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Some open problems on the analysis of the Cauchy–Fueter system in several variables

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1 Introduction

In the last few years there has been a resurgence of interest for the analysis of regular functions of a quaternionic variable and, more generally, for the analysis of monogenic functions of Clifford variables [28], of which quaternions are one of the simplest examples. There are many reasons for such a rebirth of a theory whose origin goes back to Fueter's works in the first half of this century (actually, as Professor Shapiro pointed out [16], the first works in this direction are due to Moisil and Theodoresco [19], [20]). On one hand Clifford analysis has acquired particular relevance in theoretical physics (see e.g. [5]), and, in particular, quaternionic quantum field theory is now being actively developed (see [5]); on the other hand, Clifford analysis has become increasily relevant to questions in classical harmonic analysis, as it has been clearly demonstrated by the work of McIntosh [18] (see also [28]) and more source of interest is the very recent novel approach due to Kravchenko and Shapiro, [16].

Up to now, however, very little attention has been paid (an exception being [26]) to the case of several variables (for example several quaternionic variables); this fact is most likely due to the intrinsic difficulty of studying objects (regular functions) which seem to lack some of the most elementary properties (as an example, the composition of two regular functions is not even regular, nor is their product). A couple of years ago, however, a group of mathematicians (C.A. Berenstein, P. Loustaunau and the authors) realized that an abstract algebraic treatment of the Cauchy–Fueter system could provide the necessary tool for the study of regular functions in several quaternionic variables.

In this paper, after a brief introduction to the subject, we will describe the results which we have obtained so far, and which seem to suggest the existence of a quaternionic analog of the theory of hyperfunctions. The core of the paper, however, is the last section, in which we will state several open problems, together with some ideas which may play a role in their solution.

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2 Regular functions of a quaternionic variable

In this section we will collect the fundamental definitions and properties on which the theory of regular functions is based; these notations will be of great importance in the next sections when we will treat the case of several variables. Our notation will follow [32] and [12], to which the interested reader is referred for more details. We will denote by $\mathbb{H}$ the four–dimensional real associative algebra of the quaternions, with the standard basis $\{1, i, j, k\}$ satifying the usual relations:

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

A quaternion will be an element

$$q = x_0 + x_1 i + x_2 j + x_3 k$$

so that

$$\mathbb{H} = \{q + \sum_{i=0}^{3} x_i e_i \mid e_0 = 1,\ e_1 = i,\ e_2 = j,\ e_3 = k\}.$$

The (left) Cauchy–Fueter operator is defined by

$$\frac{\partial}{\partial \bar{q}} = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3} = 0 \tag{1}$$

and we can use it to give the following

Definition 2.1 *Let $U \subseteq \mathbb{H}$ be an open and let $f : U \longrightarrow \mathbb{H}$ be differentiable in the real sense (as a function from $U \subseteq \mathbb{R}^4$ to $\mathbb{R}^4$). We say that f is (left)–regular if*

$$\frac{\partial f}{\partial \bar{q}} = 0 \quad \text{on } U.$$

We will denote by $\mathcal{R}(U)$ the right $I\!H$–vector space of regular functions. A corresponding notion of right regularity can be defined by means of the right Cauchy–Fueter operator given by

$$\frac{\partial_r}{\partial \bar{q}} = \frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1} i + \frac{\partial}{\partial x_2} j + \frac{\partial}{\partial x_3} k = 0.$$

The theory of right–regular functions is completely parallel to the theory of left regular ones.

Remark 2.2 The Cauchy–Fueter operator was introduced by Fueter [13] and [14] (but see also [19] and [20]) who was interested in constructing, over $I\!H$, a space of "holomorphic" functions. As he quickly found out, a naive definition of $I\!H$–holomorphicity (or regularity as he called it) based on the existence of the $I\!H$–derivative, does not work as the only functions $f : I\!H \longrightarrow I\!H$ which admits left (or right) $I\!H$-derivative (i.e. $\lim_{h\to 0, h\in I\!H} h^{-1}(f(q+h) - f(q))$) are linear functions of the form $f(q) = qa + b$ (or $f(q) = aq + b$). His next idea was to construct a quaternionic version of the Cauchy–Riemann system: such a version is the system (1).
It can be shown that much of what is known for holomorphic functions can be replicated for regular ones. In this section we will explore some of these properties. To begin with, the classical Cauchy kernel $1/z$ is replaced by the so–called Cauchy–Fueter kernel

$$G(q) := \frac{q^{-1}}{|q|^2} = \frac{\bar{q}}{|q|^4},$$

which is easily seen to be both left and right regular.

Let us note that, if

$$Dq := dx_1 \wedge dx_2 \wedge dx_3 - i dx_0 \wedge dx_2 \wedge dx_3 + j dx_0 \wedge dx_1 \wedge dx_3 - k dx_0 \wedge dx_1 \wedge dx_2$$

and

$$\nu := dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3$$

the formula

$$d(gDqf) = dg \wedge Dqf - gDq \wedge df = \left\{ \left(\frac{\partial_r g}{\partial \bar{q}} \right) f + g \left(\frac{\partial_l f}{\partial \bar{q}} \right) \right\} \nu,$$

allows us to prove the following two theorems.

Theorem 2.3 *(Cauchy–Fueter I). Let $f : U \subseteq I\!H \to I\!H$ be a regular function on U. If $V \subseteq U$ is any open set with smooth boundary, such that $\bar{V} \subseteq U$, then*

$$\int_{\partial V} Dqf(q) = 0$$

Theorem 2.4 *(Cauchy–Fueter II). Let $f : U \subseteq I\!H \longrightarrow I\!H$ be a regular function on U. Let $V \subset U$ be an open set with smooth boundary, relatively compact in U. If $q_0 \in V$, then*

$$f(q_0) = \frac{1}{2\pi^2} \int_{\partial V} G(q - q_0) Dqf(q).$$

Remark 2.5 Let Δ be the laplacian in $I\!R^4$. Then

$$\Delta = \frac{\partial}{\partial q}\frac{\partial}{\partial \bar{q}} = \frac{\partial}{\partial \bar{q}}\frac{\partial}{\partial q},$$

so the Cauchy–Fueter system is elliptic.

From this last property follows that all the main results on holomorphic functions like the Liouville Theorem or the Poincaré Lemma or the Maximum Modulus Principle can be also proved in this more general setting. An important difference with the theory of holomorphic functions is that it is not possible to express a regular function in terms of powers of q. In fact, the function q^n is not regular not even for $n = 1$. A Taylor series can be written in terms of particular homogeneous polynomials as follows. Let

$$P_\nu(q) = \frac{1}{n!} \sum_{1 \le \lambda_1, \ldots, \lambda_n \le 3} (x_0 e_1 - x_{\lambda_1}) \ldots (x_0 e_{\lambda_n} - x_{\lambda_n}),$$

where the sum is on all $\frac{n!}{n_1! n_2! n_3!}$ possible permutations of n_i elements equal to i, $i = 1, 2, 3$. We have the following theorem:

Theorem 2.6 *Let $f : U \subseteq I\!H \longrightarrow I\!H$, $f \in \mathcal{R}(U)$. Let $q_0 \in U$ and $\delta <$ dist $(q_0, \partial U)$. Then, if $|q - q_0| < \delta$, we have*

$$f(q) = \sum_{n=0}^{+\infty} \sum_{\nu \in \sigma_n} P_\nu(q - q_0) a_\nu,$$

where

$$a_\nu = (-1)^n \partial_\nu f(q_0) = \frac{1}{2\pi^2} \int_{|q - q_0| = \delta} G_\nu(q - q_0) Dqf(q).$$

with $\partial_\nu = \partial^\nu / \partial x_1^{n_1} \partial x_2^{n_2} \partial x_3^{n_3}$.

The existence of such an expansion is essentially due to the fact that it is possible to multiply two given regular functions using the so–called C–K product to get a regular function, at least when they are entire. An analogous C–K product is not known in the case of several variables; this fact prevents us from defining a Taylor series for regular functions defined on $\mathbb{H}^n$. We would like to point out that it is not known if it is possible to prove an analogous theorem to the case of regular functions of several quaternionic variables. The theory for regular functions in one variable is so similar to the corresponding theory in the complex case that it is also possible to prove a Mittag–Leffler theorem, that is the vanishing of $H^1(U, \mathcal{R})$ for every open set U. This allows the introduction of the notion of hyperfunctions:

Definition 2.7 *Let U be an open set in $\tilde{\mathbb{H}} = \{q = \sum_{i=0}^{3} x_i e_i \; |x_0 = 0\}$ and let V be an open set in $\mathbb{H}$ such that U is relatively closed in V. The right vector space on $\mathbb{H}$ defined by*

$$\mathcal{F}(U) := \frac{\mathcal{R}_l(V \backslash U)}{\mathcal{R}_l(V)}$$

is said to be the space of $\mathbb{H}$–hyperfunctions.

The correspondence that assigns to every open set U in $\tilde{\mathbb{H}}$ the vector space $\mathcal{F}(U)$ defines a flabby sheaf on $\tilde{\mathbb{H}}$. The classical Fantappié–Köthe–Martineau duality holds also in this case, namely

$$\mathcal{F}_K \cong \frac{\mathcal{R}_l(U \backslash K)}{\mathcal{R}_l(U)} \cong (\mathcal{G}(K))'$$

where $\mathcal{F}_K$ is the space of $\mathbb{H}$–hyperfunctions with support in K and $(\mathcal{G}(K))'$ denotes the space of left linear continuos functionals on $\mathcal{G}(K) = \text{ind}\lim_{U \supset K} \mathcal{R}_r(U)$ and the duality is topological as was proved in [29]. Since $\mathcal{F}_K$ is reflexive it holds also

$$\mathcal{G}(K) \cong (\mathcal{F}_K)'$$

3 The Cauchy–Fueter system in several variables

As we have seen in section 2, the construction of a hyperfunction theory based on boundary values of regular functions of one quaternionic variable is an extremely natural affair, but the methods used do not lend themselves to easy multi–dimensional generalizations.

The first step consists in suitably defining regular functions in several quaternionic variables. Unfortunately, the non–commutativity of quaternions plays there a rather strong role. Let us consider two quaternionic variables q_1 and q_2. If $f = f(q_1)$ and $g = g(q_2)$ are, respectively, left–regular in q_1 and q_2, then the product fg is not left–regular in both of them, since

$$\frac{\partial(fg)}{\partial\bar{q}_1} = \frac{\partial f}{\partial\bar{q}_1}g + \sum_{i=0}^{3} e_i f \frac{\partial g}{\partial x_i}$$

$$\frac{\partial(fg)}{\partial\bar{q}_2} = \frac{\partial f}{\partial\bar{q}_2}g + \sum_{i=0}^{3} e_i f \frac{\partial g}{\partial y_i}$$

and therefore left–regularity in q_1, q_2 fails. For this reason, Nono defined in [22] a notion of biregularity for functions $f(q_1, q_2)$ which requires f to be left–regular in q_1 and right–regular in q_2. A reasonably simple theory can be established for these functions; however no generalization along these lines is possible to three or more variables. We have taken, on the other hand, the direction of defining regular functions of several variables as those functions $f: \mathbb{H}^n \longrightarrow \mathbb{H}$ who are left–regular in each variable or, which is equivalent, which satisfy

$$\frac{\partial f}{\partial\bar{q}_1} = \ldots = \frac{\partial f}{\partial\bar{q}_n} = 0.$$

Constructing examples for such functions is not as simple as for biregular functions, since even simple products of regular polynomials are not regular. For example if we set $q_1 = x_0 + ix_1 + jx_2 + kx_3$ and $q_2 = y_0 + iy_1 + jy_2 + ky_3$, we have that even

$$(x_1 - ix_0)(y_2 - jy_0) + (y_2 - jy_0)(x_1 - ix_0)$$

is not regular regular in q_1 and q_2. On the other hand,

$$(x_1 - ix_0)(y_1 - iy_0) + (y_1 - iy_0)(x_1 - ix_0)$$

is regular and, more generally, examples can be constructed by using symmetric products of regular polynomials in which the factors involve the same imaginary unit. This is equivalent to the request that the two polynomials

$(x_i + e_i x_0)$, $(y_i + e_i y_0)$ are holomorphic with respect to two complex variables belonging to the same complex plane.

The point of view that we have taken, however, is to use the ellipticity of the Cauchy–Fueter system and to consider regular functions as vector solutions of the multi-dimensional Cauchy–Fueter system. From this point of view, a regular function $f : \mathbb{H}^n \longrightarrow \mathbb{H}$ is a differentiable function $\vec{f} : \mathbb{R}^{4n} \longrightarrow \mathbb{R}^4$ which satisfies a $4n \times 4$ system

$$\begin{bmatrix} \partial/\partial x_{01} & -\partial/\partial x_{11} & -\partial/\partial x_{21} & -\partial/\partial x_{31} \\ \partial/\partial x_{11} & \partial/\partial x_{01} & -\partial/\partial x_{31} & \partial/\partial x_{21} \\ \partial/\partial x_{21} & \partial/\partial x_{31} & \partial/\partial x_{01} & -\partial/\partial x_{11} \\ \partial/\partial x_{31} & -\partial/\partial x_{21} & \partial/\partial x_{11} & \partial/\partial x_{01} \\ \dots & \dots & \dots & \dots \\ \partial/\partial x_{0n} & -\partial/\partial x_{1n} & -\partial/\partial x_{2n} & -\partial/\partial x_{3n} \\ \partial/\partial x_{1n} & \partial/\partial x_{0n} & -\partial/\partial x_{3n} & \partial/\partial x_{2n} \\ \partial/\partial x_{2n} & \partial/\partial x_{3n} & \partial/\partial x_{0n} & -\partial/\partial x_{1n} \\ \partial/\partial x_{3n} & -\partial/\partial x_{2n} & \partial/\partial x_{1n} & \partial/\partial x_{0n} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = \vec{0} \qquad (2)$$

where we have written $q_t = x_{0t} + i x_{1t} + j x_{2t} + k x_{3t}$ and $f = f_0 + i f_1 + j f_2 + k f_3$. Note that in view of the ellipticity of the Cauchy–Fueter system, it does not matter whether one consider $\vec{f}$ in $\mathcal{A}$ (real analytic functions), $\mathcal{E}$ ($\mathcal{C}^\infty$ functions), $\mathcal{D}'$ (distributions) or even in $\mathcal{B}$ (hyperfunctions). We will denote by $\mathcal{R}$ the sheaf of regular functions. The advantage of this point of view is that it allows us to disengage from the use of quaternions, and enables us to bring in the powerful algebraic treatment championed in [24]. For the sake of completeness, let us review the fundamentals of such an approach. Let $R = \mathbb{C}[z_1, \dots, z_n]$ be the ring of polynomials with complex coefficients in n variables, and let $P = [P_{ij}]$ be an $r_1 \times r_0$ matrix in R so that $P(D) = [P_{ij}(D)]$ is a matrix of linear constant coefficients differential operators, where $D = (-i\frac{\partial}{\partial x_1}, \dots, -i\frac{\partial}{\partial x_n})$. If $\mathcal{S}$ is a sheaf of generalized functions (in the case we are interested in, $\mathcal{S}$ will either be the sheaf $\mathcal{B}$ of hyperfunctions, or the sheaf $\mathcal{E}$ of infinitely differentiable functions), we will denote by $\mathcal{S}^P$ the kernel of the map

$$P(D) : \mathcal{S}^{r_0} \longrightarrow \mathcal{S}^{r_1},$$

i.e. the sheaf of $\mathcal{S}$–solutions to the system

$$P(D)\vec{f} = \vec{0}. \qquad (3)$$

Now the multiplication by the matrix P^t maps R^{r_1} to R^{r_0} and we know that its cokernel

$$M = \frac{R^{r_0}}{P^t R^{r_1}}$$

contains all the relevant information on (3), in view of the well known isomorphism

$$\mathcal{H}om_R(M, \mathcal{S}) \cong \mathcal{S}^P.$$

In particular we know that Hilbert's syzygy theorem (proved in its most general form by Palamodov [24]) shows that there is a finite free resolution

$$0 \longleftarrow M \longleftarrow R^{r_0} \stackrel{P^t}{\longleftarrow} R^{r_1} \stackrel{P_2^t}{\longleftarrow} R^{r_2} \longleftarrow \ldots \longleftarrow R^{r_{m-1}} \stackrel{P_m^t}{\longleftarrow} R^{r_m} \longleftarrow 0 \quad (4)$$

for which $m \leq n$.
The dualized of (4) is the complex

$$0 \longrightarrow R^{r_0} \stackrel{P}{\longrightarrow} R^{r_1} \stackrel{P_2}{\longrightarrow} R^{r_2} \longrightarrow \ldots \longrightarrow R^{r_{m-1}} \stackrel{P_m}{\longrightarrow} R^{r_m} \longrightarrow 0 \quad (5)$$

whose cohomology groups are denoted by $Ext^p(M, R)$. Interestingly enough. even though (4) and (5) are not uniquely defined, the Ext groups (they are actually R–modules) are uniquely determined by M.
Computing the resolution (4) amounts to computing the generators for the kernel of polynomial homomorphisms or, in algebraic jargon, their syzygies. This can be done explicitely (at least in principle) by using Gröbner bases techniques, [3], and if the module M is graded (i.e. if the matrix has homogeneous columns) one can actually compute a minimal lenght free resolution.

There are several reasons why (5) is an important object, and among them the following result of Ehrenpreis et al. (see [15] for the complete references).

Theorem 3.1 *If Ω is a convex open (or convex compact) set in $\mathbb{R}^n$, and $\mathcal{S}$ is a sheaf on $\mathbb{R}^n$, then the sequence*

$$0 \longrightarrow \mathcal{S}^P(\Omega) \longrightarrow \mathcal{S}(\Omega)^{r_0} \stackrel{P(D)}{\longrightarrow} \mathcal{S}(\Omega)^{r_1} \stackrel{P_2(D)}{\longrightarrow} \ldots \stackrel{P_m(D)}{\longrightarrow} \mathcal{S}(\Omega)^{r_m} \longrightarrow 0$$

is exact, which in particular means that $P_2(D)$ is a compatibility system for $P(D)$.

On the other hand, the Ext modules also carry much relevant information: $Ext^0(M,R)=0$ means that P is one–to–one, i.e. the columns of P are linearly independent over R (and the system corresponding to P is determined [15]). Also, it is known that $Ext^1(M,R)=0$ is equivalent to the fact that the Hartogs' phenomenon holds for $\mathcal{S}^P$ (i.e. for any compact set $K \subseteq \mathbb{R}^n$ such that $\mathbb{R}^n \backslash K$ is connected, every element in $\mathcal{S}^P(\mathbb{R}^n\backslash K)$ extends to an element in $\mathcal{S}^P(\mathbb{R}^n)$); thanks to the pioneering work of Palamodov, we also know that there are several important analytic consequences of the vanishing of the higher Ext–modules (still related to various results on the removability of special singularities); the reader may refer to [11] and [24] for more details. Palamodov's work also makes evident the strong geometric flavor of this entire construction; in fact, if V is the characteristic variety associated to the operator $P(D)$, i.e. if V is the set of common zeroes of the maximal size minors of $[P_{ij}]$, then $dim(V)=n-p-1$ if and only if $Ext^j(M,R)=0$ for all $j \leq p$ and $Ext^{p+1}(M,R) \neq 0$.

But let us now go back to the specific case in which $P(D)\vec{f}=\vec{0}$ is the Cauchy–Fueter system given by (2). In this case the polynomial matrix is given by

$$\begin{bmatrix} \xi_{01} & -\xi_{11} & -\xi_{21} & -\xi_{31} \\ \xi_{11} & \xi_{01} & -\xi_{31} & \xi_{21} \\ \xi_{21} & \xi_{31} & \xi_{01} & -\xi_{11} \\ \xi_{31} & -\xi_{21} & \xi_{11} & \xi_{01} \\ \dots & \dots & \dots & \dots \\ \xi_{0n} & -\xi_{1n} & -\xi_{2n} & -\xi_{3n} \\ \xi_{1n} & \xi_{0n} & -\xi_{3n} & \xi_{2n} \\ \xi_{2n} & \xi_{3n} & \xi_{0n} & -\xi_{1n} \\ \xi_{3n} & -\xi_{2n} & \xi_{1n} & \xi_{0n} \end{bmatrix}$$

and in a series of papers [1], [4], [6], we have extensively studied the resolution for the n–dimensional Cauchy–Fueter system (or, better, for its module $\mathcal{M}_n$).

Let us summarize here our most significant results in this direction. We refer the reader to our papers for their proofs.

Theorem 3.2 *The projective dimension of the Cauchy–Fueter module $\mathcal{M}_n$ is $2n-1$, i.e. $\mathcal{M}_n$ has a free resolution of length $2n-1$ and it is not possible to find a shorter resolution for $\mathcal{M}_n$.*

Theorem 3.3 *The characteristic variety $Ch(\mathcal{M}_n)$ has dimension $2n+1$ and* *therefore*

$$Ext^j(\mathcal{M}_n, R) = 0 \qquad j = 0, 1, \ldots, 2n-2$$

while

$$Ext^{2n-1}(\mathcal{M}_n, R) \neq 0.$$

Since, as it is well known, the Cauchy–Fueter system is elliptic, one can deduce the following important consequences:

Theorem 3.4 *The sheaf $\mathcal{R} = \mathcal{B}^P = \mathcal{E}^P$ of regular functions has flabby dimension equal to $2n-1$.*

Theorem 3.5 *If U is any open set in $\mathbb{H}^n$, then*

$$H^j(U, \mathcal{R}) = 0, \qquad j \geq 2n-1.$$

The reader can see that these last two statements are the quaternionic analogues of well known results of Malgrange [30].

Another general consequence of the ellipticity of the system is the following result (see also [31]):

Theorem 3.6 *Let K be a compact convex set in $\mathbb{H}^n$. Then*

$$[H^0(K, \mathcal{R})]' \cong H_K^{2n-1}(\mathbb{H}^n, \mathcal{B}^Q).$$

For the case $n = 1$, $\mathcal{B}^Q \cong \mathcal{R}$, and for the case $n = 2$, Q isnothing but the system associated to $\partial/\partial q_2 - \partial/\partial q_1$. The situation is more complex for $n \geq 3$.

As we pointed out earlier, the vanishing of the *Ext*–modules has relevant applications to the removability of singularities of regular functions. We have the following two results:

Theorem 3.7 *Let Ω be a convex connected open set in $\mathbb{H}^n$, and let K be a compact subset of Ω. Let $\Sigma_1, \ldots, \Sigma_{2n-2}$ be closed half–spaces in $\mathbb{R}^{4n}$ and set $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_{2n-2}$. Then every regular function f in $\Omega \backslash (K \cup \Sigma)$ extends to a regular function in $\Omega \backslash \Sigma$, which coincides with f in $\Omega \backslash (K' \cup \Sigma)$ for K' a compact subset of Ω.*

Theorem 3.8 *Let L be a subspace of $\mathbb{H}^n = \mathbb{R}^{4n}$ of dimension $2n+2$. Then for every compact set $K \subseteq L$, and every connected open set Ω, relatively compact in K, every regular function defined in the neighborhood of $K \backslash \Omega$ can be extended to a regular function defined in a neighborhood of K.*

Partial improvements of these general results have recently been obtained in [21], where some classical ideas of Severi are used to remove compact singularities in $\mathbb{H} \times \mathbb{R}$ and $\mathbb{H} \times \mathbb{C}$; also, in [21] we have shown how some subclasses of $\mathcal{R}(\mathbb{H}^2)$ admit stronger singularity removability properties. It may be worth pointing out that Theorems 3.7 and 3.8 can be improved for a special subclass of regular functions defined in the recent [21]. These abstract methods also provide a concrete way to represent all regular functions on any open convex set $\Omega \subseteq \mathbb{H}^n$. Indeed, the Ehrenpreis–Palamodov Fundamental Principle implies the following representation:

Theorem 3.9 *Let $\Omega \subseteq \mathbb{H}^n$ be an open convex set. Then every regular function f in Ω can be represented as*

$$f(q_1, \ldots, q_n) = \int_{Ch(\mathcal{M}_n)} \exp(\xi \cdot q) \cdot \bar{\xi} d\mu$$

where $\xi \in \mathcal{H}^n$, and $d\mu = (d\mu_1, \ldots, d\mu_n)$ is a vector of quaternionic valued densities supported on $Ch(\mathcal{M}_n)$ and satisfying, for every compact $K \subseteq \Omega$,

$$\int_{Ch(\mathcal{M}_n)} \exp(\max_K(\xi \cdot q))|d\mu| < +\infty.$$

Note that this representation corresponds to a particular choice for the noetherian operator associated to the module $\mathcal{M}_n$. Additional representations for regular functions in several quaternionic variables are also given in the recent [25].

4 Open problems

4.1 The matrix P_2 which appears in Theorem 3.2 is particularly important because it allows us to write explicitely the compatibility conditions for the non–homogeneous system

$$\begin{cases} \frac{\partial f}{\partial \bar{q}_1} = g_1 \\ \vdots \\ \frac{\partial f}{\partial \bar{q}_n} = g_n \end{cases}$$

When $n = 2$, these conditions were first explicitely computed using CoCoA, and are provided in [1], where they were expressed in terms of the 8 real coordinates; this gives an 8×8 matrix, explicitely described in [1]. It is however possible to read these conditions in purely quaternionic terms. In this case the 8×8 real matrix becomes the very simple 2×2 quaternionic matrix

$$\begin{bmatrix} \frac{\partial^2}{\partial \bar{q}_2 \partial q_1} & -\frac{\partial^2}{\partial \bar{q}_1 \partial q_1} \\ \frac{\partial^2}{\partial \bar{q}_2 \partial q_2} & -\frac{\partial^2}{\partial \bar{q}_1 \partial q_2} \end{bmatrix}$$

which can be interpreted by saying that, on every convex set $\Omega \subseteq I\!H^2$, the system

$$\begin{cases} \frac{\partial f}{\partial \bar{q}_1} = g_1 \\ \frac{\partial f}{\partial \bar{q}_2} = g_2 \end{cases}$$

has a solution if and only if

$$\begin{cases} \frac{\partial^2 g_1}{\partial \bar{q}_2 \partial q_1} - \frac{\partial^2 g_2}{\partial \bar{q}_1 \partial q_1} = 0 \\ \frac{\partial^2 g_1}{\partial \bar{q}_2 \partial q_2} - \frac{\partial^2 g_2}{\partial \bar{q}_1 \partial q_2} = 0 \end{cases}$$

It is possible to obtain such conditions from a purely analytic point of view (but, as far as we can see now, only their necessity can be established in such fashion). For $n = 3$, the matrix P_2 is a 12×40 matrix, whose entries are homogeneous second degree polynomials, and which we have calculated once again using CoCoA. This means that we now expect 10 quaternionic compatibility conditions, represented by second order differential operators, but, up to now, we have been unable to describe such conditions on a purely analytic basis. In particular, it is known that [17] for arbitrary n, the matrix P_2 is a $4n \times p(n)$ matrix, with

$$p(n) = 2\binom{n}{2} + 4\binom{n}{3}.$$

Up to now, only $\binom{n}{2} + \binom{n}{3}$ such syzygies can be obtained by purely analytical methods.

Problem: Give an analytic interpretation for the matrix P_2 of the first syzygies, for any $n \geq 2$.

4.2 In all the examples which we have explicitely computed, we have seen that (with the exception of P_2) all the matrices appearing in Theorem 3.2 have entries which are degree one polynomials. However we have been unable to prove this fact in general, nor do we understand its analytical meaning. In [21] we have shown how small modifications of the module $\mathcal{M}_n$ may cause this phenomenon to disappear.

Problem: Prove that all the matrices which appear in Theorem 3.2 have entries which are degree one polynomials, with the exception of P_2.

4.3 When $n = 2$, the syzygies are expressed by the following equations:

$$\begin{bmatrix} \bar{q}_2 q_1 & -\bar{q}_1 q_1 \\ \bar{q}_2 q_2 & -\bar{q}_1 q_2 \end{bmatrix} \begin{bmatrix} \bar{q}_1 \\ \bar{q}_2 \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} q_2 \\ -q_1 \end{bmatrix} \begin{bmatrix} \bar{q}_2 q_1 & -\bar{q}_1 q_1 \\ \bar{q}_2 q_2 & -\bar{q}_1 q_2 \end{bmatrix} = \vec{0}$$

so that the matrices P and Q (which are explicitely constructed in [1], and expressed in real terms) originate, respectively, the sheaf of regular functions and the sheaf of anti–regular functions. For $n = 3$ the matrix Q is not so simple anymore, and in general we do not know the relationship between the sheaves $\mathcal{R}$ and $\mathcal{B}^Q$, with the exception of the interesting fact that P and Q have the same characteristic variety; this is an immediate consequence of our Theorem 3.3 and the general theory of Palamodov [24].

Problem: Establish a relationship between $\mathcal{R}$ and $\mathcal{B}^Q$.

4.4 The results of section 3 point to the possibility of developing a quaternionic hyperfunction theory, and in particular to the possibility to construct a (flabby?) sheaf via the relative cohomology of $\mathcal{R}$. In order for this theory to be developed, we need to prove a quaternionic analog of the classical theorem on the pure n–codimensionality of $\mathbb{R}^n$ in $\mathbb{C}^n$, with respect to $\mathcal{O}$; such a theorem is equivalent to the fact that

$$H^j_{\mathbb{R}^n}(\mathbb{C}^n, \mathcal{O}) = 0 \qquad \forall j \neq n$$

and

$$H^n_{\mathbb{R}^n}(\mathbb{C}^n, \mathcal{O}) \neq 0.$$

The role of the pair $(\mathbb{C}^n, \mathbb{R}^n)$ is clear, but we do not have a quaternionic analogue, partly because we do not have a full understanding of the Cauchy–Kowalewski phenomenon for general matrix systems. For the case $n = 1$, we have proved [12], that $H^j_{\mathbb{H}}(\mathbb{H}, \mathcal{R}) = 0$, for $j \neq 1$ and $H^1_{\mathbb{H}}(\mathbb{H}, \mathcal{R})$ is in fact a sheaf of hyperfunctions.

Problem: Find a $(2n+1)$–dimensional variety S in $\mathbb{H}^n = \mathbb{R}^{4n}$ such that

$$H^j_S(\mathbb{H}^n, \mathcal{R}) = 0 \qquad \forall j \neq 2n-1$$

and

$$H^{2n-1}_S(\mathbb{H}^n, \mathcal{R}) \neq 0.$$

4.5 When $n = 2$, Theorem 3.8 shows that compact singularities contained in subspaces of dimension $6, 7, 8$ can always be removed. On the other hand, it is obviously impossible to extend this result to singularities contained in subspaces of dimension 4 (just consider the origin in $\mathbb{H} \hookrightarrow \mathbb{H}^2$). The case of dimension 5 cannot be solved by purely algebraic methods, because the module $\mathcal{M}_2$ associated to the Cauchy–Fueter system has $Ext^2(\mathcal{M}_2, R) = 0$ but $Ext^3(\mathcal{M}_2, R) \neq 0$. In [21] we have shown that at least for some subclasses of $\mathcal{R}$, even the case of dimension 5 can be answered in the affirmative.
Problem: Can Theorem 3.8 be extended, when $n = 2$, to include all subspaces of dimension 5?

4.6 Theorem 3.1 holds when the open set $\Omega \subseteq \mathbb{R}^n$ is convex. This hypothesis, that is requested to assure that the space $\mathcal{S}(\Omega)$ is LAU, is too strong. In fact, (as it was proved in [11]), when Ω is not convex the theorem still holds under the weaker hypothesis of P–convexity. This is equivalent to a suitable relation between the system P and the topological structure of Ω. It has not been investigated, up to now, what is the characterization of P–convexity when P is the Cauchy–Fueter system; this characterization will probably depend on the possibility to define a notion of a tangential Cauchy–Fueter operator.
Problem: Give a geometric characterization of the P–convex sets.

4.7 Another interesting problem is related to a possible construction of a Weyl algebra $\mathbb{H}[x, \partial]$, where x and ∂ are suitable variables and operators respectively. The answer is not trivial, in fact we have, apparently, a double non commutativity in both the coefficients and the variables. In the literature, as far as we know, some examples of Weyl algebras on a general ring are provided. Unfortunately it seems that they do not work in our case, in fact we have to keep in mind that in order to study systems of differential equations, we need to have a Weyl algebra $A_1(\mathbb{H})$ such that the spaces $\mathcal{R}(U)$ and $\mathcal{F}(U)$ can be considered as $A_1(\mathbb{H})$–modules.
Problem: Construct a quaternionic Weyl algebra.

4.8 The theory of regular functions in one quaternionic variable seems to be completely understood. In spite of this, the notion of exponential function seems to cause some troubles when one tries to restore a Fourier transform theory for (compactly supported) $\mathbb{H}$–hyperfunctions. It is well known that in order to have a regular function, it is necessary to define the exponential as follows

$$\exp(q) := \sum_{n=0}^{+\infty} \sum_{\nu} P_\nu(q).$$

By using this function (that can be considered as the C–K product of $\exp(z_1 e_1)$, ..., $\exp(z_3 e_3)$, $z_i = x_i - e_i x_0$), it is possible to define the Fourier transform of a rapidly decreasing $\mathcal{C}^\infty$ function or a distribution (it suffices to follow the ideas in [7]) and also to prove the analogue of the Paley–Wiener theorem. The usual properties of the Fourier transform hold up to some "reflection operators". The Fourier transform of a $\mathbb{H}$–hyperfunction seems to be a quite complicated object, and the Paley–Wiener theorem (that is a key tool in establish the isomorphism between the space of $\mathbb{H}$–hyperfunctionswith support in the compact K and the space $\mathrm{Exp}(K)$ of entire functions of exponential type) has not been proved.
Problem: Prove a Paley–Wiener Theorem for the Fourier transform of $\mathbb{H}$–hyperfunctions.

4.9 In Quaternionic Quantum Mechanics, physicists use a different notion of the exponential function that is the formal analogue of $\exp(z)$. Obviously this is not a regular function, but this is the function used to construct scattering matrices (see e.g. [5]). It is known that, at least in the complex case, the elements of the scattering matrix are boundary values of holomorphic functions and can be considered as microfunctions whose support gives the direction of causality of the interaction. To extend this approach in the quaternionic case would be very helpful, since it allows to consider various directions of the interactions but, at the same time, we have that the non–regularity of the exponential function used in this case is an obstruction to restore the theory.
Problem: It is possible to prove that the elements of the quaternionic scattering matrix are boundary values of regular functions (in the sense of the hyperfunctions which we have introduced in[12])?

4.10 Both physical and mathematical considerations seem to imply that it

may be interesting to carry out our kind of analysis in situations that, at least formally, seem to be analogous to the quaternionic case. In [16] Shapiro and Kravchenko study the so–called Moisil–Theodoresco system which is the 3–dimensional analogue (and historical precursor) of the Cauchy–Fueter system. Since functions in the kernel of the Moisil–Theodoresco system have an intrinsic physical interest and since they have a natural regular extension to all of $\mathbb{H}$, we are interested in the following problems :
Problem: Apply our algebraic analysis to the Moisil–Theodoresco system.
Problem: Is it possible to use these results to get information on the Cauchy–Fueter system?

4.11 Recently, the authors with a group of collaborators have started to study some hypercomplex operators that are the linearization of some ultra–hyperbolic operators. The importance of these operators (wave operators) in Physics is well known. In [8] we have introduced an operator whose kernel contains, as a particular case the electromagnetic fields, i.e. functions with values in the space of biquaternions, solutions of the Maxwell's system. In the forthcoming [9] we provide the analysis of an operator that linearizes the Klein–Gordon equation using octonions. In this setting there are several open problems.
Problem: Given a hypercomplex algebra study all the possible families of operators linearizing the wave operator.
Problem: Study the fundamental solution of such operators and provide integral formulas for the functions in their kernel.

4.12 The original idea of Fueter was to consider holomorphic functions of two variables as special cases of regular functions of one quaternionic variable. An analogous approach can be taken by considering regular functions of two quaternionic variables as special cases of regular functions of some other kind of variable. For example, regular functions in two quaternionic variables can be seen as regular functions of one biquaternionic variable ([8]) or as regular functions of one suitable Clifford variable [27]. So we can pose the following general
Problem: Given an hypercomplex algebra $\mathcal{A}$ and the notion of $\mathcal{A}$–regularity, find a hypercomplex algebra $\mathcal{C}$ and a corresponding notion of $\mathcal{C}$–regularity such that every $\mathcal{A}$–regular function of several variables is a particular case of $\mathcal{C}$–regular function of one variable.

4.13 An octonionic analogue of the theory of regular functions has been developed by Sce and Dentoni in [10] and more recently revisited by Nono in [23]. There are at least two possible ways in which this approach can be followed up. On one hand, Sce, Dentoni and Nono defined a Cauchy–Fueter–Cayley operator which linearizes the laplacian in analogy of what happen in $\mathbb{C}$ and $\mathbb{H}$. In [8] however we have shown how biquaternions can be utilized to factorize more interesting ultra–hyperbolic operators and therefore it seems reasonable to introduce a notion of bioctonions to linearize similar higher dimension operators. Such an approach is currently being taken in [9]. On the other hand, it has been suggested by Ehrenpreis that it may be interesting to utilize our approach to develop a theory of regular functions of several octonion variables.
Problem: Develop such a theory, compute the flabby dimension of the sheaf of octonionic regular functions, and construct an analogue of the Moisil–Theodoresco operator in such a context.

The list of problems in this framework is far from being complete. It is our hope that this paper will stimulate further researches in the area.

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