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On the "grouping" phenomenon for holomorphic solutions of infinite order differential equations

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1. Introduction

Purpose of this paper is to clarify some interpolation questions related to the space $\mathcal{H}'(\Omega)$ of analytic functionals carried by a compact set K contained in the convex open set $\Omega \subseteq \mathbb{C}$. In particular, I will address the issue of "grouping of terms" which arises when one looks for exponential series representations of solutions, in the space $\mathcal{H}(\Omega)$ of holomorphic functions on Ω, of systems of infinite order differential equations.

Virtually all of the material contained in this paper originates from [2], [5] and [6] (but see also the recent monograph [1]); on the other hand, we are presenting this material in a unified fashion as a background to a joint work with T. Kawai [7] on the application of these ideas to some very classical overconvergence results ([3], [4]). Of some interest, hopefully, is the final construction of a large class of examples of infinite order differential operators for which "grouping" is necessary.

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2. Basic Definitions

Let p a subharmonic function, $p : \mathbb{C} \longrightarrow \mathbb{R}$, which satisfies the following conditions:
(i) $p(z) \geq 0$, $\log(1 + |z^2|) = O(p(z))$;
(ii) there exists $C > 0$ such that if $|z - \zeta| \leq 1$, then

$$p(\zeta) \leq Cp(z) + C.$$

Such a function will be said to be a *weight*, and following Hörmander, see [1], we define the space

$$A_p = A_p(\mathbb{C}) = \{f \in \mathcal{H}(\mathbb{C}) \ : \ \text{for some } A, B > 0, |f(z)| \leq Ae^{Bp(z)}\}.$$

Remark 1. The main example we are interested in, is the case in which $p(z) = |z|$. In this case $A_p(\mathbb{C})$ is the space $Exp(\mathbb{C})$ of entire functions of exponential type, which is isomorphic (via Fourier–Borel Transform) to the space $\mathcal{H}'(\mathbb{C})$ of analytic functionals (Paley–Wiener theorem).

Remark 2. For every $B > 0$, one can consider the Banach space

$$A_{p,B} = \{f \in \mathcal{H}(\mathbb{C}) \ : \ \|f\|_B := \sup_{\mathbb{C}} |f(z)e^{-Bp(z)}| < +\infty\}.$$

By considering an increasing, diverging sequence $\{B_n\}_{n\geq 1}$ of positive numbers, one introduces an LF–topology (inductive limit of a sequence of Fréchet spaces) on A_p by the equality

$$A_p = \operatorname{ind}\lim_n A_{p,B_n}.$$

Note that such topology is obviously independent of the sequence $\{B_n\}$. In this topology, closure is equivalent to sequential closure.

Simultaneously with A_p, one may consider the space

$$A_{p,0} = A_{p,0}(\mathbb{C}) = \{f \in \mathcal{H}(\mathbb{C}) \ : \ \forall \varepsilon > 0 \ \exists A_\varepsilon > 0 \text{ such that } |f(z)| \leq A_\varepsilon e^{\varepsilon p(z)}\}.$$

Functions in this space are said to be of minimal type with respect to $p(z)$. If, as before, $p(z) = |z|$, then $A_{p,0}$ is the space $Exp_0(\mathbb{C})$ of functions of infraexponential type (order 1 and type 0), which is isomorphic, (again via the Fourier–Borel transform) to the space $(\mathcal{H}(\{0\}))'$ of analytic functionals carried by the origin or, equivalently, to the space $\mathcal{B}_{\{0\}}$ of hyperfunctions supported by the origin.

Remark 3. $A_{p,0}$ is a Fréchet space whose norms are given by

$$\|f\|_m = \max_{\mathbb{C}} \left\{ \{|f(z)| \exp\left(-\frac{p(z)}{m}\right)\right\}$$

One immediately sees that $A_{p,0}$ is an FS–space.

Remark 4. If, instead of the space $\mathcal{H}(\mathbb{C})$ of entire functions, one considers the space $\mathcal{H}(\Omega)$ of holomorphic functions on a convex open set $\Omega \subseteq \mathbb{C}$, the spaces A_p and $A_{p,0}$

are not sufficient anymore to describe analytic functionals, and some modifications are needed. Indeed, the space $\mathcal{H}'(\mathbb{C})$ of analytic functionals carried by some compact contained in Ω is isomorphic to the space $Exp(\Omega)$ of *entire* functions for which there exists a compact set $K \subseteq \Omega$ and a constant $A > 0$ such that $|f(z)| \leq A\exp(H_K(z))$ where

$$H_K(z) = \sup_{w \in K} Re(z \cdot w)$$

is the so–called supporting function of K. The arguments are given in the next sections for the spaces $A_p(\mathbb{C})$ and $Exp(\mathbb{C})$, need to be suitably modified, if one wants to apply them to the spaces $Exp(\Omega)$. Unfortunately, in this latter case, the estimates and the results are not so clear as in the A_p case. Note, as well, that $Exp(\Omega)$ is not an algebra anymore.

3. Interpolation

A multiplicity variety in $\mathbb{C}$ is a collection $V = \{(z_k, m_k), m_k \geq 1\}$ of pairs with $z_k \in \mathbb{C}$, $m_k \in \mathbb{N}$, and $|z_k| \to +\infty$. Weierstrass' theorem ensures that, given a multiplicity variety V, there is always an entire function f such that $V = V(f)$. The space $A(V) := \mathcal{H}(\mathbb{C})/I(f)$ of holomorphic functions on V can be identified with the space

$$A(V) = \{\{a_{k,l}\} \ : \ a_{k,l} \in \mathbb{C},\ 0 \leq l \leq m_k - 1,\ l \in \mathbb{N}\}.$$

There is a natural surjective restriction map

$$\rho : \ \mathcal{H}(\mathbb{C}) \longrightarrow A(V)$$

defined by

$$\rho : \ g \longrightarrow \left\{ \frac{1}{l!} \frac{\partial^l g}{\partial z^l}(z_k) \right\}_{k,l}.$$

The fundamental problem which arises in the theory of mean–periodic functions (i.e. functions which are solutions of a system of convolution equations) is the identification of $\rho(A_p(\mathbb{C}))$. To this purpose, we begin by defining a first "candidate" for it.

Definition. The space $A_p(V)$ is defined as the set of all sequences $\{a_{k,l}\}$ in $A(V)$ such that, for some $A, B > 0$ and all $k \geq 1$,

$$\sum_{0 \leq l \leq m_k} |a_{k,l}| \leq A \exp(Bp(z)).$$

Remark 5. Using Cauchy's inequalities for holomorphic functions, and the property of weight functions, it is immediate to deduce that

$$\rho(A_p(\mathbb{C})) \subseteq A_p(V).$$

Remark 6. With reference to Remark 4, one can also define the space $Exp_\Omega(V)$ as the space of all the sequences $\{a_{k,l}\}$ in $A(V)$ for which there exists a compact $K \subseteq \Omega$ and a constant $A_\varepsilon > 0$ such that

$$\sum_{0 \leq l \leq m_k} |a_{k,l}| \leq A \exp(H_K(z_k)).$$

Also in this case it is easy to verify that $\rho(Exp(\Omega)) \subseteq Exp_\Omega(V)$.

Definition. A multiplicity variety V is said to be interpolating (with respect to the weight p) if

$$\rho(A_p(\mathbb{C})) = A_p(V).$$

This definition (and a similar one which could be given for $Exp(\Omega)$) immediately raises two questions which we will address in the rest of the paper:
(a) find conditions for V to be interpolating;
(b) describe $\rho(A_p(\mathbb{C}))$.

While a complete solution to (a) and (b) does not exists for the most general case, things are fairly simple when one considers the case $p(z) = |z|$. In fact, one has the following result (see [1] and [2] for many related results as well):

Theorem 1. *Let $f \in Exp(\mathbb{C})$. Then $V(f)$ is interpolating if and only if there are constants $A, B > 0$ such that*

$$\left| \frac{1}{m_k!} \frac{\partial^{m_k} f}{\partial z^{m_k}}(z_k) \right| \geq A e^{-B|z_k|} \tag{1}$$

for all $k = 1, 2, \ldots$.

Remark 7. Theorem 1 is not true (without additional assumptions) for weights which are not radial and satifying the doubling condition $p(2z) = O(p(z))$. For every weight p, however, condition (1) is still sufficient.

Remark 8. The reason why $p(z) = |z|$ behaves differently is the fact that, in $Exp(\mathbb{C})$, we have the Lindelöf theorem which claims that if $g, f \in Exp(\mathbb{C})$ and $\frac{g}{f}$ is entire, then $\frac{g}{f}$ is still in $Exp(\mathbb{C})$. This can be rephrased by saying that the ideal $I(f)$ is closed in $Exp(\mathbb{C})$. This property is not true anymore for different weights (unless conditions on f are imposed), nor it is true that any non–principal ideal in $Exp(\mathbb{C})$ is closed.

An (almost) immediate consequence of Theorem 1 is the following necessary condition for interpolation, which we will be using shortly:

Theorem 2. *Let* $f \in Exp(\mathbb{C})$. *Then if* $V(f)$ *is interpolating, there are positive constants* A, B *such that, if we set* $d_k := \min\{1, \min_{i \neq k} |z_i - z_k|\}$, *then*

$$d_k^{m_k} \geq Ae^{-B|z_k|}. \tag{2}$$

Remark 9. Once again, this theorem is not valid in arbitrary A_p spaces (or in $Exp(\Omega)$), unless additional conditions are imposed on f.

In order to describe $\rho(Exp(\mathbb{C}))$ for the case in which the variety V is non–interpolating, we need to discuss an important property of entire functions of exponential growth, which is essentially the basis for Lindelöf theorem.

Theorem 3. *Let* $f \in Exp(\mathbb{C})$, $f(0) = 1$. *Then there exists a sequence of radii* $\{R_n\}$, $R_n \uparrow +\infty$, $R_{n+1} \leq 4R_n$, *such that for some positive constants* A, B

$$\min_{|z|=R_n} |f(z)| \geq A\exp(-BR_n).$$

The proof of this result is essentially a consequence of the minimum modulus theorem for holomorphic functions. The result is crucial is establishing the closure of the ideal generated by f in $Exp(\mathbb{C})$, and also plays a crucial role in constructing the image $\rho(Exp(\mathbb{C}))$. Since, in the sequel, we will be considering infinite order differential equations, we now consider an element $P(z) \in Exp_0(\mathbb{C}) \subseteq Exp(\mathbb{C})$, and we take a sequence $\{R_n\}$ of radii such that

$$\min_{|z|=R_n} |P(z)| \geq A\exp(-BR_n),$$

as given by theorem 3. Finally, we denote by

$$V = \{\{\ \alpha_k, m_k\},\ k = 1, 2, \ldots,\ m_k \geq 1\}$$

the multiplicity variety associated to $P(z) = 0$, and we assume that the roots α_k have been ordered so that $\ldots \leq |\alpha_k| \leq |\alpha_{k+1}| \leq \ldots$.

Let

$$V_n = \{(\alpha_k, m_k)\ :\ k_n \leq k < k_{n+1}\}$$

be the subset of V for which $\alpha_k \in C_n$, where

$$C_1 = \{z\ :\ |z| < R_1\}$$

and

$$C_n = \{z\ :\ R_{n-1} < |z| < R_n\}, \qquad n \geq 2.$$

Now, given any function φ holomorphic on C_n, one can construct the interpolating function

$$\varphi_n(z) := \frac{1}{2\pi i}\int_{\partial C_n} \frac{\varphi(\zeta)}{P(\zeta)} \cdot \frac{P(\zeta) - P(z)}{\zeta - z} d\zeta$$

defined and holomorphic on C_n.

One immediately sees that

$$\varphi_n(z) = \varphi(z) - \frac{P(z)}{2\pi i}\int_{\partial C_n} \frac{\varphi(\zeta)}{P(\zeta)} \cdot \frac{d\zeta}{\zeta - z} =$$

$$= \varphi(z) + P(z)\psi_n(z)$$

with ψ_n holomorphic on C_n. As a consequence we have that

$$\rho_n(\varphi_n) = \rho_n(\varphi) = \{a_{k,l}^{(n)}\} = \{a_{kl} \ : \ \alpha_k \in C_n\}.$$

If we now define, on $A(V_n)$ the norm

$$\|a^{(n)}\|_n := \|a_{k,l}^{(n)}\|_n = \inf\{\|\varphi\|_\infty \ : \ \varphi \in A(C_n), \ \rho_n(\varphi) = \{a_{k,l}^{(n)}\}\},$$

we see that, for any $\varphi \in Exp(\mathbb{C})$,

$$\|\rho_n(\varphi)\|_n \leq A \exp(B|z|), \quad z \in C_n,$$

for some positive constants A, B, which do not depend on n. Now, if $\underline{a} \in A(V)$, $\underline{a}$ can be written as a sequence

$$\underline{a} = \{a_n\}_n, \ a_n \in A(V_n),$$

and we can define

$$A_*(V) = \{\underline{a} = \{a_n\} \in A(V) \ : \ \exists D > 0 \text{ s.t. } \|\underline{a}\|_D := \sup_n \|a_n\|_n e^{-DR_n} < +\infty\}$$

and the following crucial result can be established [2]:

Theorem 4. *There is a topological isomorphism between $\rho(Exp(\mathbb{C}))$ and $A_*(V)$, where $A_*(V)$ is endowed with its natural inductive limit topology. In particular,*

$$\frac{Exp(\mathbb{C})}{I(P)} \cong A_*(V).$$

4. Exponential representation

The preliminating material which we have discussed in the previous sections, will now allow us to prove the representation theorem for holomorphic (entire) solutions for an infinite order differential equation. We recall, in this respect, that the space of infinite order differential operators is, in fact, isomorphic to the space $Exp_0(\mathbb{C})$ of entire functions of infraexponential type. Specifically, any infraexponential type function $P(z)$ acts as a symbol for a differential operator $P(d/dz)$, and we have the following result:

Theorem 5. *Let $P \in Exp_0(\mathbb{C})$, and let*

$$V = \{z \ : \ P(z) = 0\} = \{(\alpha_k, m_k) \ : \ |\alpha_1| \leq |\alpha_2| \leq \ldots\}.$$

Then there exists a sequence a sequence of indices $k_1 < k_2 < \ldots$ such that every entire solution f of

$$P(D)f = 0$$

can be written an

$$f(z) = \sum_{n \geq 1} \left(\sum_{k_n \leq k < k_{n+1}} P_k(z) e^{\alpha_k z} \right),$$

where P_k is a polynomial of degree less than m_k, and where the sequence $\{P_k\}$ of such polynomials satisfies growth conditions necessary to the convergence of the grouped series on the right hand side of (3).

Proof. First, we consider the map

$$\beta := \rho \cdot \mathcal{F} : \ \mathcal{H}'(\mathbb{C}) \longrightarrow \frac{Exp(\mathbb{C})}{I(P)}$$

where $\mathcal{F}$ is the Fourier–Borel isomorphism between $\mathcal{H}'(\mathbb{C})$ and $Exp(\mathbb{C})$. It is immediate to see that β is a continuous surjective map between DFS–spaces. Then, by a well known result from functional analysis, its transpose map

$$\beta^t : \left(\frac{Exp(\mathbb{C})}{I(P)} \right)' \longrightarrow \mathcal{H}(\mathbb{C})$$

has closed image, and is a topological isomorphism onto its own image. Moreover, $Im(\beta^t) = (ker\beta)^{\perp}$ and since $\mathcal{F}(ker\beta) = I(P)$, we obtain (since $\mathcal{F}$ is an isomorphism) that $ker\beta = P(D)(\mathcal{H}'(\mathbb{C}))$, and so, finally,

$$(ker\beta)^{\perp} = \{f \in \mathcal{H}(\mathbb{C}) \ : \ P(D)f = 0\}$$

is the kernel in $\mathcal{H}(\mathbb{C})$ of the infinite order differential operator $P(D)$. Since we know that $Exp(\mathbb{C})/I(P)$ is topologically isomorphic to $A_*(V)$, our proof will be concluded by simply describing the dual $(A_*(V))'$ of $A_*(V)$. To do so, let $\underline{b} \in (A_*(V))'$; then $\underline{b} = \{b_n\}_{n\geq 1}$ with $b_n \in (A(V_n))'$ for any n, and with norms $\|\cdot\|'_n$ in these dual spaces given by

$$\|b_n\|'_n := \max\{\langle a_n, b_n\rangle \ : \ \|a_n\|_n \leq 1\},$$

where

$$\langle a_n, b_n\rangle := \sum_{k,l} a_{k,l}^{(n)} b_{k,l}^{(n)}$$

with (hopefully) obvious meaning of the symbols. On the other hand, if $\underline{a} = \{a_n\}_{n\geq 1}$ is an element of $A_*(V)$, then there exists $C > 0$ such that

$$\sup_n \|a_n\|_n e^{-CR_n} < +\infty,$$

and therefore if $\underline{b}$ is an element of $(A_*(V))'$, then we have that for every $D > 0$,

$$\|\underline{b}\|' := \sum_{n\geq 1} \|b_n\|'_n e^{DR_n} < +\infty$$

and now the duality bracket is obviously defined by

$$\langle \underline{a}, \underline{b}\rangle = \sum_{n\geq 1} \langle a_n, b_n\rangle.$$

Consider now a solution f of the differential equation

$$P(D)f = 0;$$

then there exists a unique $\underline{b} \in (A_*(V))'$ such that $f = \beta^t(\underline{b})$. Then, for every $S \in \mathcal{H}'(\mathbb{C})$,

$$\langle S, f\rangle = \langle S, \beta^t(\underline{b})\rangle = \langle \beta(S), \underline{b}\rangle = \langle \rho(\mathcal{F}(S)), \underline{b}\rangle,$$

with the series at the right side which (by the definition of the topology in $\mathcal{H}'(\mathbb{C})$) converges uniformely for every bounded family of analytic functionals S. In particular, for z in a fixed compact set in $\mathbb{C}$, we can take $S = \delta_z$. Then

$$\rho(\mathcal{F}(\delta_z)) = \rho(e^{z\omega}) = \left\{\frac{z^j e^{\alpha_k z}}{j!}\right\}_{k\geq 1,\ 0\leq j\leq m_k}$$

so that

$$\langle \delta_z, f\rangle = f(z) = \sum_{n\geq 1}\left(\sum_{k_n\leq k<k_{n+1};\ 0\leq j<m_k} b_{k,j}\frac{z^j e^{\alpha_k z}}{j!}\right)$$

with uniform convergence of the series on each compact set in $\mathbb{C}$.

■

Now that we have completed the proof, we would like (in order to prepare for the examples in the next section) to see whether it is possible to make the estimates on the coefficients $\{b_{k,j}\}$ a little more explicit. For the case in which V is an interpolating variety, the situation is quite simple, since $A_*(V) = A_p(V)$ ($= Exp(V)$ in our case), and therefore there are no more groupings. Indeed, if V is interpolating, it possible to choose the circles $\{R_n\}$ in such a way that (discounting multiplicities), only one zero of $P(z)$ falls between R_n and R_{n+1}, for every n. In this case we see that

$$((Exp(V))' = \left\{ \underline{b} = \{b_{k,j}\}\ s.t.\ \forall D > 0,\ \sum_{k,j} e^{B|\alpha_k|}(\sum_{o\leq j<m_k} |b_{k,j}|) < +\infty \right\}$$

which is as explicit as we can get, and gives us the necessary and sufficient condition for the convergence of the series (with no groupings).

In the case in which the variety is non–interpolating, groupings may become necessary, and the precise description of the bounds becomes much subtler. The method which Berenstein and Taylor used in [2] (see also [1]) consists in using Newton's divided differences as a way to compute explicitely a choice for an interpolating function on V. The method is a bit complicated, but our purposes will be satisfied if we quote the result, at least for the case of groupings whose points all have multiplicity one (some further modifications would be necessary for the general case).

Assume we have, in V_n, the distinct simple points

$$\alpha_1^{(n)},\ \alpha_2^{(n)}, \ldots,\ \alpha_{k_n}^{(n)}.$$

Construct the $t_n \times t_n$ square matrix $J = J^{(n)}$ (where the suffix $\cdot^{(n)}$ has been eliminated troughout its entries for the sake of readibility):

$$J = \begin{bmatrix} 1 & 1 & 1 & \dots \\ 0 & \alpha_2 - \alpha_1 & \alpha_3 - \alpha_1 & \dots \\ 0 & 0 & (\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1) & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix};$$

it should be noted that the construction of this matrix can be described explicitely in terms of the divided differences of different orders at the points in V_n, [1]. Given this matrix, the estimates on $\underline{b} = \{b_n\}$ become quite explicit. Specifically, $\underline{b}$ is an element of $(A_*(V))'$ if for any $D > 0$, and every choice of points $\beta_n \in V_n$,

$$\sum_{n \geq 1} \left(\sum_{j=1}^{t_n} |c_j^{(n)}| \right) e^{D|\beta_n|} < +\infty,$$

where the vector $\underline{c}^{(n)}$ is defined by $\underline{c}^{(n)} = J \cdot b_n$.

5. Examples of the necessity of groupings.

In this final section I will show how to construct a large class of examples of differential operators of infinite order such that the exponential representation of their entire solutions requires necessarily the use of groupings. Examples of this kind (but with different growth conditions) have been given by Schwartz and by Leont'ev (see [8] or [1] and the references therein). Our simple, but explicit, construction has been, however, the starting point for the forthcoming work of Kawai and the author [7], in which we examine a new treatment of Bernstein–Ostrowski overconvergence phenomena [3], [4] based on the properties of infinite order differential operators. In principle, the idea for the construction for such an example is simple. Since $ker(P(D)) = (A_*(V))'$, and $(A_*(V))' = Exp(V)$ if and only if V is interpolating, then one may assume that the "grouping" phenomenon will occur whenever the variety V is not interpolating, and it is therefore sufficient to construct a variety V, whose points are the only (simple) zeroes of a function P of infraexponential type, and such that V is not interpolating. However, the condition of V being interpolating is only sufficient to guarantee that no groupings are necessary. In fact, while we have no examples to the contrary in the case of varieties of infraexponential type, we know that if $p(z) = |Imz| + \log(1 + |z|)$, it is possible to find (and characterize) classes of varieties which, even though they are not interpolating, they still define convolutions operators whose solutions can all be represented as exponential series, whose convergence does not require grouping procedures. These varieties are called "weakly interpolating", [2], and, for the case of $p(z) = |z|$, it is known that the notion of interpolating and weakly interpolating varieties coincide.

To construct the variety we need, consider a sequence α_k, $|\alpha_k| \uparrow +\infty$ of zero density (i.e. $\lim_{n\to+\infty} \frac{n}{\alpha_n} = 0$) so that we know that there exists $P_1 \in Exp_0(\mathbb{C})$ such that $V(P_1) = \{\alpha_k\}$. We now choose, for every k, a quantity ε_k such that for any $A, B > 0$, $|\varepsilon_k| < Ae^{-B|\alpha_k|}$. This can be easily done by taking, for example, $\varepsilon_k = e^{-|\alpha_k|^2}$. Now consider the new sequence of points

$$\beta_k := \alpha_k + \varepsilon_k.$$

Obviously

$$\lim_{n\to+\infty} \frac{n}{\beta_n} = \lim_{n\to+\infty} \frac{n}{\alpha_n + \varepsilon_n} = 0,$$

and so we know that a second function P_2 exists in $Exp_0(\mathbb{C})$, such that $V(P_2) = \{\beta_k\}$. Now the function

$$P = P_1 \cdot P_2$$

is still in $Exp_0(\mathbb{C})$, which is an algebra, and $V(P) = \{\alpha_k\} = \{\beta_k\}$. On the other hand, we may use Theorem 2 to immediately see that V is not an interpolating variety, since condition (2) is violated for $d_k = \varepsilon_k$. I claim that the operator whose symbol is P is such that groupings are necessary to represent all the solutions to $P(D)f = 0$. To see this, we will assume that each sequence $\{\alpha_k\}$ and $\{\beta_k\}$ is, by itself, interpolating. This request is not strictly necessary, but will make our computations easier. In order to ensure that $\{\alpha_k\}$ is interpolating, it will be enough (see [1], for a discussion of this well known result of Levinson) to ask that, for example,

$$|\alpha_k - \alpha_h| \geq 2|k - h|$$

for all indexes h and k. In this way we are also guaranteed that an analogous estimate holds for the sequence $\{\beta_k\}$, which is therefore automatically interpolating. This shows that if we apply Theorem 5 to the equation

$$P(D)f = P_1(D)(P_2(D)f) = 0,$$

then the groupings produced by the proof (but which, a priory, may still not be necessary) are composed of two roots each, namely we are grouping the pairs $\{(\alpha_k, \beta_k)\}$ for any value of k (the reader may want to compare this example with the one given by Bernstein in the second chapter of [3]). Thus every entire solution of

$$P(D)f = 0$$

can be represented by the grouped series

$$f(z) = \sum_{k=1}^{+\infty} \left(A_k e^{\alpha_k z} + B_k e^{\beta_k z}\right)$$

where the coefficients A_k and B_k must satisfy suitable growth conditions to ensure that the series is convergent in the topology of $\mathcal{H}(\mathbb{C})$.

In order to determine the exact nature of these growth conditions, we proceed to construct the matrices $J^{(k)}$ for this specific case (note that here we have $t_k = 2$ for all values of $k = 1, 2, \ldots$). We immmediately see that it is

$$J^{(k)} = \begin{bmatrix} 1 & 1 \\ 0 & \alpha_k - \beta_k \end{bmatrix}$$

and that the corresponding element b_n is given by $b_k = (A_k, B_k)$ and $\underline{b} = \{b_k\}$ is an element of $(A_*(V))'$ if and only if $\vec{c} = J^{(k)} \cdot b_k$ satisfies (4). A simple matrix multiplication shows that

$$\vec{c} = (A_k + B_k, (\alpha_k - \beta_k)B_k)$$

and therefore the estimates (4) give (see Remark 10 below)

$$\lim_{k\to+\infty} \frac{\log|A_k+B_k|}{|\alpha_k|} = \frac{\log|A_k+B_k|}{|\beta_k|} = -\infty \tag{5}$$

and

$$\lim_{k\to+\infty} \frac{\log|\alpha_k-\beta_k||\beta_k|}{|\alpha_k|} = \frac{\log|\alpha_k-\beta_k||\beta_k|}{|\beta_k|} = -\infty. \tag{6}$$

Remark 10. Note that (4) can be rewritten as

$$\sum_{n\geq 1} \|\underline{c}^{(n)}\| e^{D|\beta_n|} < +\infty$$

for any $D > 0$ and any choice of points $\beta_n \in V_n$. This condition is obviously equivalent to

$$\|\underline{c}^{(n)}\| e^{D|\beta_n|} = O(1).$$

Since all norms are equivalent in finite dimensionale spaces, $\|\underline{c}^{(n)}\|$ can be interpreted as either l^1 or l^∞ norm; if we now take the logarithm, we obtain (5) and (6).

Remark 11. Condition (6) looks misleadingly asymmetric. However, since

$$|A_k| \leq |A_k+B_k| + |-B_k|,$$

we see that (6) actually is equivalent to a condition in which $|B_k|$ is replaced by $|A_k|$.

Now note that since $|\alpha_k-\beta_k| = e^{-|\alpha_k|^2}$, we obtain

$$\log|\alpha_k-\beta_k| = -|\alpha_k|^2,$$

and therefore condition (6) becomes

$$\lim_{k\to+\infty} -|\alpha_k||\beta_k| = -\infty,$$

which is satisfied (for example) by any sequence $\{\beta_k\}$ bounded away from zero.

At this point we see that the choice $\{A_k = 1\}$, $\{B_k = 1\}$ satisfies both (5) and (6); as a consequence the series

$$f(z) = \sum_{k=1}^{+\infty} (e^{\alpha_k z} - e^{\beta_k z})$$

is an entire solution to

$$P(D)f = 0.$$

However, if we were to remove the condition of "grouping" in the series, we would lose convergence immediately. Indeed, if the series were convergent without groupings, its general term would be bounded at every point. But now, by simply taking $z = 1$, we have, with $A_k = 1$,

$$|A_k e^{\alpha_k z}| = |e^{\alpha_k}| \longrightarrow +\infty$$

as $k \to +\infty$.

This simple argument shows that groupings cannot be eliminated a priori from the statement of Theorem 5. In the terminology of [3] (but see also [7]) we say that the abscissa of convergence of the series $\sum_{k=1}^{+\infty} a_k e^{\gamma_k z}$ (for $a_k = (-1)^{k+1}$, $\gamma_{2k-1} = \alpha_k$, $\gamma_{2k} = \beta_k$) is zero, while its abscissa of overconvergence is $-\infty$.

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