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Zeta functions of prehomogeneous vector spaces and weakly spherical homogeneous spaces

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0. Let G be a reductive $\mathbb{Q}$-group and H a $\mathbb{Q}$-subgroup of G. We consider the homogeneous space $X = H\backslash G$. Let P be a proper parabolic $\mathbb{Q}$-subgroup of G. We call $X = H\backslash G$ *P-spherical* (resp. *weakly spherical*) if the P-action is prehomogeneous (resp. if it is P-spherical for some P). One can associate a family of zeta functions with a weakly spherical homogeneous space and it is a conjecture that the associated zeta functions satisfy certain functional equations similar to those satisfied by Eisenstein series.

If $G = GL(n)$, then the notion of weakly spherical homogeneous space is closely related to the notion of prehomogeneous vector space and it is quite fruitful to investigate zeta functions of prehomogenous vector spaces from the view point of weakly spherical homogeneous spaces.

The aim of this note is to explain this new point of view through an example. First we give the functional equations satisfied by the zeta functions attached to the weakly spherical space $Spin(10)\backslash GL(16)$. Then we explain briefly how the result on $Spin(10)\backslash GL(16)$ can be used to obtain the explicit functional equation of the zeta functions attached to the prehomogeneous vector space $(Spin(10) \times GL(3), \text{half-spin} \otimes \square)$. This space is one of the most complicated ones among irreducible regular prehomogeneous vector spaces and the explicit formula for the functional equation of the zeta function has not been known before.

1. The starting point is the following simple observation:

Lemma 1.1 *Let $P_{e_1,\dots,e_r}$ be the standard (namely, upper triangular) parabolic subgroup of $GL(n)$ corresponding to the partition $n = e_1 + \cdots + e_r$. Let H be a connected $\mathbb{Q}$-subgroup of $GL(n)$ Then the following are equivalent:*

(1) $(H \times P_{e_1,\dots,e_{r-1}}, M(n,m))$ $(m = e_1 + \cdots + e_{r-1})$ *is a prehomogeneous vector space.*

(2) $X = H\backslash GL(n)$ *is $P_{e_1,\dots,e_r}$-spherical.*

In the case of $H = Spin(10)$, we have the following.

Lemma 1.2 (Kimura [KKO]) *We identify $H = Spin(10)$ with the image of the half-spin representation in $GL(16)$. Let P be a standard parabolic subgroup of $GL(16)$. Then, $X = Spin(10)\backslash GL(16)$ is P-spherical if and only if P contains one of $P_{1,1,1,13}$, $P_{1,1,13,1}$, $P_{1,13,1,1}$, and $P_{13,1,1,1}$.*

Combining these two lemma, we see that $(Spin(10) \times P_{1,1,1}, \text{half-spin} \otimes \square)$ is a prehomogeneous vector space. Since the calculation of the explicit functional equation for $(Spin(10) \times GL(3), \text{half-spin} \otimes \square)$ can easily be reduced to the calculation for $(Spin(10) \times P_{1,1,1}, \text{half-spin} \otimes \square)$, we concentrate our attention to this prehomogeneous space and the weakly spherical homogeneous space $Spin(10)\backslash GL(16)$.

2. Let P be one of $P_{1,1,1,13}$, $P_{1,1,13,1}$, $P_{1,13,1,1}$, and $P_{13,1,1,1}$. Denote by Ω_P the open P-orbit in X. Let $\mathcal{X}(P)$ be the group of rational characters of P and $\mathcal{X}_H(P)$ the subgroup of $\mathcal{X}(P)$ of characters corresponding to relative P-invariants on X. Then there exist 4 algebraically independent relative invariants on X and $\mathcal{X}_H(P)$ is of finite index in $\mathcal{X}(P)$. For $\chi \in \mathcal{X}_H(P)$, we fix a relative invariant f^χ satisfying

$$f^\chi(xp) = \chi(p) f^\chi(x) \quad (x \in X,\ p \in P).$$

We may assume that

$$f^{\chi\psi} = f^\chi f^\psi \quad (\chi, \psi \in \mathcal{X}_H(P)).$$

Put

$$\mathcal{E}(P) = \mathrm{Hom}(\mathcal{X}_H(P), \{\pm 1\}).$$

For $\epsilon \in \mathcal{E}(P)$, put

$$\Omega_{P,\epsilon} = \left\{ x \in \Omega_{P,\mathbb{R}} \;\middle|\; \operatorname{sgn} f^\chi(x) = \frac{f^\chi(x)}{|f^\chi(x)|} = \epsilon(\chi)\ (\forall \chi \in \mathcal{X}_{\mathcal{H}}(\mathcal{P})) \right\}.$$

Then

$$\Omega_{P,\mathbb{R}} = \bigcup_{\epsilon \in \mathcal{E}(P)} \Omega_{P,\epsilon}$$

gives the decompositon into connected components.

We put

$$\mathfrak{a}^*_{P,\mathbb{C}} = \mathcal{X}_H(P) \otimes \mathbb{C} = \mathcal{X}(P) \otimes \mathbb{C}.$$

Choose a system of generators $\{\chi_1, \chi_2, \chi_3, \chi_4\}$ of $\mathcal{X}_H(P)$ so that f^{χ_i} is regular everywhere on X. For $\lambda = \sum_{i=1}^4 \lambda_i \chi_i \in \mathfrak{a}^*_{P,\mathbb{C}}$ ($\mathrm{Re}(\lambda_i) > 0$) and $\epsilon \in \mathcal{E}(P)$, the function

$$|f(x)|_\epsilon^\lambda = \begin{cases} \prod_{i=1}^4 |f^{\chi_i}(x)|^{\lambda_i} & \text{if } x \in \Omega_{P,\epsilon}, \\ 0 & \text{otherwise.} \end{cases}$$

does not depend on the choice of system of generators and is extended to a distribution with meromorphic parameter $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$.

Let N be the unipotent radical of P and put

$$\Delta_P(p) = d(pnp^{-1})/dn \in \mathcal{X}(P),$$

where dn is the invariant gauge form on N. We define $\delta = \delta_P \in \mathfrak{a}^*_{P,\mathbb{C}}$ by

$$\delta = \delta_P = \frac{1}{2}\Delta_P.$$

Now we can define the zeta functions $E_\epsilon(P; x, \lambda)$ for $x \in X_\mathbb{Q}$ and $\epsilon \in \mathcal{E}(P)$ by

$$E_\epsilon(P; x, \lambda) = \sum_{y \in x \cdot \Gamma \cap \Omega_{P,\epsilon}/\Gamma \cap P} \mu(y) \, |f(y)|_\epsilon^{-(\lambda+\delta)},$$

where $\Gamma = GL(16, \mathbb{Z})$ and $\mu(y)$ is the normalized volume of $P_{y,\mathbb{R}}/\Gamma \cap P_y$ (P_y is the isotropy subgroup of P at y).

Theorem 2.3 (1) *Let $\mathfrak{a}^{*+}_P$ be the open cone in $\mathfrak{a}^*_{P,\mathbb{R}} = \mathcal{X}(P) \otimes \mathbb{R}$ generated by characters corresponding to everywhere regular relative invariants and put*

$$\mathfrak{a}^{*+}_{P,\mathbb{C}} = \mathfrak{a}^{*+}_P + \sqrt{-1}\mathfrak{a}^*_{P,\mathbb{R}}.$$

Then $E_\epsilon(P; x, \lambda)$ are absolutely convergent if $\lambda \in \delta + \mathfrak{a}^{*+}_{P,\mathbb{C}}$.

(2) *The series $E_\epsilon(P; x, \lambda)$ have analytic continuations to meromorphic functions of λ in $\mathfrak{a}^*_{P,\mathbb{C}}$.*

To describe the functional equations satisfied by $E_\epsilon(P; x, \lambda)$, we need some notational preliminaries. Let S_4 be the symmetric group in 4 letters. For $P = P_{e_1,e_2,e_3,e_4}$ and $\sigma \in S_4$, we define

$${}^\sigma P = P_{e_{\sigma^{-1}(1)}, e_{\sigma^{-1}(2)}, e_{\sigma^{-1}(3)}, e_{\sigma^{-1}(4)}}.$$

Moreover σ defines an isomomorphism between the standard Levi subgroups of P and ${}^\sigma P$ by permutation of the diagonal entries and induces a linear isomorphism

$$\sigma : \mathfrak{a}^*_{P,\mathbb{C}} \to \mathfrak{a}^*_{{}^\sigma P,\mathbb{C}}.$$

For $p \in P$, we write

$$p = \begin{pmatrix} p_1 & & & * \\ & p_2 & & \\ & & p_3 & \\ 0 & & & p_4 \end{pmatrix}.$$

We define the normalized zeta functions by

$$\hat{E}_\epsilon(P;x,\lambda) = \prod_{1\le\mu<\nu\le 4} \hat{\zeta}\left(z_\mu - z_\nu + \frac{e_\mu + e_\nu}{2}\right) E_\epsilon(P;x,\lambda),$$

where $\lambda = \sum_{\mu=1}^4 z_\mu \det p_\mu \in \mathfrak{a}_{P,\mathbb{C}}^*$ and $\hat{\zeta}(z) = \pi^{-z/2}\Gamma(z/2)\zeta(z)$ ($\zeta(z)$ = the Riemann zeta function).

Theorem 2.4 *For $\sigma \in S_4$ and $\epsilon \in \mathcal{E}({}^\sigma P)$, the following functional equation holds :*

$$\hat{E}_\epsilon({}^\sigma P; x, \sigma\lambda) = \sum_{\eta\in\mathcal{E}(P)} C_{\epsilon,\eta}(P;\sigma,\lambda)\hat{E}_\eta(P;x,\lambda),$$

where $C_{\epsilon,\eta}(P;\sigma,\lambda)$ are meromorphic functions of λ independent of $x \in X_\mathbb{Q}$ and have elementary expressions in terms of the gamma fucntion and the exponential function.

3. Let us collect here some data, which are necessary to make the functional equations explicit.

For simplicity, we put

$$P_1 = P_{1,1,1,13},\ P_2 = P_{1,1,13,1},\ P_3 = P_{1,13,1,1},\ P_4 = P_{13,1,1,1}.$$

We write

$$\delta(p) = \delta_P(p) = \delta_1 \det p_1 + \delta_2 \det p_2 + \delta_3 \det p_3 + \delta_4 \det p_4.$$

Then

$$(\delta_1,\delta_2,\delta_3,\delta_4) = \begin{cases} \left(\frac{15}{2},\frac{13}{2},\frac{11}{2},-\frac{3}{2}\right) & \cdots\ P = P_1, \\ \left(\frac{15}{2},\frac{13}{2},-\frac{1}{2},-\frac{15}{2}\right) & \cdots\ P = P_2, \\ \left(\frac{15}{2},\frac{1}{2},-\frac{13}{2},-\frac{15}{2}\right) & \cdots\ P = P_3, \\ \left(\frac{3}{2},-\frac{11}{2},-\frac{13}{2},-\frac{15}{2}\right) & \cdots\ P = P_4. \end{cases}$$

We can choose 4 characters $\chi_0,\chi_1,\chi_2,\chi_3$ such that $\chi_0^{\pm1},\chi_1,\chi_2,\chi_3$ generate the semigroup $\mathcal{X}_H^+(P)$ consiting of characters corresponding to everywhere regular relative P-invariants. These characters are given by the following and form a basis of the free abelian group $\mathcal{X}_H(P)$:

Case $P = P_1$:

$$\chi_0(p) = p_1p_2p_3\cdot\det p_4,\ \chi_1(p) = p_1^2p_2^2,\ \chi_2(p) = p_1^4p_2^2p_3^2,\ \chi_3(p) = p_1^4p_2^4p_3^4.$$

Case $P = P_2$:

$$\chi_0(p) = p_1p_2\cdot\det p_3p_4,\ \chi_1(p) = p_1^2p_2^2,\ \chi_2(p) = p_1^4p_2^2\det p_3^2,\ \chi_3(p) = p_1^6p_2^6\cdot\det p_3^4.$$

Case $P = P_3$:

$$\chi_0(p) = p_1 \cdot \det p_2 \cdot p_3 p_4,\ \chi_1(p) = p_4^{-2} p_3^{-2},\ \chi_2(p) = p_4^{-4} p_3^{-2} \cdot \det p_2^{-2},\ \chi_3(p) = p_4^{-6} p_3^{-6} \cdot \det p_2^{-4}.$$

Case $P = P_4$:

$$\chi_0(p) = \det p_1 \cdot p_2 p_3 p_4,\ \chi_1(p) = p_3^{-2} p_4^{-2},\ \chi_2(p) = p_2^{-2} p_3^{-2} p_4^{-4},\ \chi_3(p) = p_2^{-4} p_3^{-4} p_4^{-4}.$$

We write

$$\lambda = z_1 \det p_1 + z_2 \det p_2 + z_3 \det p_3 + z_4 \det p_4 \in \mathfrak{a}^*_{P,\mathbb{C}}$$

and put

$$\alpha_i = z_i - z_{i+1} \quad (i = 1, 2, 3).$$

Then, by the fact that $\chi_0^{\pm 1}, \chi_1, \chi_2, \chi_3$ generate the semigroup $\mathcal{X}_H^+(P)$, we have

$$\mathfrak{a}^{*+}_{P,\mathbb{C}} = \left\{ \lambda \in \mathfrak{a}^*_{P,\mathbb{C}} \,\middle|\, (*) \right\},$$

where

$$(*) : \begin{cases} \mathrm{Re}(\alpha_3) > \mathrm{Re}(\alpha_1) > 0, \mathrm{Re}(\alpha_2) > 0 & \cdots\ P = P_1, \\ 2\mathrm{Re}(\alpha_2) + \mathrm{Re}(\alpha_1) > \mathrm{Re}(\alpha_3) > \mathrm{Re}(\alpha_1) > 0 & \cdots\ P = P_2, \\ 2\mathrm{Re}(\alpha_2) + \mathrm{Re}(\alpha_3) > \mathrm{Re}(\alpha_1) > \mathrm{Re}(\alpha_3) > 0 & \cdots\ P = P_3, \\ \mathrm{Re}(\alpha_1) > \mathrm{Re}(\alpha_3) > 0, \mathrm{Re}(\alpha_2) > 0 & \cdots\ P = P_4. \end{cases}$$

The group S_4 acts on $\mathcal{E}(P)$ naturally. The following is the simplest case of the functional equation given in Theorem 2.4:

Case $\sigma = (i, i+1)$ $(i = 1, 2, 3)$ and ${}^\sigma P = P$: If $\sigma\epsilon = \epsilon$, then we have

$$\hat{E}_\epsilon(P; x, \sigma\lambda) = \hat{E}_\epsilon(P; x, \lambda).$$

If $\sigma\epsilon \neq \epsilon$, then we have

$$\begin{pmatrix} \hat{E}_\epsilon(P; x, \sigma\lambda) \\ \hat{E}_{\sigma\epsilon}(P; x, \sigma\lambda) \end{pmatrix} = \begin{pmatrix} \sec \frac{\pi(z_{i+1} - z_i)}{2} & \tan \frac{\pi(z_{i+1} - z_i)}{2} \\ \tan \frac{\pi(z_{i+1} - z_i)}{2} & \sec \frac{\pi(z_{i+1} - z_i)}{2} \end{pmatrix} \begin{pmatrix} \hat{E}_\epsilon(P; x, \lambda) \\ \hat{E}_{\sigma\epsilon}(P; x, \lambda) \end{pmatrix}.$$

Put

$$\Gamma_P(\lambda) = \prod_{1 \le \mu < \nu \le 4} \Gamma_{\mathbb{R}} \left(z_\mu - z_\nu + \frac{e_\mu + e_\nu}{2} \right), \quad \Gamma_{\mathbb{R}}(z) = \pi^{-z/2} \Gamma\left(\frac{z}{2}\right) \zeta(z).$$

In the following, for $\epsilon \in \mathcal{E}(P)$, we put

$$\epsilon_i = \epsilon(\chi_i) \quad (i = 0, 1, 2, 3).$$

Case $\sigma = (2,3)$, $P = P_{1,1,13,1}$, ${}^{\sigma}P = P_{1,13,1,1}$: In this case, unless $\epsilon_0 = \eta_0$ and $\epsilon_2 = \eta_2$, we have

$$C_{\epsilon,\eta}(P;\sigma,\lambda) = 0.$$

If $\epsilon_0 = \eta_0$ and $\epsilon_2 = \eta_2$, we have

$$\begin{aligned} C_{\epsilon,\eta}(P;\sigma,\lambda) &= \text{const} \times \exp(\text{linear form of } \lambda) \times \frac{\Gamma_{\sigma P}(\sigma\lambda)}{\Gamma_P(\lambda)} \\ &\times A^{(4,4)}_{\eta_1,\eta_2\epsilon_1}\left(\frac{\alpha_1 + 2\alpha_2 - \alpha_3}{4} - 2\right) \times \begin{cases} A^{(2,4)}_{\eta_3,\eta_2\epsilon_3}\left(\frac{\alpha_3-\alpha_1-6}{4}\right) & \text{if } \eta_2 = +1, \\ A^{(3,3)}_{\eta_3,\eta_2\epsilon_3}\left(\frac{\alpha_3-\alpha_1-6}{4}\right) & \text{if } \eta_2 = -1, \end{cases} \end{aligned}$$

where

$$\begin{pmatrix} A^{(p,q)}_{++}(z) & A^{(p,q)}_{+-}(z) \\ A^{(p,q)}_{-+}(z) & A^{(p,q)}_{--}(z) \end{pmatrix} = \Gamma_{\mathbb{R}}(2z+2)\Gamma_{\mathbb{R}}(2z+p+q) \begin{pmatrix} \cos\pi\left(x + \frac{q+1}{2}\right) & \sin\frac{p\pi}{2} \\ \sin\frac{q\pi}{2} & \cos\pi\left(x + \frac{p+1}{2}\right) \end{pmatrix}.$$

Note that this is the gamma matrix of the functional equation satisfied by

$$\left|x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2\right|^z.$$

Case $\sigma = (3,4)$, $P = P_{1,1,1,13}$, ${}^{\sigma}P = P_{1,1,13,1}$: In this case, unless $\epsilon_0 = \eta_0$ and $\epsilon_1 = \eta_1$, we have

$$C_{\epsilon,\eta}(P;\sigma,\lambda) = 0.$$

If $\epsilon_0 = \eta_0$ and $\epsilon_1 = \eta_1$, we have

$$\begin{aligned} C_{\epsilon,\eta}(P;\sigma,\lambda) &= \text{const} \times \exp(\text{linear form of } \lambda) \times \frac{\Gamma_{\sigma P}(\sigma\lambda)}{\Gamma_P(\lambda)} \\ &\times B^{(3,4)}_{(\eta_1\eta_2,\eta_1\eta_3),(\eta_1\epsilon_2\epsilon_3,\eta_1\epsilon_2)}\left(\frac{\alpha_1 - 1}{2}, \frac{\alpha_3 - \alpha_1 - 6}{4}\right), \end{aligned}$$

where

$$B^{(3,4)}_{\epsilon,\eta}(z_1,z_2) = \Gamma_{\mathbb{R}}(2z_1 + 2z_2 + 3)\Gamma_{\mathbb{R}}(2z_1 + 2z_2 + 8)\Gamma_{\mathbb{R}}(2z_2 + 2)\Gamma_{\mathbb{R}}(2z_2 + 7) \times b^{(3,4)}_{\epsilon,\eta}(z_1,z_2)$$

and the functions $b^{(3,4)}_{\epsilon,\eta}(z_1,z_2)$ are given by the following table:

$\epsilon\backslash\eta$	$++$	$+-$	$-+$	$--$
$++$	$\cos\pi(s_1+s_2)\sin\pi s_2$	0	0	0
$+-$	0	$-\sin\pi(s_1+s_2)\sin\pi s_2$	0	$-\sin\pi s_2$
$-+$	0	1	$\cos\pi(s_1+s_2)\cos\pi s_2$	$\sin\pi(s_1+s_2)$
$--$	0	$-\cos\pi s_2$	$-\cos\pi(s_1+s_2)$	$-\sin\pi(s_1+s_2)\cos\pi s_2$

Note that the matrix $\left(B_{\epsilon,\eta}^{(3,4)}\right)$ is the gamma matrix of the local functional equation of the prehomogeneous vector space $(SO(3,4) \times P_{1,1}, M(7,2))$.

4. Now we apply the result of the preceding section to the prehomogeneous vector space $(Spin(10) \times P_{1,1,1}, \text{half-spin}\otimes\square)$. First we recall the relation between prehomogeneous vector spaces and weakly spherical homogeneous spaces given in Lemma 1.1. The weakly spherical space $Spin(10)\backslash GL(16)$ we have just studied is closely related to the prehomogeneous vector space $(Spin(10) \times P_{1,1,1}, \text{half-spin} \otimes \square)$. One can express the zeta functions associated with the latter space in terms of the Riemann zeta function and $E_\epsilon(P_{1,1,1,13}; x, \lambda)$. Moreover there exists a similar relation between $E_\epsilon(P_{13,1,1,1}; x, \lambda)$ and the zeta functions associated to the prehomogeneous vector space dual to $(Spin(10) \times P_{1,1,1}, \text{half-spin} \otimes \square)$. Then it can be shown that the functional equation obtained from the general theory of prehomogeneous vector spaces is nothing but the functional equation corresponding to the cyclic permutation $\binom{1234}{2341}$, which connects $E_\epsilon(P_{13,1,1,1}; x, \lambda)$ to $E_\eta(P_{1,1,1,13}; x, \lambda)$. It is rather hard to calculate the functional equation explicitly; however, if we decompose the functional equation as

$$E(P_{1,1,1,13}; *, *) \xrightarrow{(3,4)} E(P_{1,1,13,1}; *, *) \xrightarrow{(2,3)} E(P_{1,13,1,1}; *, *) \xrightarrow{(1,2)} E(P_{13,1,1,1}; *, *),$$

then, as the formulas in §**3** shows, the functional equations corresponding to the transpositions $(1,2)$, $(2,3)$, $(3,4)$ are simple enough for explicit calculation. These simple functional equations are not visible if we stick to the view point of prehomogeneous vector spaces.

5. The detail of the proof of the above results will apear in [S5]. The result in §2 can be extended to arbitrary weakly spherical homogeneous spaces of $GL(n)$ (see [S2] and [S3]). This generalization is based on Lemma 1.1 and the theory of zeta functions associated with prehomogeneous vector spaces ([S1]). To extend the result further to reductive groups other than $GL(n)$, we can no longer appeal to the theory of prehomogeneous vector spaces. We are sure that the key of further generalization is the study of the regularization of period integrals of Eisenstein series ([S4]).

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