\in k^\times/(k^\times)^2} \frac{\alpha(1)}{\alpha(\theta)}\chi_\theta(-1) \int_{k^\times} \widehat{M_{\hat\varphi}}(x)\omega\chi_\theta(x)|x|^{s}\, d^\times x \\
=&2^{-1}|2|^{\frac{1}{2}} \sum_{\theta\in k^\times/(k^\times)^2} \frac{\alpha(1)}{\alpha(\theta)}\chi_\theta(-1)\varepsilon'(1-s,\omega^{-1}\chi_\theta,\psi) \int_{k^\times} M_{\hat\varphi}(x)\omega^{-1}\chi_\theta(x)|x|^{1-s}\, d^\times x \\
=&2^{-1}|2|^{\frac{1}{2}}\omega(-1) \sum_{\theta\in k^\times/(k^\times)^2} \frac{\alpha(1)}{\alpha(\theta)}\varepsilon'(s,\omega\chi_\theta,\psi)^{-1} \int_{k^\times} \hat\varphi(x)\omega^{-2}(x)|x|^{1-2s}\, d^\times x
\end{aligned}$$

It follow that

$$\sum_{\theta\in k^\times/(k^\times)^2} \frac{\alpha(1)}{\alpha(\theta)}\varepsilon'(s,\omega\chi_\theta,\psi)^{-1} = 2\,|2|^{-2s}\,\omega(\tfrac{1}{4})\varepsilon'(2s,\omega^2,\psi)^{-1}\varepsilon'(s+\tfrac{1}{2},\omega,\psi).$$

Corollary.

$$\varepsilon(\tfrac{1}{2},\chi_\beta,\psi) = \frac{\alpha(1)}{\alpha(\beta)}$$

Proof. Put $\omega = \chi_\beta$ in Proposition 2 and consider the limit of the equation

$$\sum_{\theta\in k^\times/(k^\times)^2} \frac{\alpha(1)}{\alpha(\theta)}\frac{\varepsilon'(2s,1,\psi)}{\varepsilon'(s,\chi_{\beta\theta},\psi)} = 2\,|2|^{-2s}\,\varepsilon'(s+\tfrac{1}{2},\chi_\beta,\psi)$$

for $s \to 0$.

§§2. The space of symmetric matrices of rank n.

Let $V = \mathrm{Sym}_n(k)$, $G = \mathrm{GL}_n$. V is a prehomogeneous vector space under the action of an algebraic group G. The set of open orbits of V is denoted by $\mathcal{O} = \mathcal{O}_V$. If k is non-archimedean, $\mathcal{O}$ is parametrised by the determinant and the Minkowski-Hasse invariant.

For each $\eta \in \mathcal{O}$, we put $d_\eta = \det \eta \in k^\times/(k^\times)^2$, $\Delta_\eta = (-1)^{[\frac{n}{2}]} d_\eta$. We denote the Minkowski-Hasse invariant of η by ϵ_η.

For $\Phi \in \mathcal{S}(V)$, we put

$$\widehat{\Phi}(x) = \int_V \Phi(y)\psi(\mathrm{tr}\, xy)dy.$$

Here the measure dx is the self dual measure for the Fourier transform, i.e., $dx = |2|^{\frac{n(n-1)}{4}} \prod_{i=1}^n dx_{ii} \prod_{i<j} dx_{ij}$.

The purpose of this section is to prove some local functional equations for the prehomogeneous vector space V.

Theorem 1. *We put $\rho = \frac{n+1}{2}$. If $\Phi \in \mathcal{S}(V)$, $n = 2m+1$, then*

$$\begin{aligned}\int_V \Phi(X)\omega(\det X)\,|\det X|^{s-\rho}\, dX =& \varepsilon'(s-m,\omega)^{-1} \prod_{r=1}^m \varepsilon'(2s-2m-1+2r,\omega^2)^{-1} \\ & \times |2|^{-2ms+\frac{m(2m+1)}{2}} \omega(\tfrac{1}{4})^m \langle -1,-1\rangle^{\frac{m(m+1)}{2}} \\ & \times \sum_{\eta\in\mathcal{O}} \langle (-1)^m, d_\eta\rangle \epsilon_\eta \\ & \times \int_{V_\eta} \widehat{\Phi}(X)\omega^{-1}(\det X)|\det X|^{-s}\, dX.\end{aligned}$$

If $n = 2m$,

$$\begin{aligned}\int_V \Phi(X)\omega(\det X)\,|\det X|^{s-\rho}\, dX =& \varepsilon'(s-m+\tfrac{1}{2},\omega)^{-1} \textstyle\prod_{r=1}^m \varepsilon'(2s-2m+2r,\omega^2)^{-1} \\ & \times |2|^{-2ms+\frac{m(2m-1)}{2}} \omega(\tfrac{1}{4})^m \\ & \times \sum_{\eta\in\mathcal{O}} \frac{\alpha(\Delta_\eta)}{\alpha(1)} \varepsilon'(s+\tfrac{1}{2},\omega\chi_{\Delta_\eta}) \\ & \times \int_{V_\eta} \widehat{\Phi}(X)\omega^{-1}(\det X)|\det X|^{-s} dX.\end{aligned}$$

Proof. From the prehomogeneity of V, there is a meromorphic function $c_\eta(\omega,s)$ such that

$$\int_V \Phi(X)\omega(\det X)\,|\det X|^{s-\rho}\, dX = \sum_{\eta\in\mathcal{O}} c_\eta(\omega,s) \int_{V_\eta} \widehat{\Phi}(X)\omega(\det X)|\det X|^{-s}\, dX.$$

We have to prove that if $n = 2m+1$,

$$c_\eta(\omega, s) = \varepsilon'(s-m,\omega)^{-1} \prod_{r=1}^{m} \varepsilon'(2s-2m-1+2r,\omega^2)^{-1}$$
$$\times |2|^{-2ms+\frac{m(2m+1)}{2}} \omega(\tfrac{1}{4})^m \langle (-1)^m\xi, d_\eta\rangle \langle -1,-1\rangle^{\frac{m(m+1)}{2}} \epsilon_\eta,$$

and that if $n = 2m$,

$$c_\eta(\omega, s) = \varepsilon'(s-m+\tfrac{1}{2},\omega)^{-1} \textstyle\prod_{r=1}^{m} \varepsilon'(2s-2m+2r,\omega^2)^{-1}$$
$$\times |2|^{-2ms+\frac{m(2m-1)}{2}} \omega(\tfrac{1}{4})^m \frac{\alpha(\Delta_\eta)}{\alpha(1)} \varepsilon'(s+\tfrac{1}{2}, \omega\chi_{\Delta_\eta}, \psi).$$

We proceed by induction on n. When $n = 1$, it is nothing but Tate's local functional equation for $\omega = \chi_\xi$: We will use this functional equation as the form

$$\int_{\rho(k^\times)^2} \Phi(x)\omega(x)|x|^s\, d^\times x$$
$$= c_k^{-1} \sum_{\beta,\theta\in k^\times/(k^\times)^2} \langle\theta,\rho\beta\rangle\, \varepsilon'(s,\omega\chi_\theta,\psi)^{-1} \int_{\beta(k^\times)^2} \hat{\Phi}(x)\omega^{-1}(x)|x|^{1-s}\, d^\times x.$$

Now we assume $n \geq 1$. We assume the functional equation is true for n. We put $V = \mathrm{Sym}_n(k)$, $V' = \mathrm{Sym}_{n+1}(k)$, $\rho = \frac{n+1}{2}$, and $\rho' = \frac{n+2}{2}$. We denote the set of GL_n orbits of V by $\mathcal{O}$ and the set of GL_{n+1} orbits of V' by $\mathcal{O}'$.

We may assume $\Phi\begin{pmatrix} T & y \\ {}^t y & x\end{pmatrix} = \phi_1(T)\phi_2(y)\phi_3(x)$ for some $\phi_1 \in \mathcal{S}(V)$, $\phi_2 \in \mathcal{S}(k^n)$, $\phi_3 \in \mathcal{S}(k)$. Note that $\hat{\Phi}\begin{pmatrix} T & y \\ {}^t y & x\end{pmatrix} = |2|^{\frac{n}{2}}\hat{\phi}_1(T)\hat{\phi}_2(2y)\hat{\phi}_3(x)$.

$$\int_{T'\in V'} \Phi(T')\omega(\det T')|\det T'|^{s-\rho'}\, dT'$$
$$= \int_{T\in V}\int_{y\in k^n}\int_{x\in k} \Phi\begin{pmatrix} (T+yx^{-1}\,{}^t y) & y \\ {}^t y & x\end{pmatrix} \omega(x\det T)|x\det T|^{s-\rho'}\, dx\, dy\, dT$$
$$= \sum_{\eta\in\mathcal{O}} c_\eta(\omega, s-\tfrac{1}{2}) \int_{T\in V_\eta}\int_{y\in k^n}\int_{x\in k} \hat{\phi}_1(T)\psi(-x^{-1}\,{}^t yTy)\phi_2(y)\phi_3(x)$$
$$\times\, \omega^{-1}(\det T)\omega(x)|x|^{s-\rho'}\, |\det T|^{-s+\frac{1}{2}}\, dx\, dy\, dT$$
$$= \sum_{\eta\in\mathcal{O}} c_\eta(\omega, s-\tfrac{1}{2}) \int_{T\in V_\eta}\int_{y\in k^n}\int_{x\in k}$$
$$\times\, \overline{\alpha_\eta(x)}|2^n x^{-n}\det T|^{-\frac{1}{2}}\hat{\phi}_1(T)\hat{\phi}_2(y)\phi_3(x)$$
$$\times\, \psi(\frac{1}{4}x\,{}^t yT^{-1}y)\omega^{-1}(\det T)|\det T|^{-s+\frac{1}{2}}\omega(x)|x|^{s-\rho'}\, dx\, dy\, dT$$

$$= \sum_{\eta\in\mathcal{O}} \sum_{\rho\in k^\times/(k^\times)^2} c_\eta(\omega, s-\tfrac{1}{2})\overline{\alpha_\eta(\rho)} \int_{T\in V_\eta}\int_{y\in k^n}\int_{x\in\rho(k^\times)^2} |2|^{\frac{n}{2}}\hat{\phi}_1(T)\hat{\phi}_2(2y)\phi_3(x)$$
$$\times\, \psi(x\,{}^t yT^{-1}y)\omega(\det T)|\det T|^{-s}\omega(x)|x|^{s-1}\,dx\,dy\,dT$$
$$= \sum_{\eta\in\mathcal{O}} \sum_{\rho,\theta,\beta\in k^\times/(k^\times)^2} c_k^{-1}c_\eta(\omega, s-\tfrac{1}{2})\overline{\alpha_\eta(\rho)}\,\langle\theta,\rho\beta\rangle\,\varepsilon'(s,\omega\chi_\theta)^{-1}$$
$$\times \int_{T\in V_\eta}\int_{y\in k^n}\int_{x\in\beta(k^\times)^2} |2|^{\frac{n}{2}}\hat{\phi}_1(T)\hat{\phi}_2(2y)\hat{\phi}_3(x+{}^t yT^{-1}y)$$
$$\times\, \omega^{-1}(x\det T)|x\det T|^{-s}\,dx\,dy\,dT$$

Therefore if we put $\eta\oplus\beta = \begin{pmatrix}\eta & \\ & \beta\end{pmatrix}$, then

$$c_{\eta\oplus\beta}(\omega,s) = c_k^{-1}c_\eta(\omega,s-\tfrac{1}{2})\textstyle\sum_{\rho,\theta}\overline{\alpha_\eta(\rho)}\,\langle\theta,\rho\beta\rangle\,\varepsilon'(s,\omega\chi_\theta)^{-1}$$

If $n=2m$ is even,

$$c_k^{-1}\sum_{\rho,\theta}\overline{\alpha_\eta(\rho)}\,\langle\theta,\rho\beta\rangle\,\varepsilon'(s,\omega\chi_\theta)^{-1}$$
$$=\langle -1,-1\rangle^{\frac{m(m+1)}{2}}\,\langle(-1)^m,d_\eta\rangle\frac{\alpha(1)}{\alpha(\Delta_\eta)}\epsilon_\eta\, c_k^{-1}\sum_{\rho,\theta}\langle\Delta_\eta,\rho\rangle\,\langle\theta,\rho\beta\rangle\,\varepsilon'(s,\omega\chi_\theta)^{-1}$$
$$=\langle -1,-1\rangle^{\frac{m(m+1)}{2}}\,\langle(-1)^m,d_\eta\rangle\frac{\alpha(1)}{\alpha(\Delta_\eta)}\epsilon_\eta\,\langle\Delta_\eta,\beta\rangle\,\varepsilon'(s,\omega\chi_{\Delta_\eta})^{-1}$$
$$=\langle -1,-1\rangle^{\frac{m(m+1)}{2}}\,\langle(-1)^m,d_{\eta\oplus\beta}\rangle\frac{\alpha(1)}{\alpha(\Delta_\eta)}\epsilon_{\eta\oplus\beta}\,\varepsilon'(s,\omega\chi_{\Delta_\eta})^{-1}.$$

It follows that

$$c_{\eta\oplus\beta}(\omega,s) = c_\eta(\omega,s-\tfrac{1}{2})\langle -1,-1\rangle^{\frac{m(m+1)}{2}}\,\langle(-1)^m,d_{\eta\oplus\beta}\rangle\tfrac{\alpha(1)}{\alpha(\Delta_\eta)}\epsilon_{\eta\oplus\beta}\,\varepsilon'(s,\omega\chi_{\Delta_\eta})^{-1}$$
$$=\varepsilon'(s-m,\omega)^{-1}\prod_{r=1}^{m}\varepsilon'(2s-2m-1+2r,\omega^2)^{-1}$$
$$\times\,|2|^{-2ms+\frac{m(2m+1)}{2}}\omega(\tfrac{1}{4})^m\tfrac{\alpha(\Delta_\eta)}{\alpha(1)}\varepsilon'(s,\omega\chi_{\Delta_\eta})$$
$$\times\,\langle -1,-1\rangle^{\frac{m(m+1)}{2}}\,\langle(-1)^m,d_{\eta\oplus\beta}\rangle\frac{\alpha(1)}{\alpha(\Delta_\eta)}\epsilon_{\eta\oplus\beta}\,\varepsilon'(s,\omega\chi_{\Delta_\eta})^{-1}$$
$$=\varepsilon'(s-m,\omega)^{-1}\prod_{r=1}^{m}\varepsilon'(2s-2m-1+2r,\omega^2)^{-1}$$
$$\times\,|2|^{-2ms+\frac{m(2m+1)}{2}}\omega(\tfrac{1}{4})^m\,\langle -1,-1\rangle^{\frac{m(m+1)}{2}}\,\langle(-1)^m,d_{\eta\oplus\beta}\rangle\,\epsilon_{\eta\oplus\beta}$$

On the other hand, if $n=2m+1$ is odd,

$$c_k^{-1}\sum_{\rho,\theta}\overline{\alpha_\eta(\rho)}\,\langle\theta,\rho\beta\rangle\,\varepsilon'(s,\omega\chi_\theta)^{-1}$$

$$=\langle -1,-1\rangle^{\frac{m(m+1)}{2}}\,\langle (-1)^m, d_\eta\rangle \frac{\alpha(1)}{\alpha(\Delta_\eta)}\epsilon_\eta\, c_k^{-1}\sum_{\rho,\theta}\overline{\alpha(\rho)}\,\langle \Delta_\eta,\rho\rangle\,\langle \theta,\rho\beta\rangle\,\varepsilon'(s,\omega\chi_\theta)^{-1}$$

$$=\langle -1,-1\rangle^{\frac{m(m+1)}{2}}\,\langle (-1)^m, d_\eta\rangle \frac{\alpha(1)}{\alpha(\Delta_\eta)}\epsilon_\eta\, c_k^{-1}\sum_{\rho,\theta}\overline{\alpha(\rho)}\,\langle \Delta_\eta\theta,\rho\rangle\,\langle \theta,\beta\rangle\,\varepsilon'(s,\omega\chi_\theta)^{-1}$$

$$=\langle -1,-1\rangle^{\frac{m(m+1)}{2}}\,\langle (-1)^m, d_\eta\rangle\overline{\alpha(\Delta_\eta)}\epsilon_\eta\, c_k^{-\frac{1}{2}}\sum_{\theta}\alpha(\Delta_\eta\theta)\,\langle \theta,\beta\rangle\,\varepsilon'(s,\omega\chi_\theta)^{-1}$$

$$=\langle -1,-1\rangle^{\frac{m(m+1)}{2}}\,\langle (-1)^m, d_\eta\rangle\overline{\alpha(\Delta_\eta)}\epsilon_\eta\, c_k^{-\frac{1}{2}}\sum_{\theta}\alpha(-\beta\theta)\,\langle \Delta_\eta\theta,\beta\rangle\,\varepsilon'(s,\omega\chi_{-\Delta_\eta\beta\theta})^{-1}$$

$$=\langle -1,-1\rangle^{\frac{m(m+1)}{2}}\,\langle (-1)^m, d_\eta\rangle\overline{\alpha(\Delta_\eta)\alpha(\beta)}\,\langle \Delta_\eta,\beta\rangle\,\epsilon_\eta\, c_k^{-\frac{1}{2}}\sum_{\theta}\frac{\alpha(1)}{\alpha(\theta)}\,\varepsilon'(s,\omega\chi_{-\Delta_\eta\beta\theta})^{-1}$$

$$=|2|^{-2s+\frac{1}{2}}\omega(\tfrac{1}{4})\langle -1,-1\rangle^{\frac{m(m+1)}{2}}\,\langle (-1)^m, d_\eta\rangle\,\frac{\alpha(\Delta_{\eta\oplus\beta})}{\alpha(1)}\epsilon_\eta\,\varepsilon'(2s,\omega^2)^{-1}\varepsilon'(s+\tfrac{1}{2},\omega\chi_{\Delta_{\eta\oplus\beta}})$$

It follows that

$$\begin{aligned}
c_{\eta\oplus\beta}(\omega,s) =& c_\eta(\omega,s-\tfrac{1}{2})\langle -1,-1\rangle^{\frac{m(m+1)}{2}}\,\langle (-1)^m, d_\eta\rangle\,\epsilon_\eta \\
&\times |2|^{-2s+\frac{1}{2}}\omega(\tfrac{1}{4})\frac{\alpha(\Delta_{\eta\oplus\beta})}{\alpha(1)}\varepsilon(2s,\omega^2)^{-1}\varepsilon'(s+\tfrac{1}{2},\omega\chi_{\Delta_{\eta\oplus\beta}}) \\
=& \varepsilon'(s-m-\tfrac{1}{2},\omega)^{-1}\prod_{r=1}^m\varepsilon'(2s-2m-2+2r,\omega^2)^{-1} \\
&\quad\times |2|^{-2m(s-\frac{1}{2})+\frac{m(2m+1)}{2}-2s+\frac{1}{2}}\omega(\tfrac{1}{4})^{m+1}\frac{\alpha(\Delta_{\eta\oplus\beta})}{\alpha(1)} \\
&\quad\times \varepsilon(2s,\omega^2)^{-1}\varepsilon'(s+\tfrac{1}{2},\omega\chi_{\Delta_{\eta\oplus\beta}}) \\
=& \varepsilon'(s-m-\tfrac{1}{2},\omega)^{-1}\prod_{r=1}^m\varepsilon'(2s-2m-2+2r,\omega^2)^{-1} \\
&\times |2|^{-2(m+1)s+\frac{(m+1)(2m+1)}{2}}\omega(\tfrac{1}{4})^{m+1}\frac{\alpha(\Delta_{\eta\oplus\beta})}{\alpha(1)}\varepsilon'(s+\tfrac{1}{2},\omega\chi_{\Delta_{\eta\oplus\beta}}).
\end{aligned}$$

Theorem 1 is thus proved.

Next we prove

Theorem 2.
If $n=2m+1$,

$$\begin{aligned}
\int_V \Phi(X)\omega(\det X)\,\epsilon_X\,|\det X|^{s-\rho}\,dX =& \varepsilon'(s,\omega\chi_{(-1)^m})^{-1}\prod_{r=1}^m\varepsilon'(2s-2m-2+2r,\omega^2)^{-1} \\
&\times |2|^{-2ms+\frac{m(2m+3)}{2}}\omega(\tfrac{1}{4})^m\,\langle -1,-1\rangle^{\frac{m(m+1)}{2}} \\
&\times \int_V\widehat{\Phi}(X)\omega^{-1}\chi_{(-1)^m}(\det X)\,|\det X|^{-s}\,dX
\end{aligned}$$

If $n=2m$,

$$\int_V \Phi(X)\omega(\det X)\,\epsilon_X\,|\det X|^{s-\rho}\,dX$$
$$=\prod_{r=1}^{m}\varepsilon'(2s-2m-1+2r,\omega^2)^{-1}|2|^{-2ms+\frac{m(2m+1)}{2}}\omega(\tfrac{1}{4})^m\frac{\alpha((-1)^m)}{\alpha(1)}$$
$$\times\sum_{\eta\in\mathcal{O}}\frac{\alpha(\Delta_\eta)}{\alpha(1)}\int_{V_\eta}\widehat{\Phi}(X)\omega^{-1}\chi_{(-1)^{m+1}}(\det X)\,\epsilon_X\,|\det X|^{-s}\,dX$$

Proof.

In fact, when n is odd, this is equivalent to Theorem 1.

As in the proof of Theorem 1, one can see there is a function $c'_\eta(\xi,s)$ such that

$$\int_V \Phi(X)\omega(\det X)\,\epsilon_X\,|\det X|^{s-\rho}\,dX=\sum_{\eta\in\mathcal{O}}c'_\eta(\omega,s)\int_{V_\eta}\widehat{\Phi}(X)\omega^{-1}(\det X)\,|\det X|^{-s}\,dX.$$

We have to prove that if $n=2m$,

$$c'_\eta(\omega,s)=\prod_{r=1}^{m}\varepsilon'(2s-2m-1+2r,\omega^2)^{-1}$$
$$\times|2|^{-2ms+\frac{m(2m+1)}{2}}\omega(\tfrac{1}{4})^m\frac{\alpha(\Delta_\eta)}{\alpha(1)}\epsilon_\eta$$

We have already proved that if $n=2m+1$, then

$$c'_\eta(\omega,s)=\varepsilon'(s,\omega\chi_{(-1)^m})^{-1}\prod_{r=1}^{m}\varepsilon'(2s-2m-2+2r,\omega^2)^{-1}$$
$$\times|2|^{-2ms+\frac{m(2m+3)}{2}}\omega(\tfrac{1}{4})^m\,\langle -1,-1\rangle^{\frac{m(m+1)}{2}}\langle(-1)^m,d_\eta\rangle.$$

As in the proof of Theorem 1, one can prove

$$c'_{\eta\oplus\beta}(\omega,s)=c_k^{-1}\sum_{\rho,\theta}c'_\eta(\omega\chi_\rho,s-\tfrac{1}{2})\overline{\alpha_\eta(\rho)}\,\langle\rho,d_\eta\rangle\,\langle\theta,\rho\beta\rangle\,\varepsilon'(x,\omega\chi_\theta)^{-1}$$

If the rank of η is $2m+1$, then

$$c'_{\eta\oplus\beta}(\omega,s)$$
$$=c_k^{-1}\sum_{\rho,\theta\in k^\times/(k^\times)^2}c'_\eta(\omega\chi_\rho,s-\tfrac{1}{2})\overline{\alpha_\eta(\rho)}\,\langle\rho,d_\eta\rangle\,\langle\theta,\rho\beta\rangle\,\varepsilon'(s,\omega\chi_\theta)^{-1}$$
$$=\prod_{r=1}^{m}\varepsilon'(2s-2m-2+2r,\omega^2)^{-1}|2|^{-2ms+\frac{m(2m+3)}{2}}\omega(\tfrac{1}{4})^m\,\langle -1,-1\rangle^{\frac{m(m+1)}{2}}\langle(-1)^m,d_\eta\rangle$$
$$\times c_k^{-1}\sum_{\rho,\theta\in k^\times/(k^\times)^2}\varepsilon'(s-\tfrac{1}{2},\omega\chi_{(-1)^m\rho})^{-1}\overline{\alpha_\eta(\rho)}\,\langle\rho,d_\eta\rangle\,\langle\theta,\rho\beta\rangle\,\varepsilon'(s,\omega\chi_\theta)^{-1}$$

$$= \prod_{r=1}^{m} \varepsilon'(2s-2m-2+2r,\omega^2)^{-1}|2|^{-2ms+\frac{m(2m+3)}{2}}\omega(\tfrac{1}{4})^m \tfrac{\alpha(1)}{\alpha(\Delta_\eta)}\epsilon_\eta$$
$$\times c_k^{-1} \sum_{\rho,\theta\in k^\times/(k^\times)^2} \overline{\alpha(\rho)}\langle\rho,(-1)^m\rangle\langle\theta,\rho\beta\rangle\varepsilon'(s-\tfrac{1}{2},\omega\chi_{(-1)^m\rho})^{-1}\varepsilon'(s,\omega\chi_\theta)^{-1}$$

Here,

$$c_k^{-1}\frac{\alpha(1)}{\alpha(\Delta_\eta)}\epsilon_\eta \sum_{\rho,\theta\in k^\times/(k^\times)^2} \overline{\alpha(\rho)}\langle\rho,(-1)^m\theta\rangle\langle\theta,\beta\rangle\varepsilon'(s-\tfrac{1}{2},\omega\chi_{(-1)^m\rho})^{-1}\varepsilon'(s,\omega\chi_\theta)^{-1}$$
$$=c_k^{-1}\frac{\alpha(1)}{\alpha(\Delta_\eta)}\epsilon_\eta \sum_{\rho,\theta\in k^\times/(k^\times)^2} \frac{\alpha((-1)^m\theta)}{\alpha(1)\alpha((-1)^m\theta\rho)}\langle\theta,\beta\rangle\varepsilon'(s-\tfrac{1}{2},\omega\chi_{(-1)^m\rho})^{-1}\varepsilon'(s,\omega\chi_\theta)^{-1}$$
$$=c_k^{-1}\frac{\alpha(1)}{\alpha(\Delta_\eta)}\epsilon_\eta \sum_{\rho,\theta\in k^\times/(k^\times)^2} \frac{\alpha((-1)^m\theta)}{\alpha(1)\alpha(\rho)}\langle\theta,\beta\rangle\varepsilon'(s-\tfrac{1}{2},\omega\chi_{\theta\rho})^{-1}\varepsilon'(s,\omega\chi_\theta)^{-1}$$
$$=c_k^{-1}2|2|^{-2s+1}\omega(\tfrac{1}{4})\varepsilon'(2s-1,\omega^2)\tfrac{\alpha(1)}{\alpha(\Delta_\eta)}\epsilon_\eta\textstyle\sum_{\theta\in k^\times/(k^\times)^2}\tfrac{\alpha((-1)^m\theta)}{\alpha(1)^2}\langle\theta,\beta\rangle$$
$$=2^{-1}|2|^{-2s+2}\omega(\tfrac{1}{4})\varepsilon'(2s-1,\omega^2)\tfrac{\alpha(1)}{\alpha(\Delta_\eta)}\epsilon_\eta\textstyle\sum_{\theta\in k^\times/(k^\times)^2}\tfrac{\alpha(\theta)}{\alpha(1)^2}\langle(-1)^m\theta,\beta\rangle$$
$$=c_k^{\frac{1}{2}}2^{-1}|2|^{-2s+2}\omega(\tfrac{1}{4})\varepsilon'(2s-1,\omega^2)\overline{\alpha(\Delta_\eta)\alpha(\beta)}\epsilon_\eta\langle(-1)^m,\beta\rangle$$
$$=|2|^{-2s+\frac{3}{2}}\omega(\tfrac{1}{4})\varepsilon'(2s-1,\omega^2)\overline{\alpha(\Delta_\eta\beta)\alpha(1)}\epsilon_\eta\langle d_\eta,\beta\rangle$$
$$=|2|^{-2s+\frac{3}{2}}\omega(\tfrac{1}{4})\varepsilon'(2s-1,\omega^2)\tfrac{\alpha(\Delta_{\eta\oplus\beta})}{\alpha(1)}\epsilon_{\eta\oplus\beta}$$

References

1. C. Moeglin, M.-F. Vigneras, J.-L. Waldspurger, *Corresondances de Howe sur un corps p-adique LN 1291*, Springer, 1987.
2. S. Rallis, G. Schiffmann, *Représentaions supercuspidales du groupe m'etaplectique*, J. Math. Kyoto Univ. (1977), 567–603.
3. S.Rallis, G. Schiffmann, *Distributions invariantes par le groupe orthogonal*, LN497 (1975), 494–642.
4. W. Scharlau, *Quadratic and Hermitian forms*, Springer, 1985.
5. J. Sweet, *A computation of Gamma matrix of a family of p-adic zeta integrals*, J. Number Theory **55**, 222–260.
6. J. Sweet, *Functional equations of p-adic zeta integrals, and representations of the metaplectic group*, preprint.
7. A. Weil, *Sur certain groupes d'opérateurs unitaires*, Acta Math. **111** (1964), 143–211.

Graduate School of Mathematics,
Kyoto University,
Kitashirakawa,
Kyoto, 606 Japan