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SINGULAR INVARIANT HYPERFUNCTIONS ON THE SPACE OF REAL SYMMETRIC MATRICES

岐阜大学工学部 室政和 (MASAKAZU MURO)

ABSTRACT. Singular invariant hyperfuncsions on the space of $n \times n$ real symmetric matrices are discussed in this paper. We construct singular invariant hyperfunctions ,i.e., invariant hyperfunctions whose supports are contained in the set $S := \{\det(x) = 0\}$, in terms of negative order coefficients of the Laurent expansions of the complex powers of the determinant function.

CONTENTS

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0. Introduction.

A complex power of a polynomial is an important material to study in contemporary mathematics. We often encounter integrals of complex powers of polynomials in various aspect; for example, zeta functions of various type, hypergeometric functions and their extensions, kernels of integral transforms and so on. There are many problems to solve. In particular, the explicit calculation of the exact order at poles and the residues of the poles with respect to the power parameter is a fundamental problem for the analysis of invariant hyperfunctions on prehomogeneous vector spaces.

In this paper, we shall study the microlocal structure of the complex power of the determinant function on the real symmetric matrix space, and compute the exact order of poles with respect to the power parameter (Theorem 2.3). Moreover, we shall determine the exact support of the principal part of the pole (Theorem 2.4).

We shall construct a suitable basis of the space of singular invariant hyperfunctions on the space of $n \times n$ real symmetric matrices $V := Sym_n(\mathbb{R})$. The hyperfunctions belonging to the basis are expressed by the coefficients of the Laurent expansion of $|\det(x)|^s$, the complex power of the determinant function. We estimate the exact order of the poles of $|\det(x)|^s$ and give the exact support of the negative-order coefficients of the Laurent expansion of $|\det(x)|^s$ at its poles.

We will give the plan of this article in the following. In §1, we shall introduce some notions and basic properties on the complex power function $P^{[\vec{a},s]}(x)$ on the space of real symmetric matrices. In the next section (§2), the main theorems are stated without proofs. In §3, we shall explain about principal symbols $\sigma_\Lambda(P^{[\vec{a},s]}(x))$ of the regular holonomic hyperfunction $P^{[\vec{a},s]}(x)$ on the Lagrangian subvariety Λ and the coefficient functions $c_i^{j,k}(\vec{a},s)$ on the connected Lagrangian component $\Lambda_i^{j,k}$. They will play a crucial role in the proofs of the main theorems. In §4, we investigate the relation formula on $c_i^{j,k}(\vec{a},s)$. In the last section (§5), the proofs of the main theorems are given.

We can obtain the same results on similar matrix spaces, for example, the space of complex Hermitian matrices or quaternion Hermitian matrices. They will appear in the future articles.

Remark 0.1. Similar results has been obtained by Blind [Bli94] by a functional analytic method.

1. Complex powers of the determinant function.

In this section, we shall explain our problem more precisely, prepare some notions and notations, and state some preliminary known result. They are well-known results, so we omit the proofs.

1.1. **Some fundamental definitions.** Let $V := Sym_n(\mathbb{R})$ be the space of $n \times n$ symmetric matrices over the real field $\mathbb{R}$ and let $\mathbf{GL}_n(\mathbb{R})$ (reap.

$\mathbf{SL}_n(\mathbb{R})$) be the general (resp. special) linear group over $\mathbb{R}$. Then the real algebraic group $\boldsymbol{G} := \mathbf{GL}_n(\mathbb{R})$ operates on the vector space $\boldsymbol{V}$ by the representation

$$\rho(g) : x \longmapsto g \cdot x \cdot {}^t g, \tag{1}$$

with $x \in \boldsymbol{V}$ and $g \in \boldsymbol{G}$. We say that a hyperfunction $f(x)$ on $\boldsymbol{V}$ is *singular* if the support of $f(x)$ is contained in the set $\boldsymbol{S} := \{x \in \boldsymbol{V};\ \det(x) = 0\}$. We call $\boldsymbol{S}$ a *singular set* of $\boldsymbol{V}$. In addition, if $f(x)$ is $\mathbf{SL}_n(\mathbb{R})$-invariant, i.e., $f(g \cdot x) = f(x)$ for all $g \in \mathbf{SL}_n(\mathbb{R})$, we call $f(x)$ a *singular invariant* hyperfunction on $\boldsymbol{V}$.

Let $P(x) := \det(x)$. Then $P(x)$ is an irreducible polynomial on $\boldsymbol{V}$, and is relatively invariant corresponding to the character $\det(g)^2$ with respect to the action of $\boldsymbol{G}$,i.e., $P(\rho(g) \cdot x) = \det(g)^2 P(x)$. The non-singular subset $\boldsymbol{V} - \boldsymbol{S}$ decomposes into $(n+1)$ open $\boldsymbol{G}$-orbits

$$\boldsymbol{V}_i := \{x \in Sym_n(\mathbb{R}); \mathrm{sgn} = (n-i, i)\}. \tag{2}$$

with $i = 0, 1, \ldots, n$. Here, $\mathrm{sgn}(x)$ for $x \in Sym_n(\mathbb{R})$ stands for the signature of the quadratic form $q_x(\vec{v}) := {}^t\vec{v} \cdot x \cdot \vec{v}$ on $\vec{v} \in \mathbb{R}^n$. We let for a complex number $s \in \mathbb{C}$,

$$|P(x)|_i^s := \begin{cases} |P(x)|^s & ,\text{if } x \in \boldsymbol{V}_i, \\ 0 & ,\text{if } x \notin \boldsymbol{V}_i. \end{cases} \tag{3}$$

Let $\mathcal{S}(\boldsymbol{V})$ be the space of rapidly decreasing functions on $\boldsymbol{V}$. For $f(x) \in \mathcal{S}(\boldsymbol{V})$, the integral

$$Z_i(f, s) := \int_{\boldsymbol{V}} |P(x)|_i^s f(x) dx, \tag{4}$$

is convergent if the real part of s is sufficiently large and is holomorphically extended to the whole complex plane. Thus we can regard $|P(x)|_i^s$ as a tempered distribution with a meromorphic parameter $s \in \mathbb{C}$. We consider a linear combination of $|P(x)|_i^s$

$$P^{[\vec{a},s]}(x) := \sum_{i=0}^{n} a_i |P(x)|_i^s, \tag{5}$$

with $s \in \mathbb{C}$ and $\vec{a} := (a_0, a_1, \ldots, a_n) \in \mathbb{C}^{n+1}$. Then $P^{[\vec{a},s]}(x)$ is a hyperfunction with a meromorphic parameter $s \in \mathbb{C}$, and depends on $\vec{a} \in \mathbb{C}^{n+1}$ linearly.

1.2. **Basic properties and some known results on complex powers.** The following theorem is easily proved by the general theory of b-functions. (see for example [Mur90]).

Theorem 1.1. 1. *$P^{[\vec{a},s]}(x)$ is holomorphic with respect to $s \in \mathbb{C}$ except for the poles at $s = -\frac{k+1}{2}$ with $k = 1, 2, \ldots$.*

2. *The possibly highest order of* $P^{[\vec{a},s]}(x)$ *at* $s = -\frac{k+1}{2}$ *is given by*

$$\begin{cases} \lfloor \frac{k+1}{2} \rfloor & ,(k = 1, 2 \ldots, n-1), \\ \lfloor \frac{n}{2} \rfloor & ,(k = n, n+1 \ldots, \text{ and } k+n \text{ is odd}), \\ \lfloor \frac{n+1}{2} \rfloor & ,(k = n, n+1 \ldots, \text{ and } k+n \text{ is even}). \end{cases} \tag{6}$$

Here, $\lfloor x \rfloor$ *means the floor of* $x \in \mathbb{R}$, *i.e., the largest integer not larger than* x.

Any negative-order coefficient of a Laurent expansion of $P^{[\vec{a},s]}(x)$ is a singular invariant hyperfunction since the integral

$$\int f(x) P^{[\vec{a},s]}(x) dx = \sum_{i=0}^{n} Z_i(f, s) \tag{7}$$

is an entire function with respect to $s \in \mathbb{C}$ if $f(x) \in C_0^\infty(\boldsymbol{V} - \boldsymbol{S})$, where $C_0^\infty(\boldsymbol{V} - \boldsymbol{S})$ is the space of compactly supported C^∞-functions on $\boldsymbol{V} - \boldsymbol{S}$. Conversely, we have the following proposition.

Proposition 1.2 ([Mur88b],[Mur90]). *Any singular invariant hyperfunction on* $\boldsymbol{V}$ *is given as a linear combination of some negative-order coefficients of Laurent expansions of* $P^{[\vec{a},s]}(x)$ *at various poles and for some* $\vec{a} \in \mathbb{C}^{n+1}$.

Proof. The prehomogeneous vector space

$$(\boldsymbol{G}, \boldsymbol{V}) := (\mathbf{GL}_n(\mathbb{R}), Sym_n(\mathbb{R}))$$

satisfies sufficient conditions stated in [Mur88b] and [Mur90]. One is the finite-orbit condition and the other is that the dimension of the space of relatively invariant hyperfunctions coincides with the number of open orbits. □

1.3. **Orbit decomposition.** The vector space $\boldsymbol{V}$ decomposes into a finite number of $\boldsymbol{G}$-orbits;

$$\boldsymbol{V} := \bigsqcup_{\substack{0 \le i \le n \\ 0 \le j \le n-i}} \boldsymbol{S}_i^j \tag{8}$$

where

$$\boldsymbol{S}_i^j := \{x \in Sym_n(\mathbb{R});\ \mathrm{sgn}(x) = (n-i-j, j)\} \tag{9}$$

with integers $0 \le i \le n$ and $0 \le j \le n-i$. A $\boldsymbol{G}$-orbit in $\boldsymbol{S}$ is called a *singular orbit.* The subset $\boldsymbol{S}_i := \{x \in \boldsymbol{V};\ \mathrm{rank}(x) = n-i\}$ is the set of elements of rank $(n-i)$. It is easily seen that $\boldsymbol{S} := \bigsqcup_{1 \le i \le n} \boldsymbol{S}_i$ and $\boldsymbol{S}_i = \bigsqcup_{0 \le j \le n-i} \boldsymbol{S}_i^j$. Each singular orbit is a stratum which not only is a $\boldsymbol{G}$-orbit but is an $\mathbf{SL}_n(\mathbb{R})$-orbit. The strata $\{\boldsymbol{S}_i^j\}_{1 \le i \le n, 0 \le j \le n-i}$ have the following closure inclusion relation

$$\overline{\boldsymbol{S}_i^j} \supset \boldsymbol{S}_{i+1}^{j-1} \cup \boldsymbol{S}_{i+1}^j, \tag{10}$$

where $\overline{\boldsymbol{S}_i^j}$ means the *closure* of the stratum $\boldsymbol{S}_i^j$.

The support of a singular invariant hyperfunction is a closed set consisting of a union of some strata S_i^j. Since the support is a closed G-invariant subset, we can express the support of a singular invariant hyperfunction as a closure of a union of the highest rank strata, which is easily rewritten by a union of singular orbits.

2. Statement of the main results.

In this section we shall give the main problems and results. When we give a complex $n+1$ dimensional vector $\vec{a} \in \mathbb{C}^{n+1}$, we can determine the exact order of poles of $P^{[\vec{a},s]}(x)$ and the exact support of the hyperfunctions appearing in the principal part of the Laurent expansion. We shall give the statement of the theorems in this section without proofs. Their proofs will be given in §5.

2.1. Main problem. When we consider complex powers of relatively invariant polynomials, we naturally ask the following questions.

Problem 2.1. What are the principal parts of the Laurent expansion of $P^{[\vec{a},s]}(x)$ at poles ? What are their exact orders of poles ? What are the supports of negative-order coefficients of a Laurent expansion of $P^{[\vec{a},s]}(x)$ at poles ?

In order to determine the exact order of $P^{[\vec{a},s]}(x)$ at $s = s_0$, we introduce the coefficient vectors

$$\boldsymbol{d}^{(k)}[s_0] := (d_0^{(k)}[s_0], d_1^{(k)}[s_0], \ldots, d_{n-k}^{(k)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^{n-k+1} \tag{11}$$

with $k = 0, 1, \ldots, n$. Here, $(\mathbb{C}^{n+1})^*$ means the dual vector space of $\mathbb{C}^{n+1}$. Each element of $\boldsymbol{d}^{(k)}[s_0]$ is a linear form on $\vec{a} \in \mathbb{C}^{n+1}$ depending on $s_0 \in \mathbb{C}$,i.e.,a linear map from $\mathbb{C}$ to $\mathbb{C}^{n+1}$,

$$d_i^{(k)}[s_0] : \mathbb{C}^{n+1} \ni \vec{a} \longmapsto \langle d_i^{(k)}[s_0], \vec{a}\rangle \in \mathbb{C}. \tag{12}$$

We denote

$$\langle \boldsymbol{d}^{(k)}[s_0], \vec{a}\rangle = (\langle d_0^{(k)}[s_0], \vec{a}\rangle, \langle d_1^{(k)}[s_0], \vec{a}\rangle, \ldots, \langle d_{n-k}^{(k)}[s_0], \vec{a}\rangle) \in \mathbb{C}^{n-k+1}. \tag{13}$$

Definition 2.1 (Coefficient vectors $\boldsymbol{d}^{(k)}[s_0]$). Let s_0 be a *half-integer*, i.e., a rational number given by $q/2$ with an integer q. We define the *coefficient vectors* $\boldsymbol{d}^{(k)}[s_0]$ for $(k = 0, 1, \ldots, n)$ by induction on k in the following way.

1. First, we set

$$\boldsymbol{d}^{(0)}[s_0] := (d_0^{(0)}[s_0], d_1^{(0)}[s_0], \ldots, d_n^{(0)}[s_0]) \tag{14}$$

such that $\langle d_i^{(0)}[s_0], \vec{a}\rangle := a_i$ for $i = 0, 1, \ldots, n$.

2. Next, we define $\boldsymbol{d}^{(1)}[s_0]$ and $\boldsymbol{d}^{(2)}[s_0]$ by

$$\boldsymbol{d}^{(1)}[s_0] := (d_0^{(1)}[s_0], d_1^{(1)}[s_0], \ldots, d_{n-1}^{(1)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^n, \tag{15}$$

with $d_j^{(1)}[s_0] := d_j^{(0)}[s_0] + \epsilon[s_0] d_{j+1}^{(0)}[s_0]$, and

$$d^{(2)}[s_0] := (d_0^{(2)}[s_0], d_1^{(2)}[s_0], \dots, d_{n-2}^{(2)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^{n-1}, \tag{16}$$

with $d_j^{(2)}[s_0] := d_j^{(0)}[s_0] + d_{j+2}^{(0)}[s_0]$. Here,

$$\epsilon[s_0] := \begin{cases} 1 & \text{, (if } s_0 \text{ is a strict half-integer),} \\ (-1)^{s_0+1} & \text{, (if } s_0 \text{ is an integer).} \end{cases} \tag{17}$$

A *strict half-integer* means a rational number given by $q/2$ with an odd integer q.

3. Lastly, by induction on k, we define all the coefficient vectors $d^{(k)}[s_0]$ for $k = 0, 1, \dots, n$ by

$$d^{(2l+1)}[s_0] := (d_0^{(2l+1)}[s_0], d_1^{(2l+1)}[s_0], \dots, d_{n-2l-1}^{(2l+1)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^{n-2l}, \tag{18}$$

with $d_j^{(2l+1)}[s_0] := d_j^{(2l-1)}[s_0] - d_{j+2}^{(2l-1)}[s_0]$, and

$$d^{(2l)}[s_0] := (d_0^{(2l)}[s_0], d_1^{(2l)}[s_0], \dots, d_{n-2l}^{(2l)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^{n-2l+1}, \tag{19}$$

with $d_j^{(2l)}[s_0] := d_j^{(2l-2)}[s_0] + d_{j+2}^{(2l-2)}[s_0]$.

Then we have the following proposition.

Proposition 2.1. *Let s_0 be a half-integer. For an integer i in $0 \le i \le n-2$ and $\vec{a} \in \mathbb{C}^{n+1}$, if $\langle d^{(i)}[s_0], \vec{a} \rangle = 0$ then $\langle d^{(i+2)}[s_0], \vec{a} \rangle = 0$. In other words, if $\langle d^{(i+2)}[s_0], \vec{a} \rangle \neq 0$ then $\langle d^{(i)}[s_0], \vec{a} \rangle \neq 0$.*

Proof. This proposition is trivial from the definition of $d^{(i)}[s_0]$. □

Corollary 2.2. *Let s_0 be a half-integer. Then we have*

1. *There exists an even integer i_0 in $0 \le i_0 \le n+1$ such that*

$$\langle d^{(i)}[s_0], \vec{a} \rangle \text{ is } \begin{cases} \neq 0 & \text{for all odd } i \text{ in } 0 \le i < i_0. \\ = 0 & \text{for all odd } i \text{ in } n \ge i > i_0 \end{cases} \tag{20}$$

2. *There exists an odd integer i_1 in $-1 \le i_1 \le n+1$ such that*

$$\langle d^{(i)}[s_0], \vec{a} \rangle \text{ is } \begin{cases} \neq 0 & \text{for all even } i \text{ in } 0 \le i < i_1. \\ = 0 & \text{for all even } i \text{ in } n \ge i > i_1. \end{cases} \tag{21}$$

Proof. We can prove this by induction on i. □

2.2. **Results on the poles of the complex power functions.** Using the above mentioned vectors $d^{(k)}[s_0]$, we can determine the exact orders of $P^{[\vec{a},s]}(x)$ at each pole.

Theorem 2.3 (Exact orders of poles). *The exact order of the poles of $P^{[\vec{a},s]}(x)$ is computed by the following algorithm.*

1. *At* $s = -\frac{2m+1}{2}(m = 1, 2, \dots)$, *the coefficient vectors* $\mathbf{d}^{(k)}[-\frac{2m+1}{2}]$ *are defined in Definition 2.1. The exact order* $P^{[\vec{a},s]}(x)$ *at* $s = -\frac{2m+1}{2}(m = 1, 2, \dots)$ *is given in terms of the coefficient vector* $\mathbf{d}^{(2k)}[-\frac{2m+1}{2}]$.
   (a) *If* $1 \leq m \leq \frac{n}{2}$, *then* $P^{[\vec{a},s]}(x)$ *has a possible pole of order not larger than* m.
      - *If* $\langle \mathbf{d}^{(2)}[-\frac{2m+1}{2}], \vec{a}\rangle = 0$, *then* $P^{[\vec{a},s]}(x)$ *is holomorphic, and the converse is true.*
      - *Generally, for integers* p *in* $1 \leq p < m$, *if* $\langle \mathbf{d}^{(2p+2)}[-\frac{2m+1}{2}], \vec{a}\rangle = 0$ *and* $\langle \mathbf{d}^{(2p)}[-\frac{2m+1}{2}], \vec{a}\rangle \neq 0$, *then* $P^{[\vec{a},s]}(x)$ *has a pole of order* p, *and the converse is true.*
      - *Lastly, if* $\langle \mathbf{d}^{(2m)}[-\frac{2m+1}{2}], \vec{a}\rangle \neq 0$, *then* $P^{[\vec{a},s]}(x)$ *has a pole of order* m, *and the converse is true.*

   (b) *If* $m > \frac{n}{2}$, *then* $P^{[\vec{a},s]}(x)$ *has a possible pole of order not larger than* $n' := \lfloor \frac{n}{2} \rfloor$
      - *If* $\langle \mathbf{d}^{(2)}[-\frac{2m+1}{2}], \vec{a}\rangle = 0$, *then* $P^{[\vec{a},s]}(x)$ *is holomorphic, and the converse is true.*
      - *Generally, for integers* p *in* $1 \leq p < n'$, *if* $\langle \mathbf{d}^{(2p+2)}[-\frac{2m+1}{2}], \vec{a}\rangle = 0$ *and* $\langle \mathbf{d}^{(2p)}[-\frac{2m+1}{2}], \vec{a}\rangle \neq 0$, *then* $P^{[\vec{a},s]}(x)$ *has a pole of order* p, *and the converse is true.*
      - *Lastly,* $P^{[\vec{a},s]}(x)$ *has a pole of order* n' *if* $\langle \mathbf{d}^{(n-1)}[-\frac{2m+1}{2}], \vec{a}\rangle \neq 0$ *(when* n *is odd) or* $\langle \mathbf{d}^{(n)}[-\frac{2m+1}{2}], \vec{a}\rangle \neq 0$ *(when* n *is even), and the converse is true.*

2. *At* $s = -m(m = 1, 2, \dots)$, *the coefficient vectors* $\mathbf{d}^{(k)}[-m]$ *are defined in Definition 2.1 with* $\epsilon[-m] = (-1)^{-m+1}$. *We obtain the exact order at* $s = -m(m = 1, 2, \dots)$ *in terms of the coefficient vectors* $\mathbf{d}^{(2k+1)}[-m]$.
   (a) *If* $1 \leq m \leq \frac{n}{2}$, *then* $P^{[\vec{a},s]}(x)$ *has a possible pole of order not larger than* m.
      - *If* $\langle \mathbf{d}^{(1)}[-m], \vec{a}\rangle = 0$, *then* $P^{[\vec{a},s]}(x)$ *is holomorphic, and the converse is true.*
      - *Generally, for integers* p *in* $1 \leq p < m$, *if* $\langle \mathbf{d}^{(2p+1)}[-m], \vec{a}\rangle = 0$ *and* $\langle \mathbf{d}^{(2p-1)}[-m], \vec{a}\rangle \neq 0$, *then* $P^{[\vec{a},s]}(x)$ *has a pole of order* p, *and the converse is true.*
      - *Lastly, if* $\langle \mathbf{d}^{(2m-1)}[-m], \vec{a}\rangle \neq 0$, *then* $P^{[\vec{a},s]}(x)$ *has a pole of order* m, *and the converse is true.*

   (b) *If* $m > \frac{n}{2}$, *then* $P^{[\vec{a},s]}(x)$ *has a possible pole of order not larger than* $n' := \lfloor \frac{n+1}{2} \rfloor$
      - *If* $\langle \mathbf{d}^{(1)}[-m], \vec{a}\rangle = 0$, *then* $P^{[\vec{a},s]}(x)$ *is holomorphic, and the converse is true.*
      - *Generally, for integers* p *in* $1 \leq p < n'$, *if* $\langle \mathbf{d}^{(2p+1)}[-m], \vec{a}\rangle = 0$ *and* $\langle \mathbf{d}^{(2p-1)}[-m], \vec{a}\rangle \neq 0$, *then* $P^{[\vec{a},s]}(x)$ *has a pole of order* p, *and the converse is true.*

- *Lastly, $P^{[\vec{a},s]}(x)$ has a pole of order n' if $\langle \mathbf{d}^{(n)}[-m], \vec{a}\rangle \neq 0$ (when n is odd) or $\langle \mathbf{d}^{(n-1)}[-m], \vec{a}\rangle \neq 0$ (when n is even), and the converse is true.*

2.3. **Results on the supports of the principal symbols.** The exact support of $P^{[\vec{a},s]}(x)$ is given by the following theorem.

Theorem 2.4 (Support of the singular invariant hyperfunctions). *Let q be a positive integer. Suppose that $P^{[\vec{a},s]}(x)$ has a pole of order p at $s = -\frac{q+1}{2}$ Let*

$$P^{[\vec{a},s]}(x) = \sum_{w=-p}^{\infty} P_w^{[\vec{a},-\frac{q+1}{2}]}(x)(s + \frac{q+1}{2})^w \tag{22}$$

be the Laurent expansion of $P^{[\vec{a},s]}(x)$ at $s = -\frac{q+1}{2}$. The support of the coefficients $P_w^{[\vec{a},-\frac{q+1}{2}]}(x)$ is contained in S if $w < 0$.

1. *Let q be an even positive integer. Then the support of $P_w^{[\vec{a},-\frac{q+1}{2}]}(x)$ for $w = -1, -2, \ldots, -p$ is contained in the closure $\overline{S_{-2w}}$. More precisely, it is given by*

$$Supp(P_w^{[\vec{a},-\frac{q+1}{2}]}(x)) = \overline{(\bigcup_{j \in \{0 \leq j \leq n+2w; \langle d_j^{(-2w)}[-\frac{q+1}{2}], \vec{a}\rangle \neq 0\}} S^j_{-2w})}. \tag{23}$$

2. *Let q be an odd positive integer. Then the support of $P_w^{[\vec{a},-\frac{q+1}{2}]}(x)$ for $w = -1, -2, \ldots, -p$ is contained in the closure $\overline{S_{-2w-1}}$. More precisely, it is given by*

$$Supp(P_w^{[\vec{a},-\frac{q+1}{2}]}(x)) = \overline{(\bigcup_{j \in \{0 \leq j \leq n+2w+1; \langle d_j^{(-2w-1)}[-\frac{q+1}{2}], \vec{a}\rangle \neq 0\}} S^j_{-2w-1})}. \tag{24}$$

Here, $Supp(-)$ means the support of the hyperfunction in $(-)$.

3. Principal symbols of invariant hyperfunctions.

In this section, we review the notion of principal symbols of simple holonomic microfunctions and coefficients with respect to the canonical basis of principal symbols. Principal symbols will play a central role in the calculus of invariant hyperfunctions on prehomogeneous vector spaces. The author calculated the Fourier transforms of complex powers of relatively invariant polynomials by putting the principal symbols to practical use in [Mur86]. In the calculation of singular invariant hyperfunctions, principal symbols and coefficients are powerful tools. So we will state the outline of the construction of principal symbols of relatively invariant hyperfunctions.

3.1. **Microfunctions on the cotangent bundle.** Let $\mathcal{B}_V$ be the sheaf of hyperfunctions on V and let $\mathcal{C}_V$ be the sheaf of microfunctions on the cotangent bundle T^*V of V. There are the natural isomorphism **sp**:

$$\mathbf{sp} : \mathcal{B}_V \longrightarrow \pi(\mathcal{C}_V) \tag{25}$$

and the exact sequence

$$0 \longrightarrow \mathcal{A}_V \longrightarrow \mathcal{B}_V \longrightarrow \pi(\mathcal{C}_V|_{T^*V-V}) \longrightarrow 0 \tag{26}$$

Here, π is the projection map from the cotangent vector space T^*V to V and $\mathcal{A}_V$ is the sheaf of real analytic functions on V. By the isomorphism (25), we can regard a hyperfunction $f(x)$ on V as a microfunction $\mathbf{sp}(f(x))$ on T^*V. In this article, we often identify the hyperfunction $f(x)$ on V with the microfunction $\mathbf{sp}(f(x))$ on T^*V through the isomorphism (25).

Remark 3.1. In this paper, the sheaf $\mathcal{C}_V$ means the sheaf of microfunctions on T^*V, not on $T^*V - V$. It was originally denoted by $\breve{\mathcal{C}}_V$ when Sato introduced the notion of microfunction originally. Roughly speaking, the sheaf of microfunctions $\mathcal{C}_V$ on T^*V is the union of the sheaf of hyperfunctions $\mathcal{B}_V$ and the sheaf $\mathcal{C}_V|_{T^*V-V}$. When the notion of microfunction was introduced as a singular part of a hyperfunctions, it often meant the sheaf $\mathcal{C}_V|_{T^*V-V}$. However, in this article, we always means the sheaf $\mathcal{C}_V$ the one on the whole space T^*V.

3.2. **Holonomic systems for relatively invariant hyperfunctions.** We consider an invariant hyperfunctions on V under the action of G as a solution to a holonomic system. Let $f(x)$ be a hyperfunction on V. We say that $f(x)$ is a *χ^s-invariant hyperfunction* if

$$f(\rho(g)x) = \chi(g)^s f(x), \tag{27}$$

for all $g \in G$, where $s \in \mathbb{C}$ and $\chi(g) := \det(g)^2$. Then, it is a hyperfunction solution to the following system of linear differential equations $\mathcal{M}_s$ by taking an infinitesimal action of G,

$$\mathcal{M}_s : (\langle d\rho(A)x, \frac{\partial}{\partial x}\rangle - s\delta\chi(A))u(x) = 0 \;\text{ for all } A \in \mathfrak{G}. \tag{28}$$

Here, $\mathfrak{G}$ is the Lie algebra of G; $d\rho$ is the infinitesimal representation of ρ; $\delta\chi$ is the infinitesimal character of χ. The system of linear differential equation (28) is a regular holonomic system and hence the solution space is finite dimensional. See for detail [Mur90].

The characteristic subvariety of the system (28) is denoted by $\mathrm{ch}(\mathcal{M}_s)$. It is given by

$$\mathcal{M}_s := \{(x,y) \in T^*V; \langle d\rho(A)x, y\rangle = 0 \;\text{ for all } A \in \mathfrak{G}\}. \tag{29}$$

The characteristic variety has the irreducible component decomposition,

$$\mathcal{M}_s := \bigcup_{i=0}^{n} \Lambda_i, \tag{30}$$

with $\Lambda_i = \overline{T^*_{S_i}V}$ where $T^*_{S_i}V$ stands for the conormal bundle of the rank $(n-i)$-orbit S_i. It is a well known result that the singular support of the hyperfunction solution to $\mathcal{M}_s$ is contained in $\mathrm{ch}(\mathcal{M}_s)$.

Remark 3.2. In this article, the singular support of a hyperfunction $f(x)$ means, by definition, the support of $\mathbf{sp}(f(x))$ in T^*V, not in $T^*V - V$.

We denote the dual vector space by V^*. The cotangent vector space T^*V is naturally identified with the product space $V \times V^*$. since the group G acts on V^* by the contragredient action, $V \times V^*$ admits the G-action. The characteristic variety $\mathrm{ch}(\mathcal{M}_s)$ is an invariant subset in $V \times V^*$. and it decomposes into a finite number of orbits. See Proposition 1.1 in [Mur86].

Proposition 3.1. *The holonomic system $\mathcal{M}_s$ is simple on each Lagrangian subvariety Λ_i. The order of $\mathcal{M}_s$ on Λ_i is given by*

$$ord_{\Lambda_i}(\mathcal{M}_s) = -is - \frac{i(i+1)}{4}. \tag{31}$$

The irreducible Lagrangian subvarieties Λ_i and Λ_{i+1} have an intersection of codimension one.

Proof. The orders on Λ_i are calculated in [Mur86]. The intersections of codimension one among Λ_i's are also given there. See the holonomy diagrams in [Mur86]. □

3.3. **Principal symbols on simple Lagrangian subvarieties.** Recall the definition of the principal symbols on simple holonomic systems defined in [Mur86]. Let Λ be a non-singular Lagrangian subvariety and let $u(x)$ be a local section of a microfunction solution to a simple holonomic system $\mathcal{M}_s$ whose support is Λ. We denote by $\sigma_\Lambda(u)$ the principal symbol of $u(x)$ on Λ. It is a real analytic section of $\sqrt{|\Omega_\Lambda|} \otimes \sqrt{|\Omega_V|}^{-1}$ where $\sqrt{|\Omega_\Lambda|}$ and $\sqrt{|\Omega_V|}$ are the sheaves of half-volume elements on Λ and V,respectively. For the precise definition, see [Mur86] Definition 2.7. As explained in [Mur86], the map

$$\sigma_\Lambda : u \longmapsto \sigma_\Lambda(u) \tag{32}$$

is a linear isomorphism from the space of microfunction solutions to the space of principal symbols of the holonomic system $\mathcal{M}_s$. In other words, there is a one to one correspondence between a microfunction solution to $\mathcal{M}_s$ and its principal symbol.

When we consider a hyperfunction solution to the holonomic system $\mathcal{M}_s$, it is sufficient to handle the principal symbol on an open dense subset of $\mathrm{ch}(\mathcal{M}_s)$. We introduce the open subset Λ_i° of Λ_i.

Definition 3.1. Let Λ_i be one of the irreducible component of $\mathrm{ch}(\mathcal{M}_s)$ defined in (30). We define the subset Λ_i° by

$$\Lambda_i^\circ := \Lambda_i - \bigcup_{i \neq j} \Lambda_j \tag{33}$$

It is an open-dense subset of Λ_i.

The open subset Λ_i° consists of several open connected subsets, each of which is a $\boldsymbol{G}$-orbit. Furthermore, Λ_i° is a non-singular algebraic subvariety and an open dense subset in Λ_i.

Proposition 3.2. *The open set Λ_i° of Λ_i decomposes into the following $\boldsymbol{G}$-orbits*

$$\Lambda_i^\circ = \bigsqcup_{\substack{0\leq j\leq n-i \\ 0\leq k\leq i}} \Lambda_i^{j,k}, \tag{34}$$

with

$$\Lambda_i^{j,k} := \boldsymbol{G}\cdot (\begin{pmatrix} I_{n-i}^{(j)} & \\ & 0_i \end{pmatrix}, \begin{pmatrix} 0_{n-i} & \\ & I_i^{(k)} \end{pmatrix}). \tag{35}$$

Here, $I_p^{(q)} := \begin{pmatrix} I_q & \\ & -I_{p-q} \end{pmatrix}$ *and I_p is an identity matrix of size p. Each orbit $\Lambda_i^{j,k}$ is a connected component in Λ_i°.*

3.4. **Canonical basis of principal symbols.** When we consider the holonomic system $\mathrm{ch}(\mathcal{M}_s)$ defined by (28), $\mathcal{M}_s$ is a simple holonomic system on all Lagrangian subvarieties Λ_i $(i = 0, 1, \dots, n)$. Then the principal symbol of a microfunction solution is given as a constant multiplication of a basis of $\sqrt{|\Omega_{\Lambda_i}|} \otimes \sqrt{|\Omega_{\boldsymbol{V}}|}$.

Let Λ_i° be the open subset defined by Definition 3.1 and let $\Lambda_i^{j,k}$ be a connected component in Λ_i°. We define a non-zero real analytic section $\Omega_i^{j,k}(s)$ of $\sqrt{|\Omega_{\Lambda_i^{j,k}}|}$ by

$$\Omega_i^{j,k}(s) := |P_{\Lambda_i^{j,k}}(x,y)|^s \sqrt{|\omega_{\Lambda_i^{j,k}}(x,y)|}. \tag{36}$$

$\Omega_i^{j,k}(s)$ depends on $s \in \mathbb{C}$ holomorphically. Here, we set

$$P_{\Lambda_i^{j,k}}(x,y) := P(\pi(x,y))/(\sigma(x,y))^{m_{\Lambda_i}}|_{\Lambda_i^{j,k}}, \tag{37}$$

$$\omega_{\Lambda_i^{j,k}}(x,y) := \frac{\pi^{-1}(|dx|) \wedge d\sigma(x,y)}{\sigma(x,y)^{\mu_{\Lambda_i}}} / d\sigma(x,y)|_{\Lambda_i^{j,k}}, \tag{38}$$

where $\sigma := \sigma(x,y)$ is a function on $\boldsymbol{V} \times \boldsymbol{V}^*$ defined by $\sigma := \langle x, y\rangle/n$; π is the projection map from the subvariety

$$W := \{(x,y) \in T^*\boldsymbol{V}; \langle d\rho(A)x, y\rangle = 0 \ \text{ for all } \ A \in \mathfrak{G}\} \subset \boldsymbol{V} \times \boldsymbol{V}^* \tag{39}$$

to $\boldsymbol{V}$; m_{Λ_i} and μ_{Λ_i} are the constants such that $-m_{\Lambda_i}s - \frac{\mu_{\Lambda_i}}{2}$ is the order of $\mathcal{M}_s$ on Λ_i. In particular, $m_{\Lambda_i} = i$ and $\mu_{\Lambda_i} = \frac{i(i+1)}{2}$ in our case.

Proposition 3.3. *Let u(s,x) be a microfunction solution with a meromorphic parameter $s \in \mathbb{C}$ to the holonomic system $\mathcal{M}_s$ and let $\Lambda_i^{j,k}$ be a connected component in Λ_i°. Then we have*

1. *The principal symbol $\sigma_{\Lambda_i^{j,k}}(u(s,x))$ is written as a constant multiplication of the real analytic section of $\sqrt{|\Omega_{\Lambda_i}|} \otimes \sqrt{|\Omega_{V}|}$,*
$$\sigma_{\Lambda_i^{j,k}}(u(s,x)) = c_i^{j,k}(s)\Omega_i^{j,k}(s)/\sqrt{|dx|}. \tag{40}$$
Here, $|dx|$ is a non-zero volume element on $\boldsymbol{V}$ defined by
$$|dx| := |\bigwedge_{1 \le i \le j \le n} dx_{ij}|, \tag{41}$$
with $x = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{12} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{pmatrix} \in \boldsymbol{V}$. Conversely, if the constant multiplication term $c_i^{j,k}(s)$ is given on each $\Lambda_i^{j,k}$, then the corresponding microfunction solution $u(s,x)$ satisfying (40) *is determined uniquely.*
2. *If $u(s,x)$ depends on $s \in \mathbb{C}$ meromorphically, then $c_i^{j,k}(s)$ is a meromorphic function in $s \in \mathbb{C}$. The converse is also true.*

Proof. 1. This assertion is equivalent to the definition of a principal symbol.
2. It is clear from that the isomorphisms **sp** in (25) and σ_Λ in (32) are $\mathbb{C}[s]$-linear, where $\mathbb{C}[s]$ is the polynomial ring of s. Then $\sigma_{\Lambda_i^{j,k}}(u(s,x))$ depends on s meromorphically if and only if $u(s,x)$ is a meromorphic function on $s \in \mathbb{C}$. Since $\Omega_i^{j,k}(s)/\sqrt{|dx|}$ depends on $s \in \mathbb{C}$ holomorphically, $c_i^{j,k}(s)$ is a meromorphic function in $s \in \mathbb{C}$.

□

3.5. **Laurent expansions of coefficient functions.** Hyperfunction solutions $u(s,x)$ to $\mathcal{M}_s$ that we consider in this paper are the linear combinations
$$u(s,x) = P^{[\vec{a},s]}(x) := \sum_{i=0}^{n} a_i \cdot |P(x)|_i^s, \tag{42}$$
with $\vec{a} = (a_0, a_1, \ldots, a_n) \in \mathbb{C}^{n+1}$ introduced in (5). Since $P^{[\vec{a},s]}(x)$ is a hyperfunction with a meromorphic parameter $s \in \mathbb{C}$, the microfunction $\mathbf{sp}(P^{[\vec{a},s]}(x))$ and its principal symbols $\sigma_{\Lambda_i^{j,k}}(P^{[\vec{a},s]}(x))$ depend on $s \in \mathbb{C}$ meromorphically. In a particular case of (40) we define the coefficients of $P^{[\vec{a},s]}(x)$ on the Lagrangian connected component $\Lambda_i^{j,k}$.

Definition 3.2. Let
$$\sigma_{\Lambda_i^{j,k}}(P^{[\vec{a},s]}(x)) = c_i^{j,k}(\vec{a},s)\Omega_i^{j,k}(s)/\sqrt{|dx|}, \tag{43}$$
with $c_i^{j,k}(\vec{a},s)$ being a meromorphic function in $s \in \mathbb{C}$. We call $c_i^{j,k}(\vec{a},s)$ the *coefficient function* or the *coefficient* of $P^{[\vec{a},s]}(x)$ on $\Lambda_i^{j,k}$ with respect to the

canonical basis,

$$\Omega_i^{j,k}(s)/\sqrt{|dx|}. \tag{44}$$

Then each coefficients $c_i^{j,k}(\vec{a},s)$ depend on $\vec{a}\in\mathbb{C}^{n+1}$ linearly and on $s\in\mathbb{C}$ meromorphically.

Proposition 3.4. *Let $P^{[\vec{a}_1,s]}(x)$ and $P^{[\vec{a}_2,s]}(x)$ be two hyperfunction solutions to the holonomic system $\mathcal{M}_s$. If their coefficients coincide on each $\Lambda_i^{j,k}$:*

$$c_i^{j,k}(\vec{a}_1,s)=c_i^{j,k}(\vec{a}_2,s), \tag{45}$$

then we have $\vec{a}_1=\vec{a}_2$. In other words, two hyperfunction solutions having the same coefficients on all $\Lambda_i^{j,k}$'s coincide with each other.

Proof. Recall the following fact on the uniqueness of hyperfunction solutions to a holonomic system. It is proved in [Mur88a].

Lemma 3.5. *Let $f_1(x)$ and $f_2(x)$ be two hyperfunction solutions to the holonomic system $\mathcal{M}_s$. If $\mathbf{sp}(f_1(x))=\mathbf{sp}(f_2(x))$ on the open set $\bigcup_{i=0}^n\Lambda_i^\circ$, then $f_1(x)$ coincides with $f_2(x)$ as a hyperfunction on $\boldsymbol{V}$.*

Lemma 3.5 asserts that a microfunction solution to $\mathcal{M}_s$ is determined by the given data on $\bigcup_{i=0}^n\Lambda_i^\circ$. Therefore we only need to consider the microfunction solutions on $\bigcup_{i=0}^n\Lambda_i^\circ$ instead on the whole characteristic variety $\mathrm{ch}(\mathcal{M}_s)$.

From Proposition 3.3, if (45) is satisfied, then $\mathbf{sp}(P^{[\vec{a}_1,s]}(x))=\mathbf{sp}(P^{[\vec{a}_2,s]}(x))$ on each Lagrangian connected component $\Lambda_i^{j,k}$ and hence they coincide on the open set $\bigcup_{i=0}^n\Lambda_i^\circ$. Thus, from Lemma 3.5, we have $P^{[\vec{a}_1,s]}(x)=P^{[\vec{a}_2,s]}(x)$ which means $\vec{a}_1=\vec{a}_2$. □

For a microfunction solution on each Lagrangian connected component $\Lambda_i^{j,k}$, we have the following equivalent conditions.

Proposition 3.6. 1. *The following conditions are equivalent.*
   - (a) *The microfunction $\mathbf{sp}(P^{[\vec{a},s]}(x))|_{\Lambda_i^{j,k}}$ has a pole of order p at $s=s_0$.*
   - (b) *One of the principal symbol $\sigma_{\Lambda_i^{j,k}}(\mathbf{sp}(P^{[\vec{a},s]}(x)))$ has a pole of order p at $s=s_0$.*

2. *The following conditions are equivalent.*
   - (a) *The principal symbol $\sigma_{\Lambda_i^{j,k}}(\mathbf{sp}(P^{[\vec{a},s]}(x)))$ has a pole of order q at $s=s_0$.*
   - (b) *The coefficient $c_i^{j,k}(\vec{a},s)$ has a pole of order q at $s=s_0$.*

Proof. The first equivalence follows from that the isomorphism σ_Λ in (32) is $\mathbb{C}[s]$-linear and commutative with the action of the differential operators $\frac{\partial}{\partial s}$. The second equivalence follows from that $\Omega_i^{j,k}(s)/\sqrt{|dx|}$ is holomorphic at all $s\in\mathbb{C}$. □

Corollary 3.7. *The following conditions are equivalent.*

1. $P^{[\vec{a},s]}(x)$ *has a pole of order p at $s = s_0$.*
2. $\mathbf{sp}(P^{[\vec{a},s]}(x))|_{\cup_{i=1}^n \Lambda_i^\circ}$ *has a pole of order p at $s = s_0$.*
3. *All the coefficients in $\{c_i^{j,k}(\vec{a}, s); 0 \leq i \leq n, 0 \leq j \leq n-i, 0 \leq k \leq i\}$ has a pole of order not larger than p at $s = s_0$ and at least one coefficient of them has a pole of order p at $s = s_0$.*

Proof. The equivalence of 2. and 3. follows from Proposition 3.6 since

$$\bigcup_{i=0}^{n} \Lambda_i^\circ = \bigsqcup_{\substack{0 \leq i \leq n \\ 0 \leq j \leq n-i \\ 0 \leq k \leq i}} \Lambda_i^{j,k}.$$

We shall show that the condition 2 follows from the condition 1. If $P^{[\vec{a},s]}(x)$ has a pole of order p at $s = s_0$, then $(s - s_0)^p P^{[\vec{a},s]}(x)$ is a non-zero holomorphic function at $s = s_0$ with respect to s. Then

$$\mathbf{sp}((s - s_0)^p P^{[\vec{a},s]}(x)) = (s - s_0)^p \mathbf{sp}(P^{[\vec{a},s]}(x))$$

is also non-zero and holomorphic at $s = s_0$. Since $(s-s_0)^p \mathbf{sp}(P^{[\vec{a},s]}(x))|_{\cup_{i=0}^n \Lambda_i^\circ}$ is holomorphic at $s = s_0$, $\mathbf{sp}(P^{[\vec{a},s]}(x))|_{\cup_{i=0}^n \Lambda_i^\circ}$ has a pole of order not larger than p at $s = s_0$. If the order is strictly less than p, then $(s-s_0)^p \mathbf{sp}(P^{[\vec{a},s]}(x))|_{\cup_{i=0}^n \Lambda_i^\circ}|_{s=}$ is a zero function. Then $(s - s_0)^p \mathbf{sp}(P^{[\vec{a},s]}(x))|_{s=s_0}$ is zero, and hence $(s - s_0)^p P^{[\vec{a},s]}(x)|_{s=s_0}$ is zero by Lemma 3.5. This is a contradiction. Therefore $\mathbf{sp}(P^{[\vec{a},s]}(x))|_{\cup_{i=0}^n \Lambda_i^\circ}$ has a pole of order p at $s = s_0$. This means that that the condition 2 follows from the condition 1.

We shall show that the condition 1 follows from the condition 2. If $\mathbf{sp}(P^{[\vec{a},s]}(x))|_{\cup_{i=0}^n \Lambda_i^\circ}$ has a pole of order p at $s = s_0$, then

$$(s - s_0)^p \mathbf{sp}(P^{[\vec{a},s]}(x))|_{\cup_{i=0}^n \Lambda_i^\circ} = \mathbf{sp}((s - s_0)^p P^{[\vec{a},s]}(x))|_{\cup_{i=0}^n \Lambda_i^\circ}$$

is non-zero and holomorphic at $s = s_0$. Therefore, $(s - s_0)^p P^{[\vec{a},s]}(x)$ is non-zero and holomorphic at $s = s_0$. Thus, $P^{[\vec{a},s]}(x)$ has a pole of order p at $s = s_0$. This means that that the condition 1 follows from the condition 2. □

We define the coefficients of Laurent expansions $P^{[\vec{a},s]}(x)$ and $c_i^{j,k}(\vec{a}, s)$.

Definition 3.3. Suppose that the complex power function $P^{[\vec{a},s]}(x)$ has a pole of order p at $s = s_0$. We give the *Laurent expansion* of $P^{[\vec{a},s]}(x)$ at $s = s_0$ by

$$P^{[\vec{a},s]}(x) = \sum_{w=-p}^{\infty} P_w^{[\vec{a},s_0]}(x)(s - s_0)^w. \tag{46}$$

Here,

$$P_w^{[\vec{a},s_0]}(x) \tag{47}$$

is the Laurent expansion coefficient of degree w of $P^{[\vec{a},s]}(x)$. For the coefficient $c_i^{j,k}(\vec{a},s)$, we give the *Laurent expansion* at $s=s_0$ by

$$c_i^{j,k}(\vec{a},s)=\sum_{w=-p}^{\infty} c_{i,(\vec{a},s_0),w}^{j,k}(s-s_0)^w. \tag{48}$$

Here,

$$c_{i,(\vec{a},s_0),w}^{j,k} \tag{49}$$

is the Laurent expansion coefficient of degree w of $c_i^{j,k}(\vec{a},s)$. Since the order of the pole of $c_i^{j,k}(\vec{a},s)$ at $s=s_0$ is not larger than p, some beginning Laurent coefficients of (48) may be zero.

We can express the support of $P_w^{[\vec{a},s_0]}(x)$ in terms of the Laurent coefficients of $c_i^{j,k}(\vec{a},s)$. Namely, we have the following proposition.

Proposition 3.8. *Suppose that $P^{[\vec{a},s]}(x)$ has a pole of order p at $s=s_0$. Let* (46) *be the Laurent expansion of $P^{[\vec{a},s]}(x)$ at $s=s_0$. Then we have*

$$\mathbf{Supp}(P_w^{[\vec{a},s_0]}(x))=\overline{(\bigcup_{\left\{(i,j)\in\mathbb{Z}^2;\begin{array}{c}c_i^{j,k}(\vec{a},s)\ has\ a\ pole\ of\\ order\geq -w\ for\ some\ k\\ in\ 0\leq k\leq i\ at\ s=s_0\end{array}\right\}} S_i^j)} \tag{50}$$

Proof. For a hyperfunction $f(x)$ on $\boldsymbol{V}$, we have

$$\mathbf{Supp}(f(x))=\overline{\pi(\mathbf{Supp}(\mathbf{sp}(f(x))))},$$

by the isomorphism (25). Therefore, we have

$$\mathbf{Supp}(P_w^{[\vec{a},s_0]}(x))=\overline{\pi(\mathbf{Supp}(\mathbf{sp}(P_w^{[\vec{a},s_0]}(x))))}. \tag{51}$$

Let q be an integer in $-p\leq -q<+\infty$. If $\mathbf{sp}(P^{[\vec{a},s]}(x))|_{\Lambda_i^{j,k}}$ has a pole of order q at $s=s_0$, then the $c_i^{j,k}(\vec{a},s)$'s pole at $s=s_0$ is of order q (Proposition 3.6). We have the Laurent expansion

$$\mathbf{sp}(P^{[\vec{a},s]}(x))|_{\Lambda_i^{j,k}}=\sum_{w=-q}^{\infty}\mathbf{sp}(P_w^{[\vec{a},s_0]}(x))|_{\Lambda_i^{j,k}}\cdot(s-s_0)^w. \tag{52}$$

by (46). On the other hand, let

$$\sigma_{\Lambda_i^{j,k}}(\mathbf{sp}(P^{[\vec{a},s]}(x)))=\sum_{w=-q}^{\infty}\sigma_{i,(\vec{a},s_0),w}^{j,k}\cdot(s-s_0)^w \tag{53}$$

be the Laurent expansion of the principal symbol $\sigma_{\Lambda_i^{j,k}}(\mathbf{sp}(P^{[\vec{a},s]}(x)))$. Then we have

$$\sigma_{\Lambda_i^{j,k}}(\mathbf{sp}(P_w^{[\vec{a},s_0]}(x)))=\sigma_{i,(\vec{a},s_0),w}^{j,k} \tag{54}$$

for $-q\leq w<+\infty$.

Now we have the following Laurent expansions,

$$\sigma_{\Lambda_i^{j,k}}(P^{[\vec{a},s]}(x)) = c_i^{j,k}(\vec{a},s)\Omega_i^{j,k}(s)/\sqrt{|dx|}$$
$$= \sum_{w=-q}^{\infty} \sigma^{j,k}_{i,(\vec{a},s_0),w} \cdot (s-s_0)^w, \tag{55}$$

$$c_i^{j,k}(\vec{a},s) = \sum_{u=-q}^{\infty} c^{j,k}_{i,(\vec{a},s_0),u} \cdot (s-s_0)^u, \tag{56}$$

$$\Omega_i^{j,k}(s) = \sum_{v=0}^{\infty} \Omega^{j,k}_{i,s_0,v} \cdot (s-s_0)^v. \tag{57}$$

Note that $\Omega^{j,k}_{i,s_0,0}, \Omega^{j,k}_{i,s_0,1}, \ldots$ in (57) are non-zero linearly independent half-volume forms on $\Lambda_i^{j,k}$. Then all the Laurent-expansion coefficients

$$\sigma^{j,k}_{i,(\vec{a},s_0),w} \qquad (-q \le w \le +\infty) \tag{58}$$

in (55) are non-zero if $c^{j,k}_{i,(\vec{a},s_0),-q} \neq 0$. This means that all the Laurent-expansion coefficients of negative order of $\sigma_{\Lambda_i^{j,k}}(P^{[\vec{a},s]}(x))$ are not zero. Hence the support $\mathbf{Supp}(\mathbf{sp}(P_w^{[\vec{a},s_0]}(x)))$ contains $\Lambda_i^{j,k}$ if $-q \le w < \infty$, which shows that

$$\begin{aligned}
\mathbf{Supp}(P_w^{[\vec{a},s_0]}(x)) &= \overline{\pi(\mathbf{Supp}(\mathbf{sp}(P_w^{[\vec{a},s_0]}(x))))} \\
&= \overline{\pi(\bigcup_{\left\{(i,j,k)\in\mathbb{Z}^3;\, \begin{subarray}{l} c_i^{j,k}(\vec{a},s) \text{ has a pole of} \\ \text{order } \ge -w \text{ at } s=s_0 \end{subarray}\right\}} \Lambda_i^{j,k})} \\
&= \overline{(\bigcup_{\left\{(i,j,k)\in\mathbb{Z}^3;\, \begin{subarray}{l} c_i^{j,k}(\vec{a},s) \text{ has a pole of} \\ \text{order } \ge -w \text{ at } s=s_0 \end{subarray}\right\}} \pi(\Lambda_i^{j,k}))} \\
&= \overline{(\bigcup_{\left\{(i,j)\in\mathbb{Z}^2;\, \begin{subarray}{l} c_i^{j,k}(\vec{a},s) \text{ has a pole of} \\ \text{order } \ge -w \text{ for some } k \\ \text{in } 0 \le k \le i \text{ at } s=s_0 \end{subarray}\right\}} S_i^j)}.
\end{aligned} \tag{59}$$

Thus we have the desired result. □

Remark 3.3. Since $\mathbf{sp}(P_w^{[\vec{a},s_0]}(x))$ is a regular holonomic microfunction, we can define its principal symbol directly. However, in our case, it is obtained by differentiating a simple microfunction with a meromorphic parameter $s \in \mathbb{C}$ with respect to s , hence its principal symbol is obtained from the differentiation with respect to the parameter s.

4. Some properties of principal symbols.

We shall calculate the analytic relations combining the coefficients of a hyperfunction solution to the holonomic system $\mathcal{M}_s$. The propositions obtained in this section enables us to estimate the order of poles of coefficients in the next section.

4.1. Relations of coefficients on contiguous Lagrangian subvarieties. We shall use the following two relations (60) and (61) in the proofs of the main theorem.

Proposition 4.1. *The coefficients on Λ_j° and Λ_{i+1}° have the following relation. These relations depend on $s \in \mathbb{C}$ meromorphically.*

$$\begin{bmatrix} c_{i+1}^{j,k+1}(\vec{a},s) \\ c_{i+1}^{j,k}(\vec{a},s) \end{bmatrix} = \frac{\Gamma(s+\frac{i+2}{2})}{\sqrt{2\pi}} \begin{bmatrix} \exp(-\frac{\pi}{2}\sqrt{-1}(s+\frac{i+2}{2})) & \exp(+\frac{\pi}{2}\sqrt{-1}(s+\frac{i+2}{2})) \\ \exp(+\frac{\pi}{2}\sqrt{-1}(s+\frac{i+2}{2})) & \exp(-\frac{\pi}{2}\sqrt{-1}(s+\frac{i+2}{2})) \end{bmatrix}$$
$$\times \begin{bmatrix} \exp(+\frac{\pi}{4}\sqrt{-1}(i-2k)) & 0 \\ 0 & \exp(-\frac{\pi}{4}\sqrt{-1}(i-2k)) \end{bmatrix}$$
$$\times \begin{bmatrix} c_i^{j+1,k}(\vec{a},s) \\ c_i^{j,k}(\vec{a},s) \end{bmatrix} \tag{60}$$

Proof. See the Theorem 2.13 of [Mur86]. The above relations are the case of $Sym_n(\mathbb{R})$. □

Proposition 4.2. *The coefficients functions on Λ_j° and Λ_{i+2}° have the following relations.*

$$\begin{bmatrix} c_{i+2}^{j,k+2}(\vec{a},s) \\ c_{i+2}^{j,k+1}(\vec{a},s) \\ c_{i+2}^{j,k}(\vec{a},s) \end{bmatrix}$$
$$= \frac{\Gamma(s+\frac{i+2}{2})\Gamma(s+\frac{i+3}{2})}{2\pi}$$
$$\times \begin{bmatrix} -\sqrt{-1}\exp(-\pi\sqrt{-1}(s+k)) & 0 & \sqrt{-1}\exp(+\pi\sqrt{-1}(s+k)) \\ \exp(\frac{1}{2}\pi\sqrt{-1}(i-2k)) & -2\cos(\frac{1}{2}\pi(2s+i)) & \exp(-\frac{1}{2}\pi\sqrt{-1}(i-2k)) \\ \sqrt{-1}\exp(+\pi\sqrt{-1}(s-k+i)) & 0 & -\sqrt{-1}\exp(-\pi\sqrt{-1}(s-k+i)) \end{bmatrix}$$
$$\times \begin{bmatrix} c_i^{j+2,k}(\vec{a},s) \\ c_i^{j+1,k}(\vec{a},s) \\ c_i^{j,k}(\vec{a},s) \end{bmatrix} \tag{61}$$

These relations depend on $s \in \mathbb{C}$ meromorphically.

Proof. These formulas are obtained by applying the relation formula (60) twice. □

4.2. Laurent expansions of coefficient matrices.

Definition 4.1. 1. We define the *coefficient matrix* $\boldsymbol{c}_i^{\bullet,k}(\vec{a},s)$ and $\boldsymbol{c}_i^{j,\bullet}(\vec{a},s)$ by the $1\times(n-i)$-matrix

$$\boldsymbol{c}_i^{\bullet,k}(\vec{a},s)=(c_i^{0,k}(\vec{a},s),c_i^{1,k}(\vec{a},s),\dots,c_i^{n-i,k}(\vec{a},s)) \tag{62}$$

and the $i\times 1$-matrix

$$\boldsymbol{c}_i^{j,\bullet}(\vec{a},s)={}^t(c_i^{j,0}(\vec{a},s),c_i^{j,1}(\vec{a},s),\dots,c_i^{j,i}(\vec{a},s)) \tag{63}$$

, respectively. The *coefficient matrix* $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a},s)$ is defined to be an $i\times(n-i)$ matrix

$$\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a},s)=(c_i^{j,k}(\vec{a},s))_{\substack{0\le k\le i\\ 0\le j\le n-i}}. \tag{64}$$

2. We define the *order of pole* of a coefficient matrix to be the maximum of the orders of the entries in the matrix. For example, the order of pole of $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a},s)$ is the maximum of the orders of the entries in $(c_i^{j,k}(\vec{a},s))_{\substack{0\le k\le i\\ 0\le j\le n-i}}$.

Let p be the order of poles of $P^{[\vec{a},s]}(x)$ at $s=s_0$. Then the Laurent expansion of $\boldsymbol{c}_i^{\bullet,k}(\vec{a},s)$ $\boldsymbol{c}_i^{j,\bullet}(\vec{a},s)$ and $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a},s)$ are written in the following form.

$$\boldsymbol{c}_i^{\bullet,k}(\vec{a},s)=\sum_{w=-p}^{\infty}\boldsymbol{c}_{i,(\vec{a},s_0),w}^{\bullet,k}(s-s_0)^w, \tag{65}$$

$$\boldsymbol{c}_i^{j,\bullet}(\vec{a},s)=\sum_{w=-p}^{\infty}\boldsymbol{c}_{i,(\vec{a},s_0),w}^{j,\bullet}(s-s_0)^w, \tag{66}$$

$$\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a},s)=\sum_{w=-p}^{\infty}\boldsymbol{c}_{i,(\vec{a},s_0),w}^{\bullet,\bullet}(s-s_0)^w. \tag{67}$$

Some beginning Laurent expansion coefficients may be zero in these Laurent expansions because the order of poles of these coefficients are not larger than the order of $P^{[\vec{a},s]}(x)$.

4.3. Properties of Laurent expansion coefficients of coefficient matrices.

Proposition 4.3. *Let s_0 be a half-integer satisfying $s_0\le -1$ and let i_0 be an integer in $0\le i_0\le n-1$.*

1. *We suppose that i_0 is even and s_0 is a strict half-integer or that i_0 is odd and s_0 is an integer. Then $\boldsymbol{c}_{i_0}^{\bullet,\bullet}(\vec{a},s)$ and $\boldsymbol{c}_{i_0+1}^{\bullet,\bullet}(\vec{a},s)$ have poles of the same order at $s=s_0$.*

2. *Suppose that one coefficient $c_{i_0}^{j_0,k_0}(\vec{a},s)$ has a pole of order p at $s=s_0$. Then all the coefficients $c_{i_0}^{j_0,k}(\vec{a},s)$ in $0\le k\le i_0$ have poles of the same order p at $s=s_0$. Their Laurent-expansion coefficients of degree $-p$ satisfy the relations*

$$(-1)^{2s_0+i_0+1}c^{j_0,k}_{i_0,(\vec{a},s_0),-p}=c^{j_0,k+1}_{i_0,(\vec{a},s_0),-p} \tag{68}$$

for all $0\le k\le i_0-1$.

Proof. 1. Note that $s_0+\frac{i_0+2}{2}$ is a strict half-integer in both cases. We consider the relation (60) in a neighborhood of $s=s_0$. Then the relation matrix between $\begin{bmatrix} c_{i+1}^{j-1,k+1}(\vec{a},s)\\ c_{i+1}^{j-1,k}(\vec{a},s)\end{bmatrix}$ and $\begin{bmatrix} c_i^{j,k}(\vec{a},s)\\ c_i^{j-1,k}(\vec{a},s)\end{bmatrix}$ depends on $s\in\mathbb{C}$ holomorphically and is invertible near $s=s_0$. The inverse matrix also depends on s holomorphically, and hence $\boldsymbol{c}_{i_0}^{\bullet,\bullet}(\vec{a},s)$ and $\boldsymbol{c}_{i_0+1}^{\bullet,\bullet}(\vec{a},s)$ have poles of the same order at $s=s_0$.

2. In the formula (60), we substitute $i:=i_0-1$. Then $\begin{bmatrix} c_{i_0}^{j-1,k+1}(\vec{a},s)\\ c_{i_0}^{j-1,k}(\vec{a},s)\end{bmatrix}$ can be written as a linear combination of $c_{i_0-1}^{j,k}(\vec{a},s)$ and $c_{i_0-1}^{j-1,k-1}(\vec{a},s)$ with coefficients of meromorphic functions of s. Then the equation (68) is naturally obtained from the form of linear combinations by (60). □

Definition 4.2. Let s_0 be a half-integer not larger than -1. By Proposition 4.3, the orders of poles of $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a},s)$ and $\boldsymbol{c}_i^{\bullet,k}(\vec{a},s)$ $(0\le k\le i)$ all coincide. We call it a *top order* of $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a},s)$ at $s=s_0$ and denote by

$$t_i=t_i(\vec{a},s_0) \tag{69}$$

the order of them. Indeed, t_i varies depending not only on s_0 but also on $\vec{a}$.

By using the top order, we can describe the relation

$$(-1)^{2s_0+i+1}\boldsymbol{c}^{\bullet,k}_{i,(\vec{a},s_0),-t_i(\vec{a},s_0)}=\boldsymbol{c}^{\bullet,k+1}_{i,(\vec{a},s_0),-t_i(\vec{a},s_0)}. \tag{70}$$

This is implied from Proposition 4.3 and the definition of t_i.

Definition 4.3. Let s_0 be a half integer not larger than -1 and let i,j,and k be integers contained in $0\le i\le n-2$,$0\le j\le n-i-2$ and $0\le k\le i$,respectively.

1. Let q be an integer. The *condition* $(\boldsymbol{Cond})^{j,k}_{i,(\vec{a},s_0),q}$ for the coefficients on Λ_i° means that the relation for the coefficients

$$c^{j,k}_{i,(\vec{a},s_0),-q}+(-1)^i c^{j+2,k}_{i,(\vec{a},s_0),-q}=0 \tag{71}$$

is satisfied. The *condition* $(\boldsymbol{Cond})^{j,0,*}_{0,(\vec{a},s_0),q}$ for the coefficients on Λ_0° means that the relation for the coefficients

$$c^{j,0}_{0,(\vec{a},s_0),-q}+(-1)^{s_0+1}c^{j+1,0}_{0,(\vec{a},s_0),-q}=0 \tag{72}$$

is satisfied. Here,s_0 must be an integer and q is always 0.

2. Let q be an integer. The *condition* $(\boldsymbol{Cond})^{\bullet,k}_{i,(\vec{a},s_0),q}$ means that the conditions $(\boldsymbol{Cond})^{j,k}_{i,(\vec{a},s_0),q}$ are satisfied for all integers j in $0 \le j \le n-i-2$. The *condition* $(\boldsymbol{Cond})^{\bullet,0,*}_{0,(\vec{a},s_0),q}$ stands for that the conditions $(\boldsymbol{Cond})^{j,0,*}_{0,(\vec{a},s_0),q}$ are satisfied for all integers j in $0 \le j \le n-2$. The *condition* $(\boldsymbol{Cond})^{j,\bullet}_{i,(\vec{a},s_0),q}$ means that the conditions $(\boldsymbol{Cond})^{j,k}_{i,(\vec{a},s_0),q}$ are satisfied for all integers j in $0 \le k \le i$. The *condition* $(\boldsymbol{Cond})^{j,\bullet,*}_{0,(\vec{a},s_0),q}$ stands for that the conditions $(\boldsymbol{Cond})^{j,0,*}_{0,(\vec{a},s_0),q}$.
3. Let q be an integer. The *condition* $(\boldsymbol{Cond})^{\bullet,\bullet}_{i,(\vec{a},s_0),q}$ means that the conditions $(\boldsymbol{Cond})^{j,k}_{i,(\vec{a},s_0),q}$ are satisfied for all integers j and k in $0 \le j \le n-i-2$ and $0 \le k \le i$, respectively. The *condition* $(\boldsymbol{Cond})^{\bullet,\bullet,*}_{0,(\vec{a},s_0),q}$ is equivalent to the condition $(\boldsymbol{Cond})^{\bullet,0,*}_{0,(\vec{a},s_0),q}$.
4. The *condition* $(\boldsymbol{Cond})^{-,-}_{i,(\vec{a},s_0),\mathbf{top}}$ means that the condition $(\boldsymbol{Cond})^{-,-}_{i,(\vec{a},s_0),q}$ when q is the maximum of the orders of poles of the coefficients appearing in the relation formula. For example, the condition $(\boldsymbol{Cond})^{j,k}_{i,(\vec{a},s_0),\mathbf{top}}$ means the relation (71) where q is the maximum of the orders of poles at $s = s_0$ of the two coefficients $c_i^{j,k}(\vec{a},s)$ and $c_i^{j+2,k}(\vec{a},s)$. The *condition* $(\boldsymbol{Cond})^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathbf{top}}$ means the condition $(\boldsymbol{Cond})^{\bullet,\bullet}_{i,(\vec{a},s_0),q}$ where q is the maximum of the orders of poles at $s = s_0$ in the entries of $(c_i^{j,k}(\vec{a},s))_{\substack{0\le k\le i\\ 0\le j\le n-i}}$.
5. The *condition* $\overline{(\boldsymbol{Cond})^{-,-,-}_{i,(\vec{a},s_0),-}}$ means the negation of the condition $(\boldsymbol{Cond})^{-,-,-}_{i,(\vec{a},s_0),-}$.

Proposition 4.4. *Let $\vec{a} \in \mathbb{C}^{n+1}$ and let s_0 be a half-integer not larger than -1. Then $(\boldsymbol{Cond})^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathbf{top}}$ is equivalent to that there exists an integers k in $0 \le k \le i$ such that $(\boldsymbol{Cond})^{\bullet,k}_{i,(\vec{a},s_0),\mathbf{top}}$ is satisfied.*

Proof. From Definition 4.2 and Definition 4.3, we have that $(\boldsymbol{Cond})^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathbf{top}}$ is equivalent to $(\boldsymbol{Cond})^{\bullet,\bullet}_{i,(\vec{a},s_0),-t_i(\vec{a},s_0)}$, and that $(\boldsymbol{Cond})^{\bullet,k}_{i,(\vec{a},s_0),\mathbf{top}}$ is equivalent to $(\boldsymbol{Cond})^{\bullet,k}_{i,(\vec{a},s_0),-t_i(\vec{a},s_0)}$. From (70), if there exists an integer k in $0 \le k \le i$ such that $(\boldsymbol{Cond})^{\bullet,k}_{i,(\vec{a},s_0),-t_i(\vec{a},s_0)}$, then $(\boldsymbol{Cond})^{\bullet,k}_{i,(\vec{a},s_0),-t_i(\vec{a},s_0)}$, for all integers k in $0 \le k \le i$, and the converse is true. This is equivalent to the condition $(\boldsymbol{Cond})^{\bullet,\bullet}_{i,(\vec{a},s_0),-t_i(\vec{a},s_0)}$. Thus we have the desired result. □

Proposition 4.5. *There are the following relations among the conditions $(\boldsymbol{Cond})^{j,k,-}_{i,(\vec{a},s_0),-}$ and the following order of poles of coefficients $c_i^{j,k}(\vec{a},s)$.*

1. *Let s_0 be an integer not larger than -1.*
   (a) *If the conditions $(\boldsymbol{Cond})^{j,\bullet,*}_{0,(\vec{a},s_0),\mathbf{top}}$, $(\boldsymbol{Cond})^{j+1,\bullet,*}_{0,(\vec{a},s_0),\mathbf{top}}$, and $(\boldsymbol{Cond})^{j+2,\bullet,*}_{0,(\vec{a},s_0),\mathbf{top}}$, are satisfied, then we have $(\boldsymbol{Cond})^{j,\bullet}_{1,(\vec{a},s_0),\mathbf{top}}$.*

(b) *If the condition* $\overline{(\boldsymbol{Cond})^{j,\bullet,*}_{0,(\vec{a},s_0),\mathbf{top}}}$ *is satisfied, then the order of pole at* $s = s_0$ *of* $\boldsymbol{c}^{j,\bullet}_1(\vec{a},s)$ *is 1. If the condition* $(\boldsymbol{Cond})^{j,\bullet,*}_{0,(\vec{a},s_0),\mathbf{top}}$ *is satisfied, then* $\boldsymbol{c}^{j,\bullet}_1(\vec{a},s)$ *is holomorphic at* $s = s_0$*. The converses are also true.*

2. *Let* s_0 *be a half-integer not larger than* -1 *and let* i *be an integer in* $0 \leq i \leq n-2$*. We suppose that* i *is even and* s_0 *is a strict half-integer or that* i *is odd and* s_0 *is an integer.*

   (a) *If* $(\boldsymbol{Cond})^{j,\bullet}_{i,(\vec{a},s_0),\mathbf{top}}$ *and* $(\boldsymbol{Cond})^{j+2,\bullet}_{i,(\vec{a},s_0),\mathbf{top}}$ *are satisfied, then we have* $(\boldsymbol{Cond})^{j,\bullet}_{i+2,(\vec{a},s_0),\mathbf{top}}$*.*

   (b) *If the condition* $\overline{(\boldsymbol{Cond})^{j,\bullet}_{i,(\vec{a},s_0),\mathbf{top}}}$ *is satisfied and* $s_0 \leq -\frac{i+2}{2}$*, then the order of pole at* $s = s_0$ *of* $\boldsymbol{c}^{j,\bullet}_{i+2}(\vec{a},s)$ *is larger by 1 than that of* $\boldsymbol{c}^{j,\bullet}_i(\vec{a},s)$*. If the condition* $(\boldsymbol{Cond})^{j,\bullet}_{i,(\vec{a},s_0),\mathbf{top}}$ *is satisfied or* $s_0 > -\frac{i+2}{2}$*, then the order of pole at* $s = s_0$ *of* $\boldsymbol{c}^{j,\bullet}_{i+2}(\vec{a},s)$ *is not larger than that of* $\boldsymbol{c}^{j,\bullet}_i(\vec{a},s)$*. The converses are also true.*

Proof. We can prove these propositions by using the relations of coefficients (60) and (61), and the condition formula (71) and (72).

1. First we prove the relation of the coefficients $c^{j,k}_i(\vec{a},s)$ with $i = 0,1$. Note that s_0 is an integer not larger than -1.

   (a) The condition $(\boldsymbol{Cond})^{j,\bullet,*}_{0,(\vec{a},s_0),\mathbf{top}}$ is equivalent to the condition $(\boldsymbol{Cond})^{j,\bullet,*}_{0,(\vec{a},s_0),0}$. From the equation (72) and the assumptions, we have

$$
\begin{aligned}
c^{j,0}_{0,(\vec{a},s_0),0} + (-1)^{s_0+1} c^{j+1,0}_{0,(\vec{a},s_0),0} &= 0, \\
c^{j+1,0}_{0,(\vec{a},s_0),0} + (-1)^{s_0+1} c^{j+2,0}_{0,(\vec{a},s_0),0} &= 0, \\
c^{j+2,0}_{0,(\vec{a},s_0),0} + (-1)^{s_0+1} c^{j+3,0}_{0,(\vec{a},s_0),0} &= 0.
\end{aligned}
\tag{73}
$$

Note that $c^{j,0}_0(\vec{a},s) = a_j$ do not depend on s for all j in $0 \leq j \leq n$. Then the equation (73) means

$$
\begin{aligned}
c^{j,0}_0(\vec{a},s) + (-1)^{s_0+1} c^{j+1,0}_0(\vec{a},s) &= 0 \\
c^{j+1,0}_0(\vec{a},s) + (-1)^{s_0+1} c^{j+2,0}_0(\vec{a},s) &= 0 \\
c^{j+2,0}_0(\vec{a},s) + (-1)^{s_0+1} c^{j+3,0}_0(\vec{a},s) &= 0
\end{aligned}
\tag{74}
$$

By substituting $i := 0$ in the relation formula (60), we have

$$\begin{aligned}
&\begin{bmatrix} c_1^{j,1}(\vec{a}, s) \\ c_1^{j,0}(\vec{a}, s) \end{bmatrix} \\
&= \frac{\Gamma(s+1)}{\sqrt{2\pi}} \begin{bmatrix} \exp(-\frac{\pi}{2}\sqrt{-1}(s+1)) & \exp(+\frac{\pi}{2}\sqrt{-1}(s+1)) \\ \exp(+\frac{\pi}{2}\sqrt{-1}(s+1)) & \exp(-\frac{\pi}{2}\sqrt{-1}(s+1)) \end{bmatrix} \\
&\times \begin{bmatrix} c_0^{j+1,0}(\vec{a}, s) \\ c_0^{j,0}(\vec{a}, s) \end{bmatrix}
\end{aligned} \tag{75}$$

for all j in $0 \leq j \leq n-1$. Through $\Gamma(s+1)$ has a pole of order 1 when s is an integer not larger than -1, $c_1^{j,1}(\vec{a}, s)$ and $c_1^{j,0}(\vec{a}, s)$ are holomorphic at $s = s_0$ by the relations (74) and (75). By computing the values of them at $s = s_0$, we have

$$\begin{aligned}
c_{1,(\vec{a},s_0),0}^{j,0} + (-1)^{s_0+1} c_{1,(\vec{a},s_0),0}^{j+1,0} &= 0 \\
c_{1,(\vec{a},s_0),0}^{j+1,0} + (-1)^{s_0+1} c_{1,(\vec{a},s_0),0}^{j+2,0} &= 0
\end{aligned} \tag{76}$$

Hence we have

$$c_{1,(\vec{a},s_0),0}^{j,0} = c_{1,(\vec{a},s_0),0}^{j+2,0}. \tag{77}$$

This means that

$$c_{1,(\vec{a},s_0),0}^{j,0} + (-1) c_{1,(\vec{a},s_0),0}^{j+2,0} = 0, \tag{78}$$

and hence we have $(\boldsymbol{Cond})_{1,(\vec{a},s_0),0}^{j,\bullet}$. In our case, since the order of pole at $s = s_0$ of $c_1^{j,\bullet}(\vec{a}, s)$ is 0, this condition is $(\boldsymbol{Cond})_{1,(\vec{a},s_0),\mathbf{top}}^{j,\bullet}$. This is the desired result.

(b) These propositions are trivial from the above calculations.

2. Next we prove the relation of the coefficients $c_i^{j,k}(\vec{a}, s)$ with $i > 1$. Let $p_0 := t_i(\vec{a}, s_0)$ be the top order of $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ at $s = s_0$ and let $p_1 := t_{i+2}(\vec{a}, s_0)$ be the top order of $\boldsymbol{c}_{i+2}^{\bullet,\bullet}(\vec{a}, s)$ at $s = s_0$ as defined in Definition 4.2.

   (a) We first suppose that i is odd and s_0 is an integer. Then the condition $(\boldsymbol{Cond})_{i,(\vec{a},s_0),\mathbf{top}}^{\bullet,\bullet}$ means the condition $(\boldsymbol{Cond})_{i,(\vec{a},s_0),p_0}^{\bullet,\bullet}$. Therefore, from the equation (71), if i is odd, then from the assumption

$$\begin{aligned}
c_{i,(\vec{a},s_0),-p_0}^{j,k} &= c_{i,(\vec{a},s_0),-p_0}^{j+2,k} \\
c_{i,(\vec{a},s_0),-p_0}^{j+2,k} &= c_{i,(\vec{a},s_0),-p_0}^{j+4,k}
\end{aligned} \tag{79}$$

   are satisfied for all k in $0 \leq k \leq i$. Using the relation formula (61), we can compute the elements of $c_{i+2,(\vec{a},s_0),-p_1}^{\bullet,\bullet}$. Then we have

$$c_{i+2,(\vec{a},s_0),-p_1}^{j,k} = c_{i+2,(\vec{a},s_0),-p_1}^{j+2,k} \tag{80}$$

is satisfied for all k in $0 \leq k \leq i+2$ since the relation matrix in (61) does not depend on j. This means the condition $(\boldsymbol{Cond})^{j,\bullet}_{i+2,(\vec{a},s_0),\mathbf{top}}$.
In the case that i is even and s_0 is a strict half-integer, we can prove the proposition in the same way. Namely, the condition $(\boldsymbol{Cond})^{j,\bullet}_{i,(\vec{a},s_0),\mathbf{top}}$:

$$c^{j,k}_{i,(\vec{a},s_0),-p_0} = -c^{j+2,k}_{i,(\vec{a},s_0),-p_0} \tag{81}$$

implies the condition $(\boldsymbol{Cond})^{j,\bullet}_{i+2,(\vec{a},s_0),\mathbf{top}}$:

$$c^{j,k}_{i+2,(\vec{a},s_0),-p_1} = -c^{j+2,k}_{i+2,(\vec{a},s_0),-p_1} \tag{82}$$

(b) We first suppose that i is odd and s_0 is an integer.
If $\overline{(\boldsymbol{Cond})^{j,\bullet}_{i+2,(\vec{a},s_0),\mathbf{top}}}$ is satisfied, then there exists integers k in $0 \leq k \leq i$ such that

$$c^{j,k}_{i,(\vec{a},s_0),-p_0} \neq c^{j+2,k}_{i,(\vec{a},s_0),-p_0}. \tag{83}$$

Then, remember the formula (61).

$$\begin{bmatrix} c^{j,k+2}_{i+2}(\vec{a},s) \\ c^{j,k+1}_{i+2}(\vec{a},s) \\ c^{j,k}_{i+2}(\vec{a},s) \end{bmatrix} \tag{84}$$

$$= \frac{\Gamma(s+\frac{i+2}{2})\Gamma(s+\frac{i+3}{2})}{2\pi} \tag{85}$$

$$\times \begin{bmatrix} -\sqrt{-1}\exp(-\pi\sqrt{-1}(s+k)) & 0 & \sqrt{-1}\exp(+\pi\sqrt{-1}(s+k)) \\ \exp(\frac{1}{2}\pi\sqrt{-1}(i-2k)) & -2\cos(\frac{1}{2}\pi(2s+i)) & \exp(-\frac{1}{2}\pi\sqrt{-1}(i-2k)) \\ \sqrt{-1}\exp(+\pi\sqrt{-1}(s-k+i)) & 0 & -\sqrt{-1}\exp(-\pi\sqrt{-1}(s-k+i)) \end{bmatrix} \tag{86}$$

$$\times \begin{bmatrix} c^{j+2,k}_{i}(\vec{a},s) \\ c^{j+1,k}_{i}(\vec{a},s) \\ c^{j,k}_{i}(\vec{a},s) \end{bmatrix} \tag{87}$$

Then the elements of the matrix (86) $\times$ (87) have poles of order p_0 at $s = s_0$. If $s_0 \leq -\frac{i+2}{2}$, then the gamma function (85) has a pole of order 1 at $s = s_0$. Hence, the elements of the matrix (85) $\times$ (86) $\times$ (87) has a pole of order $p_0 + 1$ at $s = s_0$. Therefore, $c^{j,\bullet}_{i+2}(\vec{a},s_0)$ has a pole of order larger by 1 than that of $c^{j,\bullet}_{i}(\vec{a},s_0)$ at $s = s_0$. This is the desired result. It is clear that the converse is true.
If $(\boldsymbol{Cond})^{j,\bullet}_{i+2,(\vec{a},s_0),\mathbf{top}}$ is satisfied, then

$$c^{j,k}_{i,(\vec{a},s_0),-p_0} = c^{j+2,k}_{i,(\vec{a},s_0),-p_0} \tag{88}$$

for all integers k in $0 \leq k \leq i$. Then the elements of the matrix (86) $\times$ (87) have poles of order $p_0 - 1$ at $s = s_0$. The gamma function (85) has a pole of order 1 at $s = s_0$ if $s_0 \leq -\frac{i+2}{2}$, and otherwise, it is holomorphic at $s = s_0$. Hence, the elements of the matrix (85) $\times$ (86) $\times$ (87) has a pole of order not larger than p_0 at $s = s_0$ if the condition $(\boldsymbol{Cond})^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathbf{top}}$ is satisfied or $s_0 > -\frac{i+2}{2}$. Therefore, if the condition $(\boldsymbol{Cond})^{j,\bullet}_{i,(\vec{a},s_0),\mathbf{top}}$ is satisfied or $s_0 > -\frac{i+2}{2}$, then $c^{j,\bullet}_{i+2}(\vec{a}, s_0)$ has a pole of order not larger than that of $c^{j,\bullet}_{i}(\vec{a}, s_0)$ at $s = s_0$. This is the desired result. It is clear that the converse is true.

In the case that i is even and s_0 is a strict half-integer, we can prove the proposition in the same way.

□

Corollary 4.6. 1. *Let s_0 be an integer not larger than -1.*

(a) *The condition $(\boldsymbol{Cond})^{\bullet,\bullet,*}_{0,(\vec{a},s_0),\mathbf{top}}$ implies the condition $(\boldsymbol{Cond})^{\bullet,\bullet}_{1,(\vec{a},s_0),\mathbf{top}}$. This means that the condition $\overline{(\boldsymbol{Cond})^{\bullet,\bullet,*}_{0,(\vec{a},s_0),\mathbf{top}}}$ follows from the condition $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{1,(\vec{a},s_0),\mathbf{top}}}$.*

(b) *If the condition $\overline{(\boldsymbol{Cond})^{\bullet,\bullet,*}_{0,(\vec{a},s_0),\mathbf{top}}}$ is satisfied, then the order of pole at $s = s_0$ of $\mathbf{c}^{\bullet,\bullet}_1(\vec{a}, s)$ is 1. If the condition $(\boldsymbol{Cond})^{\bullet,\bullet,*}_{0,(\vec{a},s_0),\mathbf{top}}$ is satisfied, then $\mathbf{c}^{\bullet,\bullet}_1(\vec{a}, s)$ is holomorphic at $s = s_0$. The converses are also true.*

2. *Let s_0 be a half-integer not larger than -1 and let i be an integer in $0 \leq i \leq n-2$. We suppose that i is even and s_0 is a strict half-integer or that i is odd and s_0 is an integer.*

(a) *The condition $(\boldsymbol{Cond})^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathbf{top}}$ implies the condition $(\boldsymbol{Cond})^{\bullet,\bullet}_{i+2,(\vec{a},s_0),\mathbf{top}}$. This means that the condition $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathbf{top}}}$ follows from the condition $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{i+2,(\vec{a},s_0),\mathbf{top}}}$.*

(b) *If the condition $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathbf{top}}}$ is satisfied and $s_0 \leq -\frac{i+2}{2}$, then the order of pole at $s = s_0$ of $\mathbf{c}^{\bullet,\bullet}_{i+2}(\vec{a}, s)$ is larger by 1 than that of $\mathbf{c}^{\bullet,\bullet}_{i}(\vec{a}, s)$. If the condition $(\boldsymbol{Cond})^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathbf{top}}$ is satisfied or $s_0 > -\frac{i+2}{2}$, then the order of pole at $s = s_0$ of $\mathbf{c}^{\bullet,\bullet}_{i+2}(\vec{a}, s)$ is not larger than that of $\mathbf{c}^{\bullet,\bullet}_{i}(\vec{a}, s)$. The converses are also true.*

Remark 4.1. Proposition 4.5-2-(a) and Corollary 4.6-2-(a) can be proved from the assumption that s_0 is a half-integer and i is an integer in $0 \leq i \leq n-2$. Indeed, this proposition is proved from the fact that the relation matrix in (61) does not depend on j.

Corollary 4.7. *Let s_0 be a half-integer. In this corollary, $(\boldsymbol{Cond})^{\bullet,\bullet}_{-1,(\vec{a},s_0),\mathbf{top}}$ means $(\boldsymbol{Cond})^{\bullet,\bullet,*}_{0,(\vec{a},s_0),\mathbf{top}}$. Then we have*

1. *When s_0 is an integer, there exists an even integer i_0 in $-2 \leq i_0 \leq n+1$ such that*
$$\begin{cases} \overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathbf{top}}} & \text{for all odd } i \text{ in } -1 \leq i < i_0 \\ (\boldsymbol{Cond})^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathbf{top}} & \text{for all odd } i \text{ in } n \geq i > i_0 \end{cases} \tag{89}$$
2. *When s_0 is a strict half-integer, there exists an odd integer i_1 in $-2 \leq i_1 \leq n+1$ such that*
$$\begin{cases} \overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathbf{top}}} & \text{for all even } i \text{ in } -1 \leq i < i_1 \\ (\boldsymbol{Cond})^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathbf{top}} & \text{for all even } i \text{ in } n \geq i > i_1 \end{cases} \tag{90}$$

Proof. We can prove this by induction on i. □

Proposition 4.8. *Let $\vec{a} \in \mathbb{C}^{n+1}$. We suppose that i is even and s_0 is a strict half-integer not larger than -1 or that i is odd and s_0 is an integer not larger than -1. We denote by $t_i(\vec{a}, s_0)$ the top order of $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ at $s = s_0$.*

1. *Let s_0 be an integer. We have*
$$\overline{(\boldsymbol{Cond})^{\bullet,\bullet,*}_{0,(\vec{a},s_0),\mathbf{top}}} \text{ is equivalent to } \langle \boldsymbol{d}^{(1)}[s_0], \vec{a} \rangle \neq 0, \tag{91}$$
and equivalently,
$$(\boldsymbol{Cond})^{\bullet,\bullet,*}_{0,(\vec{a},s_0),\mathbf{top}} \text{ is equivalent to } \langle \boldsymbol{d}^{(1)}[s_0], \vec{a} \rangle = 0. \tag{92}$$
If $\langle \boldsymbol{d}^{(1)}[s_0], \vec{a} \rangle \neq 0$, then $t_1(\vec{a}, s_0) = 1$ and
$$\langle \boldsymbol{d}^{(1)}[s_0], \vec{a} \rangle // \boldsymbol{c}^{\bullet,k}_{1,(\vec{a},s_0),-1}, \tag{93}$$
for $k = 0, 1$.
2. *Let s_0 be a half-integer and let i be an integer in $0 \leq i \leq n-2$. Then*
$$\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathbf{top}}} \text{ is equivalent to } \langle \boldsymbol{d}^{(i+2)}[s_0], \vec{a} \rangle \neq 0, \tag{94}$$
and equivalently,
$$(\boldsymbol{Cond})^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathbf{top}} \text{ is equivalent to } \langle \boldsymbol{d}^{(i+2)}[s_0], \vec{a} \rangle = 0. \tag{95}$$
If $\langle \boldsymbol{d}^{(i+2)}[s_0], \vec{a} \rangle \neq 0$, then
$$\langle \boldsymbol{d}^{(i+2)}[s_0], \vec{a} \rangle // \boldsymbol{c}^{\bullet,k}_{i+2,(\vec{a},s_0),-t_{i+2}(\vec{a},s_0)}, \tag{96}$$
for all k in $0 \leq k \leq i+2$.

Remark 4.2. 1. In Proposition 4.8-2, when $\langle \boldsymbol{d}^{(i+2)}[s_0], \vec{a} \rangle \neq 0$,
$$t_{i+2}(\vec{a}, s_0) = \begin{cases} t_i(\vec{a}, s_0) + 1 & \text{if } s_0 \leq -\frac{i+2}{2} \\ t_i(\vec{a}, s_0) & \text{if } s_0 > -\frac{i+2}{2} \end{cases} \tag{97}$$
by Proposition 4.5-2-(b).

2. If $\langle \boldsymbol{d}^{(i+2)}[s_0], \vec{a}\rangle = 0$, and $\langle \boldsymbol{d}^{(i)}[s_0], \vec{a}\rangle \neq 0$, then $t_{i+2}(\vec{a}, s_0) = t_i(\vec{a}, s_0)$ and

$$\boldsymbol{c}^{\bullet,k}_{i+2,(\vec{a},s_0),-t_{i+2}(\vec{a},s_0)} = \begin{cases} c_1 \times (\underbrace{1,0,1,0\ldots}_{n-i-1}) + c_2 \times (\underbrace{0,1,0,1\ldots}_{n-i-1}) & \text{if } i \text{ is odd,} \\ c_1 \times (\underbrace{1,0,-1,0,\ldots}_{n-i-1}) + c_2 \times (\underbrace{0,1,0,-1,\ldots}_{n-i-1}) & \text{if } i \text{ is even,} \end{cases} \tag{98}$$

for all k in $0 \le k \le i+2$. Here, c_1 and c_2 are constants.

Proof. 1. First, we shall prove Proposition 4.8-1. Since s_0 is an integer, (91) follows from the computation each of them by (72) and (15). Indeed, note that $c^{j,0}_{0,(\vec{a},s_0),0} = a_j$ from the case of $q = 0$ in (72). Then $(\boldsymbol{Cond})^{\bullet,\bullet,*}_{0,(\vec{a},s_0),0}$ means

$$a_j + (-1)^{s_0+1} a_{j+1} = 0, \qquad \text{for all } j \text{ in } 0 \le j \le n. \tag{99}$$

From (14), we have $\langle d^{(0)}_j[s_0], \vec{a}\rangle = j$. By (15), $\langle \boldsymbol{d}^{(1)}[s_0], \vec{a}\rangle = 0$ means (99). Thus we have (92), which also proves (91). This is the first assertion.

If $\langle \boldsymbol{d}^{(1)}[s_0], \vec{a}\rangle \neq 0$, then $\overline{(\boldsymbol{Cond})^{\bullet,\bullet,*}_{0,(\vec{a},s_0),0}}$ by (91). Therefore, (93) is implied from the calculation of $\boldsymbol{c}^{\bullet,k}_{1,(\vec{a},s_0),-1}$ by (60) near $s = s_0$. It is described in (75). Then, at least one matrix $\begin{bmatrix} c^{j,1}_1(\vec{a}, s) \\ c^{j,0}_1(\vec{a}, s) \end{bmatrix}$ with j in $0 \le j \le n$ has a pole of order 1. Then, we have $t_1(\vec{a}, s_0) = 1$. By taking the residues of (75), we have

$$\begin{bmatrix} c^{j,1}_{1,(\vec{a},s_0),-1} \\ c^{j,0}_{1,(\vec{a},s_0),-1} \end{bmatrix} = a \times \begin{bmatrix} (-1)^{s_0+1} & 1 \\ 1 & (-1)^{s_0+1} \end{bmatrix} \times \begin{bmatrix} c^{j+1,1}_{0,(\vec{a},s_0),0} \\ c^{j,0}_{0,(\vec{a},s_0),0} \end{bmatrix}, \tag{100}$$

with a constant a. Then we have

$$\begin{aligned} c^{j,1}_{1,(\vec{a},s_0),-1} &= a \times (c^{j,0}_{0,(\vec{a},s_0),0} + (-1)^{s_0+1} c^{j+1,1}_{0,(\vec{a},s_0),0}) \\ &= a \times (a_j + (-1)^{s_0+1} a_{j+1}) \\ c^{j,0}_{1,(\vec{a},s_0),-1} &= a \times (c^{j,0}_{0,(\vec{a},s_0),0} + (-1)^{s_0+1} c^{j+1,1}_{0,(\vec{a},s_0),0}) \\ &= a \times (a_j + (-1)^{s_0+1} a_{j+1}) \end{aligned} \tag{101}$$

for $0 \le j \le n$. On the other hand, from (15), we have

$$\begin{aligned} \langle d^{(1)}_j[s_0], \vec{a}\rangle &= \langle d^{(0)}_j[s_0], \vec{a}\rangle + (-1)^{s_0+1}\langle d^{(0)}_{j+1}[s_0], \vec{a}\rangle \\ &= (a_j + (-1)^{s_0+1} a_{j+1}) \end{aligned} \tag{102}$$

for $0 \leq j \leq n$. By (102) and (101), we obtain (93). This is the second assertion.

Thus,we have completed the proof of Proposition 4.8-1.

2. We shall prove Proposition 4.8-2 by induction on i.

First we shall prove Lemma 4.9 and Lemma 4.10 as initial conditions of the induction for the cases $i = 0$ and $i = 1$, respectively.

Lemma 4.9. *Let s_0 be a strict half-integer ≤ -1.*

$$\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{0,(\vec{a},s_0),\mathbf{top}}} \text{ is equivalent to } \langle \boldsymbol{d}^{(2)}[s_0], \vec{a}\rangle \neq 0, \tag{103}$$

and equivalently,

$$(\boldsymbol{Cond})^{\bullet,\bullet}_{0,(\vec{a},s_0),\mathbf{top}} \text{ is equivalent to } \langle \boldsymbol{d}^{(2)}[s_0], \vec{a}\rangle = 0. \tag{104}$$

If $\langle \boldsymbol{d}^{(2)}[s_0], \vec{a}\rangle \neq 0$, then $t_2(\vec{a}, s_0) = t_0(\vec{a}, s_0) + 1 = 1$ and

$$\langle \boldsymbol{d}^{(2)}[s_0], \vec{a}\rangle // \boldsymbol{c}^{\bullet,k}_{2,(\vec{a},s_0),-t_2(\vec{a},s_0)}, \tag{105}$$

for all k in $0 \leq k \leq 2$.

Proof. Since $\boldsymbol{c}^{\bullet,\bullet}_0(\vec{a}, s) = \vec{a}$, it does not depend on $s \in \mathbb{C}$. Then, its top order is 0. Hence, $(\boldsymbol{Cond})^{\bullet,\bullet}_{0,(\vec{a},s_0),\mathbf{top}}$ means $(\boldsymbol{Cond})^{\bullet,\bullet}_{0,(\vec{a},s_0),0}$. Then, by (71), we have

$$c^{j+2,0}_{0,(\vec{a},s_0),0} + c^{j,0}_{0,(\vec{a},s_0),0} = 0$$

for all j in $0 \leq j \leq n-2$. Since $c^{j,0}_0(\vec{a}, s) = c^{j,0}_{0,(\vec{a},s_0),0} = a_j = \langle d^{(0)}_j[s_0], \vec{a}\rangle$, we have

$$a_{j+2} + a_j = \langle d^{(2)}_j[s_0], \vec{a}\rangle = 0$$

for all j in $0 \leq j \leq n-2$, which means $\langle \boldsymbol{d}^{(2)}[s_0], \vec{a}\rangle = 0$. Thus, we have obtained (104).

Next, we suppose $\langle \boldsymbol{d}^{(2)}[s_0], \vec{a}\rangle \neq 0$. Using (61), we can compute $\boldsymbol{c}^{\bullet,\bullet}_{2,(\vec{a},s_0),-t_2(\vec{a},s_0)}$ from $\boldsymbol{c}^{\bullet,\bullet}_{0,(\vec{a},s_0),0}$. Namely, taking the top terms of the Laurent expansion of the equation (61), we have

$$\begin{bmatrix} c^{j,2}_{2,(\vec{a},s_0),-t_2} \\ c^{j,1}_{2,(\vec{a},s_0),-t_2} \\ c^{j,0}_{2,(\vec{a},s_0),-t_2} \end{bmatrix} = \begin{bmatrix} b_2 & 0 & b_2 \\ b_1 & 0 & b_1 \\ b_0 & 0 & b_0 \end{bmatrix} \times \begin{bmatrix} c^{j+2,0}_{0,(\vec{a},s_0),0} \\ c^{j+1,0}_{0,(\vec{a},s_0),0} \\ c^{j,0}_{0,(\vec{a},s_0),0} \end{bmatrix} \tag{106}$$

with $b_2 b_1 b_0 \neq 0$.

Then, we have

$$c^{j,k}_{2,(\vec{a},s_0),-t_2} = b_k(a_{j+2} + a_j)$$

for all j and k. From the definition,

$$\langle d^{(2)}_j[s_0], \vec{a}\rangle = a_{j+2} + a_j$$

for all j. Thus we have

$$\langle \boldsymbol{d}^{(2)}[s_0], \vec{a}\rangle // \boldsymbol{c}^{\bullet,k}_{2,(\vec{a},s_0),-t_2(\vec{a},s_0)},$$

for all k in $0 \leq k \leq 2$. □

Lemma 4.10. *Let s_0 be an integer ≤ -1.*

$$\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{1,(\vec{a},s_0),\mathbf{top}}} \text{ is equivalent to } \langle \boldsymbol{d}^{(3)}[s_0], \vec{a}\rangle \neq 0, \tag{107}$$

and equivalently,

$$(\boldsymbol{Cond})^{\bullet,\bullet}_{1,(\vec{a},s_0),\mathbf{top}} \text{ is equivalent to } \langle \boldsymbol{d}^{(3)}[s_0], \vec{a}\rangle = 0. \tag{108}$$

If $\langle \boldsymbol{d}^{(3)}[s_0], \vec{a}\rangle \neq 0$, then $t_3(\vec{a}, s_0) = t_1(\vec{a}, s_0) + 1$ and

$$\langle \boldsymbol{d}^{(3)}[s_0], \vec{a}\rangle // \boldsymbol{c}^{\bullet,k}_{3,(\vec{a},s_0),-t_3(\vec{a},s_0)}, \tag{109}$$

for all k in $0 \leq k \leq 3$.

Proof. By Proposition 4.6-1-(a), $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{1,(\vec{a},s_0),\mathbf{top}}}$ implies $\overline{(\boldsymbol{Cond})^{\bullet,\bullet,*}_{0,(\vec{a},s_0),\mathbf{top}}}$.

Using (61), we can compute $\boldsymbol{c}^{\bullet,\bullet}_{3,(\vec{a},s_0),-t_3(\vec{a},s_0)}$ from $\boldsymbol{c}^{\bullet,\bullet}_{1,(\vec{a},s_0),-t_1(\vec{a},s_0)}$. Namely, taking the top terms of the Laurent expansion of the equation (61), we have

$$\begin{bmatrix} c^{j,k+2}_{3,(\vec{a},s_0),-t_3} \\ c^{j,k+1}_{3,(\vec{a},s_0),-t_3} \\ c^{j,k}_{3,(\vec{a},s_0),-t_3} \end{bmatrix} = \begin{bmatrix} a & 0 & -a \\ b & 0 & -b \\ c & 0 & -c \end{bmatrix} \times \begin{bmatrix} c^{j+2,k}_{1,(\vec{a},s_0),-t_1} \\ c^{j+1,k}_{1,(\vec{a},s_0),-t_1} \\ c^{j,k}_{1,(\vec{a},s_0),-t_1} \end{bmatrix} \tag{110}$$

with $abc \neq 0$. Thus

$$c^{j,k+2}_{3,(\vec{a},s_0),-t_3} = a \times (-c^{j+2,k}_{1,(\vec{a},s_0),-t_1} + c^{j,k}_{1,(\vec{a},s_0),-t_1}) \tag{111}$$

for all k. On the other hand, by (18), we have

$$\langle d^{(3)}_j[s_0], \vec{a}\rangle = -\langle d^{(1)}_{j+2}[s_0], \vec{a}\rangle + \langle d^{(1)}_j[s_0], \vec{a}\rangle \tag{112}$$

Therefore, we have

$$\langle \boldsymbol{d}^{(3)}[s_0], \vec{a}\rangle // \boldsymbol{c}^{\bullet,k}_{3,(\vec{a},s_0),-t_3(\vec{a},s_0)}, \tag{113}$$

for all k by Proposition 4.8-1. Since $\boldsymbol{c}^{\bullet,\bullet}_{3,(\vec{a},s_0),-t_3(\vec{a},s_0)} \neq 0$ we have $\langle \boldsymbol{d}^{(3)}[s_0], \vec{a}\rangle \neq 0$.

Conversely, we suppose that $\langle \boldsymbol{d}^{(3)}[s_0], \vec{a}\rangle \neq 0$. Then, from Proposition 2.1,we have $\langle \boldsymbol{d}^{(1)}[s_0], \vec{a}\rangle \neq 0$. Then, from Proposition 4.8-1, we have $\overline{(\boldsymbol{Cond})^{\bullet,\bullet,*}_{0,(\vec{a},s_0),\mathbf{top}}}$ and

$$\langle \boldsymbol{d}^{(1)}[s_0], \vec{a}\rangle // \boldsymbol{c}^{\bullet,k}_{1,(\vec{a},s_0),-1}, \tag{114}$$

for all k in $0 \leq k \leq 1$.

Suppose that $(\boldsymbol{Cond})^{\bullet,\bullet}_{1,(\vec{a},s_0),\mathbf{top}}$. Then, from the definition,

$$-c^{j+2,k}_{1,(\vec{a},s_0),-t_1} + c^{j,k}_{1,(\vec{a},s_0),-t_1} = 0$$

for all j and k. Then, by (112) and (114), we obtain $\langle \boldsymbol{d}^{(3)}[s_0], \vec{a}\rangle = 0$. This is a contradiction. Therefore, we have $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{1,(\vec{a},s_0),\mathbf{top}}}$. $\square$

Next, we assume the following conditions with a fixed integer $l \geq 2$. These are the assumptions of the induction with $i = l-2$. We suppose either that l is even and s_0 is a strict half-integer ≤ -1 or that l is odd and s_0 is an integer ≤ -1.

(a) We have

$$\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{l-2,(\vec{a},s_0),\mathbf{top}}} \text{ is equivalent to } \langle \boldsymbol{d}^{(l)}[s_0], \vec{a}\rangle \neq 0, \tag{115}$$

(b) If $\langle \boldsymbol{d}^{(l)}[s_0], \vec{a}\rangle \neq 0$, then

$$\langle \boldsymbol{d}^{(l)}[s_0], \vec{a}\rangle // \boldsymbol{c}^{\bullet,k}_{l,(\vec{a},s_0),-t_l(\vec{a},s_0)}, \tag{116}$$

for all k in $0 \leq k \leq l$.

We shall prove Lemma 4.11 and Lemma 4.12 for the proof of Proposition 4.8-2 by induction on i.

Lemma 4.11. *We suppose* (115) *and* (116). *Then* $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{l,(\vec{a},s_0),\mathbf{top}}}$ *implies* $\langle \boldsymbol{d}^{(l+2)}[s_0], \vec{a}\rangle \neq 0$ *and*

$$\langle \boldsymbol{d}^{(l+2)}[s_0], \vec{a}\rangle // \boldsymbol{c}^{\bullet,k}_{l+2,(\vec{a},s_0),-t_{l+2}(\vec{a},s_0)}, \tag{117}$$

for all k in $0 \leq k \leq l+2$.

Proof. By Proposition 4.6-2-(a), $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{l,(\vec{a},s_0),\mathbf{top}}}$ implies $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{l-2,(\vec{a},s_0),\mathbf{top}}}$.

Using (61), we can compute $\boldsymbol{c}^{\bullet,\bullet}_{l+2,(\vec{a},s_0),-t_{l+2}(\vec{a},s_0)}$ from $\boldsymbol{c}^{\bullet,\bullet}_{l,(\vec{a},s_0),-t_l(\vec{a},s_0)}$. Namely, taking the top terms of the Laurent expansion of the equation (61), we have

$$\begin{bmatrix} c^{j,k+2}_{l+2,(\vec{a},s_0),-t_{l+2}} \\ c^{j,k+1}_{l+2,(\vec{a},s_0),-t_{l+2}} \\ c^{j,k}_{l+2,(\vec{a},s_0),-t_{l+2}} \end{bmatrix} = A \times \begin{bmatrix} c^{j+2,k}_{l,(\vec{a},s_0),-t_l} \\ c^{j+1,k}_{l,(\vec{a},s_0),-t_l} \\ c^{j,k}_{l,(\vec{a},s_0),-t_l} \end{bmatrix} \tag{118}$$

with a 3×3 constant matrix A. If l is odd and s_0 is an integer, then A has the following form:

$$A = \begin{bmatrix} a & 0 & -a \\ b & 0 & -b \\ c & 0 & -c \end{bmatrix} \tag{119}$$

with $abc \neq 0$, and if l is even and s_0 is a strict half-integer, then A has the following form:

$$A = \begin{bmatrix} a & 0 & a \\ b & 0 & b \\ c & 0 & c \end{bmatrix} \tag{120}$$

with $abc \neq 0$. Thus if l is odd, then

$$c^{j,k+2}_{l+2,(\vec{a},s_0),-t_{l+2}} = a \times (-c^{j+2,k}_{l,(\vec{a},s_0),-t_l} + c^{j,k}_{l,(\vec{a},s_0),-t_l}) \tag{121}$$

and if l is even, then

$$c^{j,k+2}_{l+2,(\vec{a},s_0),-t_{l+2}} = a \times (c^{j+2,k}_{l,(\vec{a},s_0),-t_l} + c^{j,k}_{l,(\vec{a},s_0),-t_l}) \tag{122}$$

for all k. On the other hand, by (18) and (19), we have

$$\langle d^{(l+2)}_j[s_0], \vec{a}\rangle = -\langle d^{(l)}_{j+2}[s_0], \vec{a}\rangle + \langle d^{(l)}_j[s_0], \vec{a}\rangle \tag{123}$$

if l is odd, and

$$\langle d^{(l+2)}_j[s_0], \vec{a}\rangle = \langle d^{(l)}_{j+2}[s_0], \vec{a}\rangle + \langle d^{(l)}_j[s_0], \vec{a}\rangle \tag{124}$$

if l is even. Therefore, whether l is odd or even, we have

$$\langle \boldsymbol{d}^{(l+2)}[s_0], \vec{a}\rangle // \boldsymbol{c}^{\bullet,k}_{l+2,(\vec{a},s_0),-t_{l+2}(\vec{a},s_0)}, \tag{125}$$

for all k by (116). Since $\boldsymbol{c}^{\bullet,\bullet}_{l+2,(\vec{a},s_0),-t_{l+2}(\vec{a},s_0)} \neq 0$ we have $\langle \boldsymbol{d}^{(l+2)}[s_0], \vec{a}\rangle \neq 0$. □

Lemma 4.12. *We suppose* (115) *and* (116). *Then* $\langle \boldsymbol{d}^{(l+2)}[s_0], \vec{a}\rangle \neq 0$ *implies* $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{l,(\vec{a},s_0),\mathbf{top}}}$.

Proof. From $\langle \boldsymbol{d}^{(l+2)}[s_0], \vec{a}\rangle \neq 0$ and Proposition 2.1,we have $\langle \boldsymbol{d}^{(l)}[s_0], \vec{a}\rangle \neq 0$. Then, from the assumption of the induction (115) and (116), we have $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{l-2,(\vec{a},s_0),\mathbf{top}}}$ and

$$\langle \boldsymbol{d}^{(l)}[s_0], \vec{a}\rangle // \boldsymbol{c}^{\bullet,k}_{l,(\vec{a},s_0),-t_l(\vec{a},s_0)}, \tag{126}$$

for all k in $0 \leq k \leq l$.

Suppose that $(\boldsymbol{Cond})^{\bullet,\bullet}_{l,(\vec{a},s_0),\mathbf{top}}$. Then, from the definition,

$$-c^{j+2,k}_{l,(\vec{a},s_0),-t_l} + c^{j,k}_{l,(\vec{a},s_0),-t_l} = 0$$

if l is odd, and

$$c^{j+2,k}_{l,(\vec{a},s_0),-t_l} + c^{j,k}_{l,(\vec{a},s_0),-t_l} = 0$$

if l is even, for all j and k. Then, by (124), (123) and (126), we obtain $\langle \boldsymbol{d}^{(l+2)}[s_0], \vec{a}\rangle = 0$. This is a contradiction. Therefore, we have $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{l,(\vec{a},s_0),\mathbf{top}}}$. □

Thus, by Lemma 4.11 and Lemma 4.12, we obtain the following result under the conditions (115) and (116). These are the results of the induction with $i = l$.

(a) We have

$$\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{l,(\vec{a},s_0),\mathbf{top}}} \text{ is equivalent to } \langle \boldsymbol{d}^{(l+2)}[s_0], \vec{a}\rangle \neq 0, \tag{127}$$

(b) If $\langle d^{(l+2)}[s_0], \vec{a}\rangle \neq 0$, then

$$\langle d^{(l+2)}[s_0], \vec{a}\rangle // c^{\bullet,k}_{l+2,(\vec{a},s_0),-t_{l+2}(\vec{a},s_0)}, \tag{128}$$

for all k in $0 \leq k \leq l+2$.

Therefore, with Lemma 4.9 and Lemma 4.10, we have completed the proof of Proposition 4.8-2 by induction on i.

□

5. Proofs of the main theorems.

In this section we shall prove Theorem 2.3 and Theorem 2.4. We have reduced the problems of "orders of poles" and of "supports of Laurent coefficients" of the hyperfunction $P^{[\vec{a},s]}(x)$ to the calculation of those of coefficients on the Lagrangian components of the microfunction $\mathbf{sp}(P^{[\vec{a},s]}(x))$ by Proposition 3.6,Corollary 3.7 and Proposition 3.8. We shall determine the orders of the coefficients applying the relations obtained in Proposition 4.1.

5.1. Some preliminary propositions.

Proposition 5.1. *Let s_0 be a half-integer in $s_0 < -\frac{n+1}{2}$.*

1. *Suppose that s_0 is a strict half-integer.*
   (a) *If the condition $(\boldsymbol{Cond})^{\bullet,\bullet}_{0,(\vec{a},s_0),0}$ is satisfied,then all the coefficient matrices $c^{\bullet,\bullet}_i(\vec{a},s)$ are holomorphic at $s = s_0$.*
   (b) *If there exists an even integer i_0 in $2 \leq i_0 \leq n$ such that the conditions $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{i_0-2,(\vec{a},s_0),\mathbf{top}}}$ and $(\boldsymbol{Cond})^{\bullet,\bullet}_{i_0,(\vec{a},s_0),\mathbf{top}}$ are satisfied, then*
      (i) *the order of pole at $s = s_0$ of $c^{\bullet,\bullet}_i(\vec{a},s)$ is $\lfloor \frac{i}{2} \rfloor$ if $i < i_0$.*
      (ii) *the order of pole at $s = s_0$ of $c^{\bullet,\bullet}_i(\vec{a},s)$ is $\frac{i_0}{2}$ if $i = i_0$.*
      (iii) *the order of pole at $s = s_0$ of $c^{\bullet,\bullet}_i(\vec{a},s)$ is not larger than $\frac{i_0}{2}$ if $i > i_0$.*
   (c) *If the condition $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{i_0,(\vec{a},s_0),\mathbf{top}}}$ is satisfied for the largest even integer i_0 in $0 \leq i_0 \leq n$, then the order of pole at $s = s_0$ of $c^{\bullet,\bullet}_i(\vec{a},s)$ is $\lfloor \frac{i}{2} \rfloor$.*
2. *Suppose that s_0 is an integer.*
   (a) *If the condition $(\boldsymbol{Cond})^{\bullet,\bullet,*}_{0,(\vec{a},s_0),0}$ is satisfied,then all the coefficient matrices $c^{\bullet,\bullet}_i(\vec{a},s)$ are holomorphic at $s = s_0$.*
   (b) *If the conditions $\overline{(\boldsymbol{Cond})^{\bullet,\bullet,*}_{0,(\vec{a},s_0),0}}$ and $(\boldsymbol{Cond})^{\bullet,\bullet}_{1,(\vec{a},s_0),\mathbf{top}}$ are satisfied, then*
      (i) *the order of pole at $s = s_0$ of $c^{\bullet,\bullet}_1(\vec{a},s)$ is 1 and $c^{\bullet,\bullet}_0(\vec{a},s)$ is holomorphic at $s = s_0$.*
      (ii) *the order of pole at $s = s_0$ of $c^{\bullet,\bullet}_i(\vec{a},s)$ is not larger than 1 if $i > 1$.*
   (c) *If there exists an odd integer i_0 in $2 \leq i_0 \leq n$ such that the conditions $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{i_0-2,(\vec{a},s_0),\mathbf{top}}}$ and $(\boldsymbol{Cond})^{\bullet,\bullet}_{i_0,(\vec{a},s_0),\mathbf{top}}$ are satisfied, then*

(i) *the order of pole at* $s = s_0$ *of* $c_i^{\bullet,\bullet}(\vec{a}, s)$ *is* $\lfloor \frac{i+1}{2} \rfloor$ *if* $i < i_0$.
(ii) *the order of pole at* $s = s_0$ *of* $c_i^{\bullet,\bullet}(\vec{a}, s)$ *is* $\frac{i_0+1}{2}$ *if* $i = i_0$.
(iii) *the order of pole at* $s = s_0$ *of* $c_i^{\bullet,\bullet}(\vec{a}, s)$ *is not larger than* $\frac{i_0+1}{2}$ *if* $i > i_0$.

(d) *If the condition* $\overline{(\boldsymbol{Cond})_{i_0,(\vec{a},s_0),\mathbf{top}}^{\bullet,\bullet}}$ *is satisfied for the largest odd integer* i_0 *in* $0 \le i_0 \le n$, *then the order of pole at* $s = s_0$ *of* $c_i^{\bullet,\bullet}(\vec{a}, s)$ *is* $\lfloor \frac{i+1}{2} \rfloor$ *for all* $0 \le i \le n$.

Proof. Note that s_0 is a half-integer in $s_0 < -\frac{n+1}{2}$.

1. We first suppose that s_0 is a strict half-integer. Then, by Corollary 4.7-2 and Corollary 4.6-2-(a), we have
   (a) If the condition $(\boldsymbol{Cond})_{0,(\vec{a},s_0),0}^{\bullet,\bullet}$ is satisfied,then $(\boldsymbol{Cond})_{i,(\vec{a},s_0),\mathbf{top}}^{\bullet,\bullet}$ for all even i in $0 \le i \le n$.
   (b) If there exists an even integer i_0 in $2 \le i_0 \le n$ such that the conditions $\overline{(\boldsymbol{Cond})_{i_0-2,(\vec{a},s_0),\mathbf{top}}^{\bullet,\bullet}}$ and $(\boldsymbol{Cond})_{i_0,(\vec{a},s_0),\mathbf{top}}^{\bullet,\bullet}$ are satisfied, then $\overline{(\boldsymbol{Cond})_{i,(\vec{a},s_0),\mathbf{top}}^{\bullet,\bullet}}$ for all even i in $0 \le i < i_0$, and $(\boldsymbol{Cond})_{i,(\vec{a},s_0),\mathbf{top}}^{\bullet,\bullet}$ for all even i in $n \ge i \ge i_0$.
   (c) If the condition $\overline{(\boldsymbol{Cond})_{i_0,(\vec{a},s_0),\mathbf{top}}^{\bullet,\bullet}}$ is satisfied for the largest even integer i_0 in $0 \le i_0 \le n$, then $\overline{(\boldsymbol{Cond})_{i,(\vec{a},s_0),\mathbf{top}}^{\bullet,\bullet}}$ for all even i in $0 \le i \le n$.

   From Corollary 4.6-2-(b), if $\overline{(\boldsymbol{Cond})_{i,(\vec{a},s_0),\mathbf{top}}^{\bullet,\bullet}}$ is satisfied for an even integer i, then the order of pole of $c_{i+2}^{\bullet,\bullet}(\vec{a}, s)$ at $s = s_0$ is larger by 1 than that of $c_i^{\bullet,\bullet}(\vec{a}, s)$ since $s_0 \le -\frac{n+1}{2}$. If $(\boldsymbol{Cond})_{i,(\vec{a},s_0),\mathbf{top}}^{\bullet,\bullet}$ is satisfied for an even integer i, then the order of pole of $c_{i+2}^{\bullet,\bullet}(\vec{a}, s)$ at $s = s_0$ is not larger than that of $c_i^{\bullet,\bullet}(\vec{a}, s)$. Since $c_0^{\bullet,\bullet}(\vec{a}, s)$ is a constant vector, we can compute the orders of poles of $c_i^{\bullet,\bullet}(\vec{a}, s)$ for all even integers i in $0 \le i \le n$. For odd integers i in $0 \le i \le n$, the order of pole of $c_i^{\bullet,\bullet}(\vec{a}, s)$ is that of $c_{i-1}^{\bullet,\bullet}(\vec{a}, s)$ by Proposition 4.3-1.

   In the following, we compute the orders of poles of $c_i^{\bullet,\bullet}(\vec{a}, s)$ in each case.
   (a) Suppose that the condition $(\boldsymbol{Cond})_{0,(\vec{a},s_0),0}^{\bullet,\bullet}$ is satisfied.
   If $\vec{a} = 0$, then all $c_i^{\bullet,\bullet}(\vec{a}, s)$ are zero. Hence all of them are holomorphic at $s = s_0$.
   Suppose that $\vec{a} \neq 0$. Since $c_0^{\bullet,\bullet}(\vec{a}, s)$ is a non-zero constant vector, its order of pole is 0 . Then, all the orders of poles of $c_i^{\bullet,\bullet}(\vec{a}, s)$ at $s = s_0$ for even i in $0 \le i \le n$ are 0. The orders of poles of $c_i^{\bullet,\bullet}(\vec{a}, s)$ at $s = s_0$ for odd i in $0 \le i \le n$ are also 0. Thus all $c_i^{\bullet,\bullet}(\vec{a}, s)$ are holomorphic at $s = s_0$.
   (b) Suppose that there exists an even integer i_0 in $2 \le i_0 \le n$ such that the conditions $\overline{(\boldsymbol{Cond})_{i_0-2,(\vec{a},s_0),\mathbf{top}}^{\bullet,\bullet}}$ and $(\boldsymbol{Cond})_{i_0,(\vec{a},s_0),\mathbf{top}}^{\bullet,\bullet}$ are satisfied. Then the order of pole of $c_i^{\bullet,\bullet}(\vec{a}, s)$ is $\frac{i}{2}$ for even i

in $0 \le i \le i_0$, and is not larger than $\frac{i_0}{2}$ for even i in $i > i_0$. For odd i, the order of pole of $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ coincides with that of $\boldsymbol{c}_{i-1}^{\bullet,\bullet}(\vec{a}, s)$ Then the order of pole of $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ is $\lfloor \frac{i}{2} \rfloor$ for all i in $0 \le i \le i_0$, and is not larger than $\frac{i_0}{2}$ for all i in $i > i_0$.

(c) Suppose that the condition $\overline{\boldsymbol{(Cond)}^{\bullet,\bullet}_{i_0,(\vec{a},s_0),\mathrm{top}}}$ is satisfied for the largest even integer i_0 in $0 \le i_0 \le n$. Then the order of pole of $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ is $\frac{i}{2}$ for even i in $0 \le i \le n$. For odd i, the order of pole of $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ coincides with that of $\boldsymbol{c}_{i-1}^{\bullet,\bullet}(\vec{a}, s)$. Then the order of pole of $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ is $\lfloor \frac{i}{2} \rfloor$ for all i in $0 \le i \le n$.

Thus we complete the proof of Proposition 5.1-1.

2. Secondly, we suppose that s_0 is an integer. Then, by Corollary 4.7-1 and Corollary 4.6-2-(a),(b), we have

(a) If the condition $\boldsymbol{(Cond)}^{\bullet,\bullet,*}_{0,(\vec{a},s_0),0}$ is satisfied,then $\boldsymbol{(Cond)}^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathrm{top}}$ for all odd i in $0 \le i \le n$.

(b) If the conditions $\overline{\boldsymbol{(Cond)}^{\bullet,\bullet,*}_{0,(\vec{a},s_0),0}}$ and $\boldsymbol{(Cond)}^{\bullet,\bullet}_{1,(\vec{a},s_0),\mathrm{top}}$ are satisfied, then $\overline{\boldsymbol{(Cond)}^{\bullet,\bullet,*}_{0,(\vec{a},s_0),0}}$ and $\boldsymbol{(Cond)}^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathrm{top}}$ for all odd i in $n \ge i \ge 1$.

(c) If there exists an odd integer i_0 in $2 \le i_0 \le n$ such that the conditions $\overline{\boldsymbol{(Cond)}^{\bullet,\bullet}_{i_0-2,(\vec{a},s_0),\mathrm{top}}}$ and $\boldsymbol{(Cond)}^{\bullet,\bullet}_{i_0,(\vec{a},s_0),\mathrm{top}}$ are satisfied, then $\overline{\boldsymbol{(Cond)}^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathrm{top}}}$ for all odd i in $0 \le i < i_0$, and $\boldsymbol{(Cond)}^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathrm{top}}$ for all odd i in $n \ge i \ge i_0$.

(d) If the condition $\overline{\boldsymbol{(Cond)}^{\bullet,\bullet}_{i_0,(\vec{a},s_0),\mathrm{top}}}$ is satisfied for the largest odd integer i_0 in $0 \le i_0 \le n$, then $\overline{\boldsymbol{(Cond)}^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathrm{top}}}$ for all odd i in $0 \le i \le n$.

From Corollary 4.6-2-(b), if $\overline{\boldsymbol{(Cond)}^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathrm{top}}}$ is satisfied for an odd integer i, then the order of pole of $\boldsymbol{c}_{i+2}^{\bullet,\bullet}(\vec{a}, s)$ at $s = s_0$ is larger by 1 than that of $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ since $s_0 \le -\frac{n+1}{2}$. If $\boldsymbol{(Cond)}^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathrm{top}}$ is satisfied for an odd integer i, then the order of pole of $\boldsymbol{c}_{i+2}^{\bullet,\bullet}(\vec{a}, s)$ at $s = s_0$ is not larger than that of $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$. Since $\boldsymbol{c}_0^{\bullet,\bullet}(\vec{a}, s)$ is a constant vector, we can compute the orders of poles of $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ for all odd integers i in $0 \le i \le n$. For even integers i in $0 \le i \le n$, the order of pole of $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ is that of $\boldsymbol{c}_{i-1}^{\bullet,\bullet}(\vec{a}, s)$ by Proposition 4.3-1.

In the following, we compute the orders of poles of $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ in each case.

(a) Suppose that the condition $\boldsymbol{(Cond)}^{\bullet,\bullet,*}_{0,(\vec{a},s_0),0}$ is satisfied.

If $\vec{a} = 0$, then all $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ are zero. Hence all of them are holomorphic at $s = s_0$.

Suppose that $\vec{a} \neq 0$. Since $\boldsymbol{c}_0^{\bullet,\bullet}(\vec{a}, s)$ is a non-zero constant vector, its order of pole is 0 . Then, all the orders of poles of $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ at $s = s_0$ for odd i in $0 \le i \le n$ are 0. The orders of poles of $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$

at $s = s_0$ for even i in $2 \leq i \leq n$ are also 0. Thus all $c_i^{\bullet,\bullet}(\vec{a}, s)$ are holomorphic at $s = s_0$.

(b) Suppose that the conditions $\overline{(\boldsymbol{Cond})_{0,(\vec{a},s_0),0}^{\bullet,\bullet,*}}$ and $(\boldsymbol{Cond})_{1,(\vec{a},s_0),\mathbf{top}}^{\bullet,\bullet}$ are satisfied.
Then the order of pole of $c_i^{\bullet,\bullet}(\vec{a}, s)$ is 0 for $i = 0$, is 1 for $i = 1$, and is not larger than 1 for odd i in $i > 1$. For even i, the order of pole of $c_i^{\bullet,\bullet}(\vec{a}, s)$ coincides with that of $c_{i-1}^{\bullet,\bullet}(\vec{a}, s)$. Then, the order of pole of $c_i^{\bullet,\bullet}(\vec{a}, s)$ is 0 for $i = 0$, is 1 for $i = 1$, and is not larger than 1 for all i in $i > 1$.

(c) Suppose that there exists an odd integer i_0 in $2 \leq i_0 \leq n$ such that the conditions $\overline{(\boldsymbol{Cond})_{i_0-2,(\vec{a},s_0),\mathbf{top}}^{\bullet,\bullet}}$ and $(\boldsymbol{Cond})_{i_0,(\vec{a},s_0),\mathbf{top}}^{\bullet,\bullet}$ are satisfied. Then the order of pole of $c_i^{\bullet,\bullet}(\vec{a}, s)$ is $\frac{i+1}{2}$ for odd i in $0 \leq i \leq i_0$, and is not larger than $\frac{i_0+1}{2}$ for odd i in $i > i_0$. For even i, the order of pole of $c_i^{\bullet,\bullet}(\vec{a}, s)$ coincides with that of $c_{i-1}^{\bullet,\bullet}(\vec{a}, s)$. Then the order of pole of $c_i^{\bullet,\bullet}(\vec{a}, s)$ is $\lfloor \frac{i+1}{2} \rfloor$ for all i in $0 \leq i \leq i_0$, and is not larger than $\frac{i_0+1}{2}$ for all i in $i > i_0$.

(d) Suppose that the condition $\overline{(\boldsymbol{Cond})_{i_0,(\vec{a},s_0),\mathbf{top}}^{\bullet,\bullet}}$ is satisfied for the largest odd integer i_0 in $0 \leq i_0 \leq n$. Then the order of pole of $c_i^{\bullet,\bullet}(\vec{a}, s)$ is $\frac{i+1}{2}$ for odd i in $0 \leq i \leq n$. For even i, the order of pole of $c_i^{\bullet,\bullet}(\vec{a}, s)$ coincides with that of $c_{i-1}^{\bullet,\bullet}(\vec{a}, s)$. Then the order of pole of $c_i^{\bullet,\bullet}(\vec{a}, s)$ is $\lfloor \frac{i+1}{2} \rfloor$ for all i in $0 \leq i \leq n$.

Thus we complete the proof of Proposition 5.1-2.

□

Proposition 5.2. *Let s_0 be a half-integer in $-\frac{n+1}{2} \leq s_0 \leq -1$.*

1. *Suppose that s_0 is a strict half-integer.*
   (a) *If the condition $(\boldsymbol{Cond})_{0,(\vec{a},s_0),0}^{\bullet,\bullet}$ is satisfied,then all the coefficient matrices $c_i^{\bullet,\bullet}(\vec{a}, s)$ are holomorphic at $s = s_0$.*
   (b) *If there exists an even integer i_0 in $2 \leq i_0 \leq -2s_0 - 3$ such that the conditions $\overline{(\boldsymbol{Cond})_{i_0-2,(\vec{a},s_0),\mathbf{top}}^{\bullet,\bullet}}$ and $(\boldsymbol{Cond})_{i_0,(\vec{a},s_0),\mathbf{top}}^{\bullet,\bullet}$ are satisfied, then*
      (i) *the order of pole at $s = s_0$ of $c_i^{\bullet,\bullet}(\vec{a}, s)$ is $\lfloor \frac{i}{2} \rfloor$ if $i < i_0$.*
      (ii) *the order of pole at $s = s_0$ of $c_i^{\bullet,\bullet}(\vec{a}, s)$ is $\frac{i_0}{2}$ if $i = i_0$.*
      (iii) *the order of pole at $s = s_0$ of $c_i^{\bullet,\bullet}(\vec{a}, s)$ is not larger than $\frac{i_0}{2}$ if $i > i_0$.*
   (c) *If there exists an even integer i_0 in $-2s_0 - 3 \leq i_0$ such that the condition $\overline{(\boldsymbol{Cond})_{i_0,(\vec{a},s_0),\mathbf{top}}^{\bullet,\bullet}}$ is satisfied, then*
      (i) *the order of pole at $s = s_0$ of $c_i^{\bullet,\bullet}(\vec{a}, s)$ is $\lfloor \frac{i}{2} \rfloor$ if $i < -2s_0 - 1$.*
      (ii) *the order of pole at $s = s_0$ of $c_i^{\bullet,\bullet}(\vec{a}, s)$ is $\frac{-2s_0-1}{2}$ if $i = -2s_0 - 1$.*
      (iii) *the order of pole at $s = s_0$ of $c_i^{\bullet,\bullet}(\vec{a}, s)$ is not larger than $\frac{-2s_0-1}{2}$ if $i > -2s_0 - 1$.*
2. *Suppose that s_0 is an integer.*

(a) *If the condition* $(\boldsymbol{Cond})^{\bullet,\bullet,*}_{0,(\vec{a},s_0),0}$ *is satisfied,then all the coefficient matrices* $\boldsymbol{c}^{\bullet,\bullet}_i(\vec{a},s)$ *are holomorphic at* $s = s_0$.

(b) *If the conditions* $\overline{(\boldsymbol{Cond})^{\bullet,\bullet,*}_{0,(\vec{a},s_0),0}}$ *and* $(\boldsymbol{Cond})^{\bullet,\bullet}_{1,(\vec{a},s_0),\mathbf{top}}$ *are satisfied, then*

 (i) *the order of pole at* $s = s_0$ *of* $\boldsymbol{c}^{\bullet,\bullet}_1(\vec{a},s)$ *is 1 and* $\boldsymbol{c}^{\bullet,\bullet}_0(\vec{a},s)$ *is holomorphic at* $s = s_0$.

 (ii) *the order of pole at* $s = s_0$ *of* $\boldsymbol{c}^{\bullet,\bullet}_i(\vec{a},s)$ *is not larger than 1 if* $i > 1$.

(c) *If there exists an odd integer* i_0 *in* $2 \le i_0 \le -2s_0 - 3$ *such that the conditions* $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{i_0-2,(\vec{a},s_0),\mathbf{top}}}$ *and* $(\boldsymbol{Cond})^{\bullet,\bullet}_{i_0,(\vec{a},s_0),\mathbf{top}}$ *are satisfied, then*

 (i) *the order of pole at* $s = s_0$ *of* $\boldsymbol{c}^{\bullet,\bullet}_i(\vec{a},s)$ *is* $\lfloor \frac{i+1}{2} \rfloor$ *if* $i < i_0$.

 (ii) *the order of pole at* $s = s_0$ *of* $\boldsymbol{c}^{\bullet,\bullet}_i(\vec{a},s)$ *is* $\frac{i_0+1}{2}$ *if* $i = i_0$.

 (iii) *the order of pole at* $s = s_0$ *of* $\boldsymbol{c}^{\bullet,\bullet}_i(\vec{a},s)$ *is not larger than* $\frac{i_0+1}{2}$ *if* $i > i_0$.

(d) *If there exists an odd integer* i_0 *in* $-2s_0 - 3 \le i_0$ *such that the condition* $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{i_0,(\vec{a},s_0),\mathbf{top}}}$ *is satisfied, then*

 (i) *the order of pole at* $s = s_0$ *of* $\boldsymbol{c}^{\bullet,\bullet}_i(\vec{a},s)$ *is* $\lfloor \frac{i+1}{2} \rfloor$ *if* $i < -2s_0 - 1$.

 (ii) *the order of pole at* $s = s_0$ *of* $\boldsymbol{c}^{\bullet,\bullet}_i(\vec{a},s)$ *is* $-s_0$ *if* $i = -2s_0 - 1$.

 (iii) *the order of pole at* $s = s_0$ *of* $\boldsymbol{c}^{\bullet,\bullet}_i(\vec{a},s)$ *is not larger than* $-s_0$ *if* $i > -2s_0 - 1$.

Proof. Note that s_0 be a half-integer in $-\frac{n+1}{2} \le s_0 \le -1$.

1. We first suppose that s_0 is a strict half-integer. Then, by Corollary 4.7-2 and Corollary 4.6-2-(a), we have

 (a) If the condition $(\boldsymbol{Cond})^{\bullet,\bullet}_{0,(\vec{a},s_0),0}$ is satisfied,then $(\boldsymbol{Cond})^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathbf{top}}$ for all even i in $0 \le i \le n$.

 (b) If there exists an even integer i_0 in $2 \le i_0 \le n$ such that the conditions $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{i_0-2,(\vec{a},s_0),\mathbf{top}}}$ and $(\boldsymbol{Cond})^{\bullet,\bullet}_{i_0,(\vec{a},s_0),\mathbf{top}}$ are satisfied, then $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathbf{top}}}$ for all even i in $0 \le i < i_0$, and $(\boldsymbol{Cond})^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathbf{top}}$ for all even i in $n \ge i \ge i_0$.

 (c) If the condition $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{i_0,(\vec{a},s_0),\mathbf{top}}}$ is satisfied for the largest even integer i_0 in $0 \le i_0 \le n$, then $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathbf{top}}}$ for all even i in $0 \le i \le n$.

 From Corollary 4.6-2-(b), if $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathbf{top}}}$ is satisfied and $s_0 \le -\frac{i+2}{2}$ for an even integer i, then the order of pole of $\boldsymbol{c}^{\bullet,\bullet}_{i+2}(\vec{a},s)$ at $s = s_0$ is larger by 1 than that of $\boldsymbol{c}^{\bullet,\bullet}_i(\vec{a},s)$ If $(\boldsymbol{Cond})^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathbf{top}}$ is satisfied or $s_0 > -\frac{i+2}{2}$ for an even integer i, then the order of pole of $\boldsymbol{c}^{\bullet,\bullet}_{i+2}(\vec{a},s)$ at $s = s_0$ is not larger than that of $\boldsymbol{c}^{\bullet,\bullet}_i(\vec{a},s)$. Since $\boldsymbol{c}^{\bullet,\bullet}_0(\vec{a},s)$ is a constant vector, we can compute the orders of poles of $\boldsymbol{c}^{\bullet,\bullet}_i(\vec{a},s)$ for all even integers i in $0 \leq i \leq n$. For odd integers i in $0 \leq i \leq n$, the order of pole of $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ is that of $\boldsymbol{c}_{i-1}^{\bullet,\bullet}(\vec{a}, s)$ by Proposition 4.3-1.

In the following, we compute the orders of poles of $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ in each case.

(a) Suppose that the condition $(\boldsymbol{Cond})^{\bullet,\bullet}_{0,(\vec{a},s_0),0}$ is satisfied.
If $\vec{a} = 0$, then all $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ are zero. Hence all of them are holomorphic at $s = s_0$.
Suppose that $\vec{a} \neq 0$. Since $\boldsymbol{c}_0^{\bullet,\bullet}(\vec{a}, s)$ is a non-zero constant vector, its order of pole is 0 . Then, all the orders of poles of $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ at $s = s_0$ for even i in $0 \leq i \leq n$ are 0. The orders of poles of $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ at $s = s_0$ for odd i in $0 \leq i \leq n$ are also 0. Thus all $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ are holomorphic at $s = s_0$.

(b) Suppose that there exists an even integer i_0 in $2 \leq i_0 \leq -2s_0 - 3$ such that the conditions $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{i_0-2,(\vec{a},s_0),\mathbf{top}}}$ and $(\boldsymbol{Cond})^{\bullet,\bullet}_{i_0,(\vec{a},s_0),\mathbf{top}}$ are satisfied. Then the order of pole of $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ is $\frac{i}{2}$ for even i in $0 \leq i \leq i_0$, and is not larger than $\frac{i_0}{2}$ for even i in $i > i_0$. For odd i, the order of pole of $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ coincides with that of $\boldsymbol{c}_{i-1}^{\bullet,\bullet}(\vec{a}, s)$ Then the order of pole of $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ is $\lfloor \frac{i}{2} \rfloor$ for all i in $0 \leq i \leq i_0$, and is not larger than $\frac{i_0}{2}$ for all i in $i > i_0$.

(c) Suppose that there exists an even integer i_0 in $i_0 \geq -2s_0 - 3$ such that the conditions $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{i_0-2,(\vec{a},s_0),\mathbf{top}}}$ is satisfied. Then the order of pole of $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ is $\frac{i}{2}$ for even i in $0 \leq i \leq -2s_0 - 1$, and is not larger than $\frac{i_0}{2}$ for even i in $i > -2s_0 - 1$. For odd i, the order of pole of $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ coincides with that of $\boldsymbol{c}_{i-1}^{\bullet,\bullet}(\vec{a}, s)$ Then the order of pole of $\boldsymbol{c}_i^{\bullet,\bullet}(\vec{a}, s)$ is $\lfloor \frac{i}{2} \rfloor$ for all i in $0 \leq i \leq -2s_0 - 1$, and is not larger than $\frac{-2s_0-1}{2}$ for all i in $i > -2s_0 - 1$.

Thus we complete the proof of Proposition 5.2-1.

2. Secondly, we suppose that s_0 is an integer. Then, by Corollary 4.7-1 and Corollary 4.6-2-(a),(b), we have
   (a) If the condition $(\boldsymbol{Cond})^{\bullet,\bullet,*}_{0,(\vec{a},s_0),0}$ is satisfied,then $(\boldsymbol{Cond})^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathbf{top}}$ for all odd i in $0 \leq i \leq n$.
   (b) If the conditions $\overline{(\boldsymbol{Cond})^{\bullet,\bullet,*}_{0,(\vec{a},s_0),0}}$ and $(\boldsymbol{Cond})^{\bullet,\bullet}_{1,(\vec{a},s_0),\mathbf{top}}$ are satisfied, then $\overline{(\boldsymbol{Cond})^{\bullet,\bullet,*}_{0,(\vec{a},s_0),0}}$ and $(\boldsymbol{Cond})^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathbf{top}}$ for all odd i in $n \geq i \geq 1$.
   (c) If there exists an odd integer i_0 in $2 \leq i_0 \leq n$ such that the conditions $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{i_0-2,(\vec{a},s_0),\mathbf{top}}}$ and $(\boldsymbol{Cond})^{\bullet,\bullet}_{i_0,(\vec{a},s_0),\mathbf{top}}$ are satisfied, then $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathbf{top}}}$ for all odd i in $0 \leq i < i_0$, and $(\boldsymbol{Cond})^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathbf{top}}$ for all odd i in $n \geq i \geq i_0$.
   (d) If the condition $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{i_0,(\vec{a},s_0),\mathbf{top}}}$ is satisfied for the largest odd integer i_0 in $0 \leq i_0 \leq n$, then $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathbf{top}}}$ for all odd i in $0 \leq i \leq n$.

From Corollary 4.6-2-(b), if $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathbf{top}}}$ is satisfied and $s_0 \leq \frac{i+2}{2}$ for an odd integer i, then the order of pole of $\boldsymbol{c}^{\bullet,\bullet}_{i+2}(\vec{a},s)$ at $s=s_0$ is larger by 1 than that of $\boldsymbol{c}^{\bullet,\bullet}_{i}(\vec{a},s)$. If $(\boldsymbol{Cond})^{\bullet,\bullet}_{i,(\vec{a},s_0),\mathbf{top}}$ is satisfied or $s_0 > \frac{i+2}{2}$ for an odd integer i, then the order of pole of $\boldsymbol{c}^{\bullet,\bullet}_{i+2}(\vec{a},s)$ at $s=s_0$ is not larger than that of $\boldsymbol{c}^{\bullet,\bullet}_{i}(\vec{a},s)$. Since $\boldsymbol{c}^{\bullet,\bullet}_{0}(\vec{a},s)$ is a constant vector, we can compute the orders of poles of $\boldsymbol{c}^{\bullet,\bullet}_{i}(\vec{a},s)$ for all odd integers i in $0 \leq i \leq n$. For even integers i in $0 \leq i \leq n$, the order of pole of $\boldsymbol{c}^{\bullet,\bullet}_{i}(\vec{a},s)$ is that of $\boldsymbol{c}^{\bullet,\bullet}_{i-1}(\vec{a},s)$ by Proposition 4.3-1.

In the following, we compute the orders of poles of $\boldsymbol{c}^{\bullet,\bullet}_{i}(\vec{a},s)$ in each case.

(a) Suppose that the condition $(\boldsymbol{Cond})^{\bullet,\bullet,*}_{0,(\vec{a},s_0),0}$ is satisfied.
If $\vec{a}=0$, then all $\boldsymbol{c}^{\bullet,\bullet}_{i}(\vec{a},s)$ are zero. Hence all of them are holomorphic at $s=s_0$.
Suppose that $\vec{a}\neq 0$. Since $\boldsymbol{c}^{\bullet,\bullet}_{0}(\vec{a},s)$ is a non-zero constant vector, its order of pole is 0 . Then, all the orders of poles of $\boldsymbol{c}^{\bullet,\bullet}_{i}(\vec{a},s)$ at $s=s_0$ for odd i in $0 \leq i \leq n$ are 0. The orders of poles of $\boldsymbol{c}^{\bullet,\bullet}_{i}(\vec{a},s)$ at $s=s_0$ for even i in $2 \leq i \leq n$ are also 0. Thus all $\boldsymbol{c}^{\bullet,\bullet}_{i}(\vec{a},s)$ are holomorphic at $s=s_0$.

(b) Suppose that the conditions $\overline{(\boldsymbol{Cond})^{\bullet,\bullet,*}_{0,(\vec{a},s_0),0}}$ and $(\boldsymbol{Cond})^{\bullet,\bullet}_{1,(\vec{a},s_0),\mathbf{top}}$ are satisfied.
Then the order of pole of $\boldsymbol{c}^{\bullet,\bullet}_{i}(\vec{a},s)$ is 0 for $i=0$, is 1 for $i=1$, and is not larger than 1 for odd i in $i>1$. For even i, the order of pole of $\boldsymbol{c}^{\bullet,\bullet}_{i}(\vec{a},s)$ coincides with that of $\boldsymbol{c}^{\bullet,\bullet}_{i-1}(\vec{a},s)$. Then, the order of pole of $\boldsymbol{c}^{\bullet,\bullet}_{i}(\vec{a},s)$ is 0 for $i=0$, is 1 for $i=1$, and is not larger than 1 for all i in $i>1$.

(c) Suppose that there exists an odd integer i_0 in $2 \leq i_0 \leq -2s_0-3$ such that the conditions $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{i_0-2,(\vec{a},s_0),\mathbf{top}}}$ and $(\boldsymbol{Cond})^{\bullet,\bullet}_{i_0,(\vec{a},s_0),\mathbf{top}}$ are satisfied. Then the order of pole of $\boldsymbol{c}^{\bullet,\bullet}_{i}(\vec{a},s)$ is $\frac{i+1}{2}$ for odd i in $0 \leq i \leq i_0$, and is not larger than $\frac{i_0+1}{2}$ for odd i in $i>i_0$. For even i, the order of pole of $\boldsymbol{c}^{\bullet,\bullet}_{i}(\vec{a},s)$ coincides with that of $\boldsymbol{c}^{\bullet,\bullet}_{i-1}(\vec{a},s)$. Then the order of pole of $\boldsymbol{c}^{\bullet,\bullet}_{i}(\vec{a},s)$ is $\lfloor\frac{i+1}{2}\rfloor$ for all i in $0 \leq i \leq i_0$, and is not larger than $\frac{i_0+1}{2}$ for all i in $i>i_0$.

(d) Suppose that there exists an odd integer i_0 in $i_0 \geq -2s_0-3$ such that the condition $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{i_0-2,(\vec{a},s_0),\mathbf{top}}}$ is satisfied. Then the order of pole of $\boldsymbol{c}^{\bullet,\bullet}_{i}(\vec{a},s)$ is $\frac{i+1}{2}$ for odd i in $0 \leq i \leq -2s_0-1$, and is not larger than $-s_0=\frac{(-2s_0-1)+1}{2}$ for odd i in $i>-2s_0-1$. For even i, the order of pole of $\boldsymbol{c}^{\bullet,\bullet}_{i}(\vec{a},s)$ coincides with that of $\boldsymbol{c}^{\bullet,\bullet}_{i-1}(\vec{a},s)$. Then the order of pole of $\boldsymbol{c}^{\bullet,\bullet}_{i}(\vec{a},s)$ is $\lfloor\frac{i+1}{2}\rfloor$ for all i in $0 \leq i \leq -2s_0-1$, and is not larger than $-s_0$ for all i in $i>-2s_0-1$.

Thus we complete the proof of Proposition 5.2-2.

$\square$

5.2. Proof of the theorem on the exact orders of the complex powers. In this section we shall give a proof of Theorem 2.3

1. First, we prove the Theorem 2.3-1.
   Suppose that $s_0 := -\frac{2m+1}{2}$ with $m = 1, 2, \ldots$. Then s_0 is a strict half-integer.
   (a) Consider the case that $1 \leq m \leq \frac{n}{2}$. Then, $-\frac{3}{2} \geq s_0 \geq -\frac{n+1}{2}$, and we can apply Proposition 5.2-1.
   If $\langle \boldsymbol{d}^{(2)}[s_0], \vec{a}\rangle = 0$, then $(\boldsymbol{Cond})^{\bullet,\bullet}_{0,(\vec{a},s_0),\mathbf{top}}$ is satisfied (Proposition 4.8-2). Then we have $(\boldsymbol{Cond})^{\bullet,\bullet}_{0,(\vec{a},s_0),0}$. By Proposition 5.2-1-(a), all the coefficients $\boldsymbol{c}^{\bullet,\bullet}_i(\vec{a}, s)$ for $0 \leq i \leq n$ are holomorphic at $s = s_0$. Then , by Corollary 3.7, $P^{[\vec{a},s]}(x)$ is holomorphic at $s = s_0$. The converses are also true.
   Let p be an integer in $1 \leq p < m$. If $\langle \boldsymbol{d}^{(2p+2)}[s_0], \vec{a}\rangle = 0$ and $\langle \boldsymbol{d}^{(2p)}[s_0], \vec{a}\rangle \neq 0$, then $(\boldsymbol{Cond})^{\bullet,\bullet}_{2p,(\vec{a},s_0),\mathbf{top}}$ and $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{2p-2,(\vec{a},s_0),\mathbf{top}}}$ (Proposition 4.8-2). Therefore, by Proposition 5.2-1-(b), the order of pole of $\boldsymbol{c}^{\bullet,\bullet}_i(\vec{a}, s)$ at $s = s_0$ is $\lfloor \frac{i}{2} \rfloor$ if $i < 2p$, is p if $i = 2p$, and is not larger than p if $i > 2p$. Then , by Corollary 3.7, the order of $P^{[\vec{a},s]}(x)$ at $s = s_0$ is p. The converses are also true.
   If $\langle \boldsymbol{d}^{(2m)}[s_0], \vec{a}\rangle \neq 0$, then $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{2m-2,(\vec{a},s_0),\mathbf{top}}}$ (Proposition 4.8-2). Therefore, by Proposition 5.2-1-(c), the order of pole of $\boldsymbol{c}^{\bullet,\bullet}_i(\vec{a}, s)$ at $s = s_0$ is $\lfloor \frac{i}{2} \rfloor$ if $i < 2m$, m if $i = 2m$, and not larger than m if $i > 2m$. Then , by Corollary 3.7, the order of $P^{[\vec{a},s]}(x)$ at $s = s_0$ is m. The converses are also true.
   (b) Consider the case that $m > \frac{n}{2}$. Then, $s_0 \leq -\frac{n+1}{2}$, and we can apply Proposition 5.1-1. Let $n' := \lfloor \frac{n}{2} \rfloor$.
   If $\langle \boldsymbol{d}^{(2)}[s_0], \vec{a}\rangle = 0$, then $(\boldsymbol{Cond})^{\bullet,\bullet}_{0,(\vec{a},s_0),\mathbf{top}}$ is satisfied (Proposition 4.8-2). Then we have $(\boldsymbol{Cond})^{\bullet,\bullet}_{0,(\vec{a},s_0),0}$. By Proposition 5.1-1-(a), all the coefficients $\boldsymbol{c}^{\bullet,\bullet}_i(\vec{a}, s)$ for $0 \leq i \leq n$ are holomorphic at $s = s_0$. Then , by Corollary 3.7, $P^{[\vec{a},s]}(x)$ is holomorphic at $s = s_0$. The converses are also true.
   Let p be an integer in $1 \leq p < n'$. If $\langle \boldsymbol{d}^{(2p+2)}[s_0], \vec{a}\rangle = 0$ and $\langle \boldsymbol{d}^{(2p)}[s_0], \vec{a}\rangle \neq 0$, then $(\boldsymbol{Cond})^{\bullet,\bullet}_{2p,(\vec{a},s_0),\mathbf{top}}$ and $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{2p-2,(\vec{a},s_0),\mathbf{top}}}$ (Proposition 4.8-2). Therefore, by Proposition 5.1-1-(b), the order of pole of $\boldsymbol{c}^{\bullet,\bullet}_i(\vec{a}, s)$ at $s = s_0$ is $\lfloor \frac{i}{2} \rfloor$ if $i < 2p$, is p if $i = 2p$, and is not larger than p if $i > 2p$, Then , by Corollary 3.7, the order of $P^{[\vec{a},s]}(x)$ at $s = s_0$ is p. The converses are also true.
   Suppose that n is odd (resp. even). If $\langle \boldsymbol{d}^{(n-1)}[s_0], \vec{a}\rangle \neq 0$, (resp. $\langle \boldsymbol{d}^{(n)}[s_0], \vec{a}\rangle \neq 0$,) then $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{n-3,(\vec{a},s_0),\mathbf{top}}}$ (resp. $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{n-2,(\vec{a},s_0),\mathbf{top}}}$) by Proposition 4.8-2. Therefore, by Proposition 5.1-1-(c), the order of pole of $\boldsymbol{c}^{\bullet,\bullet}_i(\vec{a}, s)$ at $s = s_0$ is $\lfloor \frac{i}{2} \rfloor$ for all i in $0 \leq i \leq n$. Then , by Corollary 3.7, the order of $P^{[\vec{a},s]}(x)$ at $s = s_0$ is $\lfloor \frac{n}{2} \rfloor = n'$,i.e.,

the largest order of the poles of $c_i^{\bullet,\bullet}(\vec{a},s)$ at $s=s_0$. The converses are also true.

2. Secondly, we prove the Theorem 2.3-2.
Suppose that $s_0 := -m$ with $m=1,2,\ldots$. Then s_0 is an integer.
   (a) Consider the case that $1 \le m \le \frac{n}{2}$. Then, $-1 \ge s_0 \ge -\frac{n}{2}$, and we can apply Proposition 5.2-2.
   If $\langle d^{(1)}[s_0],\vec{a}\rangle = 0$, then $(Cond)^{\bullet,\bullet,*}_{0,(\vec{a},s_0),\mathbf{top}}$ is satisfied (Proposition 4.8-1). Then we have $(Cond)^{\bullet,\bullet}_{0,(\vec{a},s_0),0}$. By Proposition 5.2-2-(a), all the coefficients $c_i^{\bullet,\bullet}(\vec{a},s)$ for $0 \le i \le n$ are holomorphic at $s=s_0$. Then, by Corollary 3.7, $P^{[\vec{a},s]}(x)$ is holomorphic at $s=s_0$. The converses are also true.
   If $\langle d^{(3)}[s_0],\vec{a}\rangle = 0$ and $\langle d^{(1)}[s_0],\vec{a}\rangle \ne 0$, then $(Cond)^{\bullet,\bullet}_{1,(\vec{a},s_0),\mathbf{top}}$ and $\overline{(Cond)^{\bullet,\bullet,*}_{0,(\vec{a},s_0),\mathbf{top}}}$ (Proposition 4.8-1). Therefore, by Proposition 5.2-2-(b), the order of pole of $c_0^{\bullet,\bullet}(\vec{a},s)$ at $s=s_0$ is 0, the order of pole of $c_1^{\bullet,\bullet}(\vec{a},s)$ at $s=s_0$ is 1, and the order of pole of $c_i^{\bullet,\bullet}(\vec{a},s)$ at $s=s_0$ is not larger than 1 if $i\ge 2$. Then, by Corollary 3.7, the order of $P^{[\vec{a},s]}(x)$ at $s=s_0$ is 1. The converses are also true.
   Let p be an integer in $2 \le p < m$. If $\langle d^{(2p+1)}[s_0],\vec{a}\rangle = 0$ and $\langle d^{(2p-1)}[s_0],\vec{a}\rangle \ne 0$, then $(Cond)^{\bullet,\bullet}_{2p-1,(\vec{a},s_0),\mathbf{top}}$ and $\overline{(Cond)^{\bullet,\bullet}_{2p-3,(\vec{a},s_0),\mathbf{top}}}$ (Proposition 4.8-2). Therefore, by Proposition 5.2-2-(c), the order of pole of $c_i^{\bullet,\bullet}(\vec{a},s)$ at $s=s_0$ is $\lfloor\frac{i+1}{2}\rfloor$ if $i<2p$, is $p=\lfloor\frac{2p+1}{2}\rfloor$ if $i=2p$, and is not larger than $p=\lfloor\frac{2p+1}{2}\rfloor$ if $i>2p$. Then, by Corollary 3.7, the order of $P^{[\vec{a},s]}(x)$ at $s=s_0$ is p. The converses are also true.
   If $\langle d^{(2m-1)}[s_0],\vec{a}\rangle \ne 0$, then $\overline{(Cond)^{\bullet,\bullet}_{2m-3,(\vec{a},s_0),\mathbf{top}}}$ (Proposition 4.8-2). Therefore, by Proposition 5.2-2-(d), the order of pole of $c_i^{\bullet,\bullet}(\vec{a},s)$ at $s=s_0$ is $\lfloor\frac{i+1}{2}\rfloor$ if $i<2m-1$, is m if $i=2m-1$, and is not larger than m if $i>2m-1$, Then, by Corollary 3.7, the order of $P^{[\vec{a},s]}(x)$ at $s=s_0$ is m. The converses are also true.
   (b) Consider the case that $m>\frac{n}{2}$. Then, $s_0 \le -\frac{n+1}{2}$, and we can apply Proposition 5.1. Let $n' := \lfloor\frac{n+1}{2}\rfloor$.
   If $\langle d^{(1)}[s_0],\vec{a}\rangle = 0$, then $(Cond)^{\bullet,\bullet,*}_{0,(\vec{a},s_0),\mathbf{top}}$ is satisfied (Proposition 4.8-1). Then we have $(Cond)^{\bullet,\bullet,*}_{0,(\vec{a},s_0),0}$. By Proposition 5.1-2-(a), all the coefficients $c_i^{\bullet,\bullet}(\vec{a},s)$ for $0\le i\le n$ are holomorphic at $s=s_0$. Then, by Corollary 3.7, $P^{[\vec{a},s]}(x)$ is holomorphic at $s=s_0$. The converses are also true.
   If $\langle d^{(3)}[s_0],\vec{a}\rangle = 0$ and $\langle d^{(1)}[s_0],\vec{a}\rangle \ne 0$, then $(Cond)^{\bullet,\bullet}_{1,(\vec{a},s_0),\mathbf{top}}$ and $\overline{(Cond)^{\bullet,\bullet,*}_{0,(\vec{a},s_0),\mathbf{top}}}$ are satisfied (Proposition 4.8-1). Therefore, by Proposition 5.1-2-(b), the order of pole of $c_i^{\bullet,\bullet}(\vec{a},s)$ at $s=s_0$ is 1 if $i\le 2$, and is not larger than 1 if $i>2$. Then, by Corollary 3.7, the order of $P^{[\vec{a},s]}(x)$ at $s=s_0$ is 1. The converses are also true.

Let p be an integer in $2 \leq p < n'$. If $\overline{\langle d^{(2p+1)}[s_0], \vec{a}\rangle = 0 \text{ and}}$ $\langle d^{(2p-1)}[s_0], \vec{a}\rangle \neq 0$, then $(\boldsymbol{Cond})^{\bullet,\bullet}_{2p-1,(\vec{a},s_0),\mathbf{top}}$ and $(\boldsymbol{Cond})^{\bullet,\bullet}_{2p-3,(\vec{a},s_0),\mathbf{top}}$ are satisfied (Proposition 4.8-2). Therefore, by Proposition 5.1-2-(c), the order of pole of $c_i^{\bullet,\bullet}(\vec{a}, s)$ at $s = s_0$ is $\lfloor \frac{i+1}{2} \rfloor$ if $i < 2p$, is p if $i = 2p$, and is not larger than p if $i > 2p$. Then , by Corollary 3.7, the order of $P^{[\vec{a},s]}(x)$ at $s = s_0$ is p. The converses are also true.
Suppose that n is odd (resp. even). If $\overline{\langle d^{(n)}[s_0], \vec{a}\rangle \neq 0, \text{ (resp.}}$ $\langle d^{(n-1)}[s_0], \vec{a}\rangle \neq 0,$) then $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{n-2,(\vec{a},s_0),\mathbf{top}}}$ (resp. $\overline{(\boldsymbol{Cond})^{\bullet,\bullet}_{n-3,(\vec{a},s_0),\mathbf{top}}}$) by Proposition 4.8-2. Therefore, by Proposition 5.1-2-(d), the order of pole of $c_i^{\bullet,\bullet}(\vec{a}, s)$ at $s = s_0$ is $\lfloor \frac{i+1}{2} \rfloor$ for all i in $0 \leq i \leq n$. Then , by Corollary 3.7, the order of $P^{[\vec{a},s]}(x)$ at $s = s_0$ is $\lfloor \frac{n+1}{2} \rfloor = n'$,i.e., the largest order of the poles of $c_i^{\bullet,\bullet}(\vec{a}, s)$ at $s = s_0$. The converses are also true.

Thus, we complete the proof of Theorem 2.3.

5.3. **Proof of the theorem on the support.** In this section we shall give a proof of Theorem 2.4.

In the proof, we let $s_0 := -\frac{q+1}{2}$. By Proposition 3.8, we have (50):

$$\mathbf{Supp}(P_w^{[\vec{a},s_0]}(x)) = (\overline{\bigcup_{\left\{\substack{c_i^{j,k}(\vec{a},s) \text{ has a pole of} \\ (i,j)\in\mathbb{Z}^2; \text{order} \geq -w \text{ for some } k \\ \text{in } 0 \leq k \leq i \text{ at } s = s_0}\right\}} S_i^j}) \tag{129}$$

Therefore, we have to calculate the orders of the coefficients $c_i^{j,k}(\vec{a}, s)$ at $s = s_0$. Since we have supposed that $P^{[\vec{a},s]}(x)$ has a pole of order p at $s = s_0$ in this proof,

$$p \leq \left\lfloor \frac{q+1}{2} \right\rfloor \leq \frac{q+1}{2} = s_0 \tag{130}$$

by (31). We let

$$U(s_0, p) := \left\{ (i,j) \in \mathbb{Z}^2; \substack{c_i^{j,k}(\vec{a}, s) \text{ has a pole of order} \geq p \text{ for some} \\ k \text{ in } 0 \leq k \leq i \text{ at } s = s_0} \right\} \tag{131}$$

1. First, suppose that q is an even integer.

Lemma 5.3. *Suppose that* $0 \leq i < -2w$. *Then we have*

$$(i,j) \notin U(s_0, -w).$$

Proof. By Proposition 5.1-1 and Proposition 5.2-1, $c_i^{\bullet,\bullet}(\vec{a}, s)$ has a pole of order strictly less than $-w$ at $s = s_0$ if $0 \leq i < -2w$. Thus we have the result. $\square$

On the other hand, if $0 \leq i < -2w$, then

$$S_i^j \bigcap (\overline{\bigcup_{\substack{i > -2w \\ 0 \leq j \leq n-i}} S_i^j}) = \emptyset.$$

Thus , if $0 \le i \le -2w$, then

$$S_i^j \bigcap \overline{(\bigcup_{(i,j)\in U(s_0,-w)} S_i^j)} = \emptyset. \tag{132}$$

Lemma 5.4. *If $i \ge -2w$ and $(i,j) \in U(s_0,-w)$, then*

$$S_i^j \subset \overline{(\bigcup_{\substack{(i,j)\in U(s_0,-w)\\ i=-2w}} S_i^j)} \tag{133}$$

Proof. Suppose that $i \ge -2w$ and that $c_i^{j,\bullet}(\vec{a},s)$ has a pole of order $\ge -w$. Then, from Proposition 4.5 and Proposition 4.3, there exists an integer j_0 in $j \le j_0 \le j+(i+2w)$ such that $\underline{c_{-2w}^{j_0,\bullet}}(\vec{a},s)$ has a pole of order $\ge -w$ at $s = s_0$. Then, we have $S_i^j \subset \overline{S_{-2w}^{j_0}}$, and hence we have the desired result. □

Therefore, we have

$$\overline{\bigcup_{\substack{(i,j)\in U(s_0,-w)\\ i\ge -2w}} S_i^j} \subset \overline{\bigcup_{\substack{(i,j)\in U(s_0,-w)\\ i=-2w}} S_i^j},$$

and hence

$$\overline{\bigcup_{\substack{(i,j)\in U(s_0,-w)\\ i\ge -2w}} S_i^j} = \overline{\bigcup_{\substack{(i,j)\in U(s_0,-w)\\ i=-2w}} S_i^j}.$$

Thus, by (132), we obtain

$$\overline{\bigcup_{(i,j)\in U(s_0,-w)} S_i^j} = \overline{\bigcup_{\substack{(i,j)\in U(s_0,-w)\\ i=-2w}} S_i^j}. \tag{134}$$

Lemma 5.5.

$$\{(i,j) \in U(s_0,-w); i = -2w\} = \{(-2w,j) \in \mathbb{Z}^2; \langle d_j^{(-2w)}[s_0], \vec{a}\rangle \ne 0\} \tag{135}$$

Proof. In order that $c_{-2w}^{\bullet,\bullet}(\vec{a},s)$ has a pole of order $-w$ at $s = s_0$, it is necessary and sufficient that $\overline{(\boldsymbol{Cond})_{-2w-2,(\vec{a},s_0),\mathbf{top}}^{\bullet,\bullet}}$, is satisfied, by Proposition 4.5-2. Then by Proposition 4.8-2, it is equivalent that $\langle \boldsymbol{d}^{(-2w)}[s_0], \vec{a}\rangle \ne 0$ and $s_0 \le w$ are satisfied. Since the condition $s_0 \le w$ is valid by (130) and the assumption $-w \le p$, we have the result. □

Therefore, when q is an even integer,

$$\begin{aligned}\mathbf{Supp}(P_w^{[\vec{a},s_0]}(x)) &= \overline{\bigcup_{(i,j)\in U(s_0,-w)} S_i^j} && \text{(by Proposition 3.8)}\\ &= \overline{\bigcup_{\substack{(i,j)\in U(s_0,-w)\\ i=-2w}} S_i^j} && \text{(by (134))}\\ &= \overline{\bigcup_{\substack{0\le j\le n+2w\\ \langle d_j^{(-2w)}[s_0],\vec{a}\rangle\neq 0}} S_{-2w}^j} && \text{(by (135))}\end{aligned}$$

This means the result (23).

2. Secondly, suppose that q is an odd integer.

Lemma 5.6. *Suppose that* $0 \le i < -2w-1$. *Then we have*

$$(i,j) \notin U(s_0,-w).$$

Proof. It is proved in the same way as the proof of Lemma 5.3. □

On the other hand, if $0 \le i < -2w-1$, then we have

$$S_i^j \bigcap (\overline{\bigcup_{(i,j)\in U(s_0,-w)} S_i^j}) = \emptyset. \tag{136}$$

Lemma 5.7. *If* $i \ge -2w-1$ *and* $(i,j) \in U(s_0,-w)$, *then*

$$S_i^j \subset (\overline{\bigcup_{\substack{(i,j)\in U(s_0,-w)\\ i=-2w-1}} S_i^j}) \tag{137}$$

Proof. Suppose that $i \ge -2w-1$ and that $c_i^{j,\bullet}(\vec{a},s)$ has a pole of order $\ge -w$. Then, from Proposition 4.5 and Proposition 4.3, there exists an integer j_0 in $j \le j_0 \le j+(i+2w+1)$ such that $c_{-2w-1}^{j_0,\bullet}(\vec{a},s)$ has a pole of order $\ge -w$ at $s=s_0$. Then , $S_i^j \subset \overline{S_{-2w-1}^{j_0}}$, and hence we have the desired result. □

Therefore, we obtain

$$\overline{\bigcup_{(i,j)\in U(s_0,-w)} S_i^j} = \overline{\bigcup_{\substack{(i,j)\in U(s_0,-w)\\ i=-2w-1}} S_i^j}, \tag{138}$$

in the same way as the proof of the case that q is even.

Lemma 5.8.

$$\{(i,j)\in U(s_0,-w); i=-2w-1\} = \{(-2w-1,j)\in\mathbb{Z}^2; \langle d_j^{(-2w-1)}[s_0],\vec{a}\rangle \neq 0\} \tag{139}$$

Proof. It is proved in the same way as the proof of Lemma 5.5. □

Therefore, when q is an even integer,

$$\begin{aligned}\mathbf{Supp}(P_w^{[\vec{a},s_0]}(x)) &= \overline{\bigcup_{(i,j)\in U(s_0,-w)} S_i^j} && \text{(by Proposition 3.8)}\\ &= \overline{\bigcup_{\substack{(i,j)\in U(s_0,-w)\\ i=-2w-1}} S_i^j} && \text{(by (138))}\\ &= \overline{\bigcup_{\substack{0\le j\le n+2w+1\\ \langle d_j^{(-2w-1)}[s_0],\vec{a}\rangle\neq 0}} S^j_{-2w-1}} && \text{(by (139))}\end{aligned}$$

This means the result (24).

Thus, we complete the proof of Theorem 2.4.

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Gifu University, Yanagito 1-1, Gifu, 501-11,JAPAN

E-mail address: muro@cc.gifu-u.ac.jp