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The generalized Whittaker functions for $SU(2,1)$

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Introduction.

In the theory of automorphic forms, Fourier expansion of modular forms is a fundamental tool for investigation. For example, coefficients of the expansion can be used for construction of L-functions. In spite of this importance, the theory of Fourier expansion of automorphic forms seems still in very primitive state.

Our concern is to have a theory of fully developed Fourier expansion of modular forms on $SU(2,1)$, the real special unitary group of signature $(2+,1-)$. To have such theory we need Whittaker functions and generalized Whittaker functions of the standard representations of $SU(2,1)$. A quite explicit result is obtained by Koseki-Oda [K-O] for Whittaker functions. The remaining problem for our purpose is to consider the generalized Whittaker functions. This is the theme of the present paper.

The peculiarity of the case of $SU(2,1)$, different from the case of $SL_2(\mathbb{R})$, is that the maximal unipotent subgroup N is not abelian. It is isomorphic to the Heisenberg group of dimension three, and has infinite dimensional irreducible unitary representations σ, which are called Stone von Neumann representations. Together with unitary characters they constitute the unitary dual of N. The Fourier expansion of automorphic forms on $SU(2,1)$ is to consider irreducible decomposition of the restriction $\pi|_N$ of automorphic representations π with respect to N. Therefore we have to handle those terms which corresponds to the Stone von Neumann representations.

Naive formulation of the problem is to investigate intertwiners in $\mathrm{Hom}_N(\pi|_N, \sigma)$ which is isomorphic to $\mathrm{Hom}_G(\pi, \mathrm{Ind}_N^G \sigma)$ by Frobenius reciprocity. But this fails in general, because the intertwining space in question is infinite dimensional. The right formulation of the problem is given by introducing a larger group R containing N.

Here is the formulation of our main result. Let P be the minimal parabolic subgroup of $SU(2,1)$ with Levi decomposition $L \ltimes N$. And let S be the maximal closed subgroup of L which acts trivially on the center $Z(N)$ of N. The group R is the semidirect product S and N. We want to investigate the intertwining space $\mathrm{Hom}_G(\pi, \mathrm{Ind}_R^G \eta)$ for certain irreducible unitary representation η of R, and the images of intertwiners: these are the space of generalized Whittaker functionals and the space of generalized Whittaker functions, respectively. Our main results are to obtain an explicit formula for the radial part of such generalized Whittaker functions with special K-type, and to show the multiplicity one theorem for the intertwining space (Theorem 7.2.1, Theorem 8.2.1).

Our main concern is the theory of automorphic forms. However the author believes that our results is also interesting for the problem of realization in generalized Gel'fand-Graev representations.

1 The structure of Lie groups and algebras

1.1 The Iwasawa decomposition

We realize the identity component of the stabilizer group $SU(2,1)$ of the Hermitian form of three variables with signature $(2+,1-)$ as follows

$$SU(2,1) := \{g \in SL(3,\mathbb{C}) \mid {}^t\bar{g} I_{2,1} g = I_{2,1}\},$$

where $I_{2,1} := \mathrm{diag}(1,1,-1)$. We denote the group by G. Let

$$G = NAK$$

be the Iwasawa decomposition of G, then

$$K \cong S(U(2)\times U(1)), \qquad N \cong H(\mathbb{R}^2),$$

$$A = \{a_r := \begin{pmatrix} \frac{r+r^{-1}}{2} & & \frac{r-r^{-1}}{2} \\ & 1 & \\ \frac{r-r^{-1}}{2} & & \frac{r+r^{-1}}{2} \end{pmatrix} \mid r \in \mathbb{R}_{>0}\}.$$

Here $H(\mathbb{R}^2)$ denotes the Heisenberg group of dimension 3.

Denote the Lie algebra The Lie algebra $\mathfrak{su}(2,1)$ of G by $\mathfrak{g}$ and let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

be the Cartan decomposition corresponding to the Cartan involution $\theta : X \mapsto I_{2,1} X I_{2,1}^{-1}$. Since G/K is Hermitian, we have a decomposition $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ such that $\mathfrak{p}_+$ is identified with the holomorphic tangent space at the origin $1 \cdot K \in G/K$, corresponding to the complex structure of G/K. Put

$$\mathfrak{a} = \mathbb{R}\cdot H, \qquad \mathfrak{n} = \mathbb{R}E_1 \oplus \mathbb{R}E_{2,+} \oplus \mathbb{R}E_{2,-},$$

where

$$H := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad E_1 := i \begin{pmatrix} 1 & & -1 \\ & 0 & \\ 1 & & -1 \end{pmatrix},$$

$$E_{2,+} := \begin{pmatrix} & -1 & \\ 1 & & -1 \\ & -1 & \end{pmatrix}, E_{2,-} := \begin{pmatrix} & -i & \\ -i & & i \\ & -i & \end{pmatrix},$$

then

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$$

gives the Iwasawa decomposition of $\mathfrak{g}$.

1.2 Root system

We fix a compact Cartan subalgebra $\mathfrak{t}$ in $\mathfrak{k}$ and its basis by

$$\mathfrak{t} = \{\mathrm{diag}(\sqrt{-1}h_1, \sqrt{-1}h_2, \sqrt{-1}h_3) \mid h_i \in \mathbb{R},\ h_1 + h_2 + h_3 = 0\},$$

$$H'_{12} = \mathrm{diag}(1. -1, 0), \qquad H'_{13} = \mathrm{diag}(1.0, -1).$$

Define linear forms β_{ij} on $\mathfrak{t}_{\mathbb{C}}(i \neq j, 1 \leq i, j \leq 3)$ by

$$\beta_{ij} : \mathfrak{t}_{\mathbb{C}} \ni \mathrm{diag}(t_1, t_2, t_3) \mapsto t_i - t_j \in \mathbb{C}.$$

Then the root system Σ associated to $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ is given by $\Sigma := \{\beta_{ij} \mid i \neq j,\ 1 \leq i, j \leq 3\}$. We fix a positive root system $\Sigma_+ := \{\beta_{ij} \mid i < j\}$.

Let $\mathfrak{g}_\beta$ be the root space associated to $\beta \in \Sigma$: $\mathfrak{g}_\beta := \{X \in \mathfrak{g} \mid [H, X] = \beta(H)X,\ \forall H \in \mathfrak{t}_{\mathbb{C}}\}$ We denote Σ_c and Σ_n the sets of compact and noncompact roots, respectively. In our choice of coordinate,

$$\Sigma_c := \{\beta \in \Sigma \mid \mathfrak{g}_\beta \subset \mathfrak{k}_{\mathbb{C}}\}, \qquad \Sigma_n := \{\beta \in \Sigma \mid \mathfrak{g}_\beta \subset \mathfrak{p}_{\mathbb{C}}\}.$$

and matrix element E_{ij} $(1 \leq i, j \leq 3)$ generates the root space $\mathfrak{g}_{\beta_{ij}}$. We put

$$X_{\beta_{ij}} = \begin{cases} E_{ij} & \mathrm{when}(i, j) \neq (2, 1); \\ -E_{ij} & \mathrm{when}(i, j) = (2, 1), \end{cases}$$

and take it as a root vector in $\mathfrak{g}_{\beta_{ij}}$, because this is natural in the meaning that complex conjugation with respect to our choice of real form of $\mathfrak{sl}(3, \mathbb{C})$ converts two root vectors $X_{\beta_{ij}}$ and $X_{\beta_{ji}}$ mutually. Put $\Sigma_{c,+} := \Sigma_c \cap \Sigma_+$ and $\Sigma_{n,+} := \Sigma_n \cap \Sigma_+$.

2 Representation theory of the group R

2.1 Representations of R with nontrivial central characters

By the Stone von Neumann theorem, the unitary dual $\widehat{N}$ of N is exhausted by unitary characters and irreducible unitary representations which is determined by its central character ψ, up to unitary equivalence. Consider an infinite dimensional one $\sigma \in \widehat{N}$ with central character ψ. Let L be the Levi subgroup of P, and $Z(N)$ the center of N. Then L acts both on $\widehat{N}$ and $Z(N)$ by conjugation. Hence the stabilizer S of σ in L is the centralizer of $Z(N)$. In particular S is independent of σ. We define the group R by semidirect product as

$$R = S \ltimes N.$$

We extend σ to an irreducible unitary representation of R.

Because the action of S on N by conjugation is faithful, S can be regarded as a subgroup of the automorphism group of N. Passing to the abelianized subgroup $N^{ab} = N/[N:N]$ of N, we have $S \hookrightarrow \mathrm{Aut}\ N \to \mathrm{Aut}\ N^{ab}$. Since N^{ab} is identified with $\mathbb{R}^{\oplus 2}$, $\mathrm{Aut}\ N^{ab} \cong GL_2(\mathbb{R})$. Composing all these identifications, we get an isomorphism between S and $SO(2)$

$$\begin{array}{ccccccc} S & \hookrightarrow & \mathrm{Aut}N & \to & GL_2(\mathbb{R}) & \supset & SO(2), \\ \mathrm{diag}(\alpha, \beta, \bar{\alpha}^{-1}) & & & \mapsto & & & R(3\theta) \end{array}$$

by putting $\alpha = e^{i\theta}, \beta = e^{-2i\theta}, \theta \in [0, 2\pi)$, where $R(3\theta)$ means the rotation of angle 3θ.

We realize σ on $L^2(\mathbb{R})$ as

$$\rho_\psi\big((x,y;t)\big).\Phi(\xi) \;=\; \psi\big(t + \langle x + 2\xi, y\rangle\big) \cdot \Phi(\xi + x),$$

where $(x,y;t) \in H(\mathbb{R})$, $\Phi \in L^2(\mathbb{R})$ and $\langle\ ,\ \rangle$ is the natural symplectic form on $\mathbb{R}^2$. This is called *the Shrödinger model* of σ.
By the theory of Weil representations, we have the canonical extension

$$\omega_\psi \times \rho_\psi : \ \widetilde{Sp}_1(\mathbb{R}) \ltimes H(\mathbb{R}^2) \to \mathrm{Aut}(L^2(\mathbb{R})).$$

Here $\widetilde{Sp}_1(\mathbb{R})$ is the two-fold covering of $SL_2(\mathbb{R})$, $(\omega_\psi, L^2(\mathbb{R}))$ its Weil representation. Identifying S, N with $SO(2)$, $H(\mathbb{R}^2)$ respectively, the semidirect product $R = S \ltimes N$ is regarded as a subgroup of $\widetilde{Sp}_1(\mathbb{R}) \ltimes H(\mathbb{R}^2)$. Let $\tilde{R}$ be the pullback $\tilde{R} := \tilde{S} \ltimes N \cong \widetilde{SO}(2) \ltimes H(\mathbb{R}^2)$ of R by the covering

$$pr \times id \ : \widetilde{Sp}_1(\mathbb{R}) \ltimes H(\mathbb{R}^2) \twoheadrightarrow SL_2(\mathbb{R}) \ltimes H(\mathbb{R}^2).$$

Then tensoring an odd character $\tilde{\chi}$ of $\widetilde{SO}(2)$ to $(\omega_\psi \times \rho_\psi)|_{\tilde{R}}$ finally, we have a representation of R

$$\tilde{\chi} \otimes (\omega_\psi \times \rho_\psi)|_{\tilde{R}} \ : \ R = S \ltimes N \ \to \ \mathrm{Aut}(L^2(\mathbb{R})).$$

We denote this representation by $(\eta, L^2(\mathbb{R}))$. A character of $\widetilde{SO}(2)$ is called odd, if it does not factors through the covering $\widetilde{SO}(2) \twoheadrightarrow SO(2)$, which is usually parameterized by some elements $\mu = m + \frac{1}{2}$ in $\frac{1}{2}\mathbb{Z}\backslash\mathbb{Z}$ $(m \in \mathbb{Z})$.

Here is a diagram explaining the above construction

$$\begin{array}{ccccc} \tilde{R} = \tilde{S} \ltimes N & & \widetilde{Sp}_1(\mathbb{R}) \ltimes H(\mathbb{R}^2) & \xrightarrow{\omega_\psi \times \rho_\psi} & \mathrm{Aut}(L^2(\mathbb{R})) \\ \downarrow & & \downarrow{\scriptstyle pr\times id} & & \\ R = S \ltimes N & \longrightarrow & SL_2(\mathbb{R}) \ltimes H(\mathbb{R}^2). & & \end{array}$$

2.2 A basis of η and the action of Lie$N$

It is well known that Hermite functions $h_n(\xi) := (-1)^n e^{\xi^2/2} \cdot \frac{d^n}{d\xi^n} e^{-\xi^2}$, $n = 1, 2, 3, \ldots$ form an orthogonal Hilbert basis of $L^2(\mathbb{R})$.

We normalize tha acton of the generators $E_1, E_{2,\pm}$ of $\mathfrak{n}$ through ρ_{ψ_s} as

$$\rho_{\psi_s}(E_1).\Phi(\xi) = 2\sqrt{-1}s \cdot \Phi(\xi)$$

$$\rho_{\psi_s}(E_{2,+}).\Phi(\xi) = -\sqrt{2}\Phi'(\xi), \ \rho_{\psi_s}(E_{2,-}).\Phi(\xi) = -2\sqrt{2}is\xi \cdot \Phi(\xi)$$

on $\Phi \in L^2(\mathbb{R})$, where ψ_s is an additive character of $\mathbb{R} \ni t \mapsto e^{ist} \in U(1)$. Then easily seen,

Proposition 2.2.1 *The subspace of smooth vectors in $L^2(\mathbb{R})$ is the Schwartz space $\mathcal{S}(\mathbb{R})$, and the action of root vectors E_1, $E_{2,+}$, $E_{2,-}$ on $\mathcal{S}(\mathbb{R})$ through the underlining Harish-Chandra module of $\eta_{\mu,\psi_s} = \tilde{\chi}_\mu \otimes \tilde{\eta}_{\psi_s}$ are as follows:*

$$\eta(E_1).h_n = 2\sqrt{-1}s \cdot h_n$$

$$\eta(E_{2,+}).h_n = \frac{1}{\sqrt{2}} h_{n+1} - \sqrt{2}n \cdot h_{n-1}, \ \eta(E_{2,-}).h_n = -\sqrt{2}is \cdot h_{n+1} - 2\sqrt{2}isn \cdot h_{n-1}.$$

□

3 Representations of maximal compact subgroup

3.1 Parameterization of irreducible K-modules

The set L_T^+ of $\Sigma_{c,+}$-dominant T-integral weights is given by $L_T^+ = \{(m,n) \in \mathbb{Z}^{\oplus 2} | m \geq n\}$. For each $\mu = (\mu_1, \mu_2) \in L_T^+$, the vector space V_μ spaned by $\{v_k^\mu \mid 0 \leq k \leq d_\mu\}$ with $\mathfrak{k}_{\mathbb{C}}$-action as

$$\begin{aligned}
\tau_\mu(Z)v_k^\mu &= (\mu_1 + \mu_2)v_k^\mu, \\
\tau_\mu(H'_{12})v_k^\mu &= \{\mu - (d_\mu - k)\beta_{12}\}(H'_{12})v_k^\mu = (2k - d_\mu)v_k^\mu, \\
\tau_\mu(H'_{13})v_k^\mu &= \{\mu - (d_\mu - k)\beta_{12}\}(H'_{13})v_k^\mu = (k + \mu_2)v_k^\mu, \\
\tau_\mu(X_{\beta_{12}})v_k^\mu &= (k+1)v_{k+1}^\mu, \\
\tau_\mu(X_{\beta_{21}})v_k^\mu &= (k - d_\mu - 1)v_{k-1}^\mu.
\end{aligned}$$

gives an irreducible K-module (τ_μ, V_μ) via the highest weight theory.

3.2 Tensor products with $\mathfrak{p}_{\mathbb{C}}$

We regard the 4-dimensional vector space $\mathfrak{p}_{\mathbb{C}}$ as a $\mathfrak{k}_{\mathbb{C}}$-module via the adjoint representation ad. Then $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are invariant subspaces, and

$$\mathfrak{p}_+ = \mathbb{C}X_{\beta_{13}} \oplus \mathbb{C}X_{\beta_{23}} \cong V_{\beta_{13}}, \qquad \mathfrak{p}_- = \mathbb{C}X_{\beta_{32}} \oplus \mathbb{C}X_{\beta_{31}} \cong V_{\beta_{32}}.$$

Given an irreducible K-module V_μ we have $V_\mu \otimes \mathfrak{p}_{\mathbb{C}} = (V_\mu \otimes \mathfrak{p}_+) \oplus (V_\mu \otimes \mathfrak{p}_-)$, and Clebsch-Gordan's theorem tells us the following decomposition of $V_\mu \otimes \mathfrak{p}_\pm$:

$$V_\mu \otimes \mathfrak{p}_+ \cong V_{\mu+\beta_{13}} \oplus V_{\mu+\beta_{23}}, \quad V_\mu \otimes \mathfrak{p}_- \cong V_{\mu+\beta_{32}} \oplus V_{\mu+\beta_{31}}$$

Here we understand $V_\nu = (0)$ if $\nu \in L_T$ is not dominant. We hence have

$$V_\mu \otimes \mathfrak{p}_{\mathbb{C}} \cong V_\mu^+ \oplus V_\mu^-;$$
$$V_\mu^+ := V_{\mu+\beta_{13}} \oplus V_{\mu+\beta_{32}}, \qquad V_\mu^- := V_{\mu-\beta_{13}} \oplus V_{\mu-\beta_{32}},$$

under the above convention.

The decompositions of $V_\mu \otimes \mathfrak{p}_{\mathbb{C}}$ induce the following projectors:

$$p_{\beta_{13}}^+(\mu) : V_\mu \otimes \mathfrak{p}_{\mathbb{C}} \to V_{\mu+\beta_{13}}, \quad p_{\beta_{23}}^+(\mu) : V_\mu \otimes \mathfrak{p}_{\mathbb{C}} \to V_{\mu-\beta_{32}},$$
$$p_{\beta_{23}}^-(\mu) : V_\mu \otimes \mathfrak{p}_{\mathbb{C}} \to V_{\mu+\beta_{32}}, \quad p_{\beta_{13}}^-(\mu) : V_\mu \otimes \mathfrak{p}_{\mathbb{C}} \to V_{\mu-\beta_{13}},$$

In terms of $\{v_k^\mu\}$, they are expressed as follows:

Proposition 3.2.1 ([K-O] Prop 2-3)

$$p_{\beta_{13}}^+(\mu)(v_k^\mu \otimes X_{\beta_{13}}) = (k+1)v_{k+1}^{\mu+\beta 13}, \quad p_{\beta_{13}}^+(\mu)(v_k^\mu \otimes X_{\beta_{23}}) = (d_\mu - k + 1)v_k^{\mu+\beta 13},$$
$$p_{\beta_{23}}^-(\mu)(v_k^\mu \otimes X_{\beta_{32}}) = -(k+1)v_{k+1}^{\mu+\beta 32}, \quad p_{\beta_{23}}^-(\mu)(v_k^\mu \otimes X_{\beta_{31}}) = (d_\mu - k + 1)v_k^{\mu+\beta 32},$$
$$p_{\beta_{23}}^+(\mu)(v_k^\mu \otimes X_{\beta_{13}}) = -v_k^{\mu-\beta 32}, \qquad p_{\beta_{23}}^+(\mu)(v_k^\mu \otimes X_{\beta_{23}}) = v_{k-1}^{\mu-\beta 32},$$
$$p_{\beta_{13}}^-(\mu)(v_k^\mu \otimes X_{\beta_{32}}) = v_k^{\mu-\beta 13}, \qquad p_{\beta_{13}}^-(\mu)(v_k^\mu \otimes X_{\beta_{31}}) = v_{k-1}^{\mu-\beta 13},$$

for $k = 1, \ldots, d_\lambda$. Here one should note that $d_{\mu\pm\beta_{13}} = d_{\mu\pm\beta_{32}} = d_\mu \pm 1$. □

4 Generalized Whittaker models

4.1 The space of the generalized Whittaker functionals

Let η be a unitary representation of R with representation space $\mathcal{S}$ defined in section 2. We call the C^∞-induced representation $\mathrm{Ind}_R^G\eta$ of η from R to G with representation space

$$C_\eta^\infty(R\backslash G) := \{f : G \to \mathcal{S}^\infty \mid f \text{ is a } C^\infty\text{-function satisfying, } f(rg) = \eta(r).f(g),\ \forall r \in R, \forall g \in G\}$$

on which G acts via right translation, *the reduced generalized Gelfand-Graev representation.* Here we used standard notation by $\mathcal{S}^\infty$ meaning the subspace consisting of all smooth vectors in $\mathcal{S}$.
We can now define the space of the generalized Whittaker functionals as the space of intertwining operators.

Definition For an irreducible admissible representation $(\pi, \mathcal{H}_\pi)$ of G, we identify the underlining $(\mathfrak{g}_{\mathbb{C}}, K)$-module of π with π itself, and call the space of intertwiners

$$I_{\pi,\eta} := \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi^*, \mathrm{Ind}_R^G\eta)$$

of $(\mathfrak{g}_{\mathbb{C}}, K)$-modules *the space of the algebraic generalized Whittaker functionals.*

4.2 Generalized Whittaker functions with fixed K-type

In order to investigate algebraic generalized Whittaker functionals $l \in I_{\pi,\eta}$, we study the functions $l(v^*) \in \mathrm{Ind}_R^G\eta$: the image of vectors v^* belonging to $(\pi^*, \mathcal{H}_\pi^*)$ by l. To describe these functions explicitly, we specify a K-type of π and consider vectors v^* belonging to this K-type.
For a K-type (τ, V_τ) of π, choose a K-equivalent injection $\iota_\tau : \tau \hookrightarrow \pi$, and pullback a generalized Whittaker functional l by this injection ι_τ,

$$\mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi^*, \mathrm{Ind}_R^G\eta) \ni l \mapsto \iota_\tau^*(l) \in \mathrm{Hom}_K(\tau^*, \mathrm{Ind}_R^G\eta|_K).$$

Here we note the isomorphism

$$\mathrm{Hom}_K(\tau^*, \mathrm{Ind}_R^G\eta|_K) \cong (\mathrm{Ind}_R^G\eta|_K \otimes \tau)^K.$$

The latter space is

$$C_{\eta,\tau}^\infty(R\backslash G/K) := \left\{\varphi : G \to \mathcal{S}(\mathbb{R}) \otimes_{\mathbb{C}} V_\tau \;\middle|\; \begin{array}{l} \varphi \text{ is a } C^\infty\text{-function satisfying,} \\ \varphi(rgk) = \eta(r)\tau(k)^{-1}.\varphi(g), \\ \forall r \in R, \forall g \in G, \forall k \in K \end{array}\right\}.$$

We study functions $F \in C_{\eta,\tau}^\infty(R\backslash G/K)$ representing $\iota_\tau^*(l)$. By definition,

$$l(v^*)(g) = \langle v^*, F(g)\rangle_K,$$

$v^* \in V_\tau^*$. Here $\langle\ ,\ \rangle_K$ means the canonical pairing of K-modules V_τ^* and V_τ.

Definition We call the above function F corresponding to $\iota_\tau^*(l)$, $l \in I_{\pi,\eta}$ *the algebraic generalized Whittaker function associated to representation π with K-type τ.* Moreover if we impose the slowly increasing condition for the A-radial part of F, such a function is called *the generalized Whittaker function.*

5 Differential equations for generalized Whittaker functions of the discrete series

5.1 The Schmid operators

We denote the space of V_λ-valued functions on G with the τ_λ-equivalence by

$$C^\infty_{\tau_\lambda}(G/K) := \{\varphi : G \to V_\lambda \mid \varphi \text{ is a } C^\infty\text{-function satisfying, } \varphi(rgk) = \tau_\lambda(k)^{-1}.\varphi(g)\ ,\forall g \in G\ ,\forall k \in K\}.$$

We can regard $\mathfrak{p}_{\mathbf{C}}$ as a K-module through the adjoint representation $\mathrm{Ad}_{\mathfrak{p}_{\mathbf{C}}}$. The differential operator

$$\begin{aligned}\nabla_{\tau_\lambda} &: C^\infty_{\tau_\lambda}(G/K) \to C^\infty_{\tau_\lambda\otimes\mathrm{Ad}_{\mathfrak{p}_{\mathbf{C}}}}(G/K)\\ \nabla_{\tau_\lambda}\varphi &:= \sum_{i=1}^{4} R_{X_i}\varphi \otimes X_i,\end{aligned}$$

is a K-homomorphism. Here $\{X_i\ (i = 1 \sim 4)\}$ is an orthonormal basis of $\mathfrak{p}$ with respect to the Killing form on $\mathfrak{g}$ and $R_X\varphi$ means the right differential of function φ by $X \in \mathfrak{g}$. The operator ∇_{τ_λ} is called *the Schmid operator*. We take as orthonormal basis of $\mathfrak{p}_{\mathbf{C}}$

$$C(X_\beta + X_{-\beta}), \qquad C\sqrt{-1}(X_\beta - X_{-\beta}),$$

where β is β_{13} or β_{23} and C is a positive constant depending on the normalization of the fixed Killing form. Then, using this basis, the Schmid operator ∇_{τ_λ} can be written as

$$\nabla_{\tau_\lambda}\varphi \;=\; 2C^2 \sum_{\beta=\beta_{13},\beta_{23}} R_{X_{-\beta}}\varphi \otimes X_\beta + 2C^2 \sum_{\beta=\beta_{13},\beta_{23}} R_{X_\beta}\varphi \otimes X_{-\beta}.$$

Here we note that $\{X_{\beta_{13}}, X_{\beta_{23}}\}$ is the set of root vectors corresponding to positive noncompact roots. The above description of ∇_{τ_λ} in two terms corresponds to the decomposition of $\mathfrak{p}_{\mathbf{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$.

Now we define two differential operators as following

$$\begin{aligned}\nabla^{\pm}_{\tau_\lambda} &: C^\infty_{\tau_\lambda}(G/K) \to C^\infty_{\tau_\lambda\otimes\mathrm{Ad}_{\mathfrak{p}_\pm}}(G/K)\\ \nabla^{+}_{\tau_\lambda}\varphi &:= R_{X_{-\beta_{13}}}\varphi \otimes X_{\beta_{13}} + R_{X_{-\beta_{23}}}\varphi \otimes X_{\beta_{23}},\\ \nabla^{-}_{\tau_\lambda}\varphi &:= R_{X_{\beta_{13}}}\varphi \otimes X_{-\beta_{13}} + R_{X_{\beta_{23}}}\varphi \otimes X_{-\beta_{23}}.\end{aligned}$$

For later use, we prepare *the $\pm\beta$-shift operators* for every positive noncompact root $\beta \in \Sigma_{n,+}$ and $\lambda \in L_T^+$.

$$\begin{aligned}\mathcal{D}^{\pm\beta}_{\tau_\lambda} &: C^\infty_{\tau_\lambda}(G/K) \to C^\infty_{\tau_{\lambda\pm\beta}}(G/K)\\ \mathcal{D}^{\pm\beta}_{\tau_\lambda}\varphi(g) &:= p^{\pm}_{\beta}(\nabla^{\pm}_{\tau_\lambda}\varphi(g))\end{aligned}$$

Here $p^{\pm}_{\beta}$ are the projectors $\tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}_\pm} \twoheadrightarrow \tau_{\lambda\pm\beta}$ defined in subsection 3.2.

All the operators constructed above can be defined similarly for $C^\infty_{\eta,\tau_\lambda}(R\backslash G/K)$.

$$\begin{aligned}\nabla_{\eta,\tau_\lambda} &: C^\infty_{\eta,\tau_\lambda}(R\backslash G/K) \;\to\; C^\infty_{\eta,\tau_\lambda\otimes\mathrm{Ad}_{\mathfrak{p}_{\mathbf{C}}}}(R\backslash G/K),\\ \nabla^{\pm}_{\eta,\tau_\lambda} &: C^\infty_{\eta,\tau_\lambda}(R\backslash G/K) \;\to\; C^\infty_{\eta,\tau_\lambda\otimes\mathrm{Ad}_{\mathfrak{p}_\pm}}(R\backslash G/K),\\ \mathcal{D}^{\pm\beta}_{\eta,\tau_\lambda} &: C^\infty_{\eta,\tau_\lambda}(R\backslash G/K) \;\to\; C^\infty_{\eta,\tau_{\lambda\pm\beta}}(R\backslash G/K).\end{aligned}$$

5.2 Yamashita's characterization

Here is a variant of a result of Yamashita which characterize the space of the algebraic minimal K-type generalized Whittaker functionals for discrete series representations. This is fundamental for our purpose. Let

$$\Sigma_I^+ := \{\beta_{12}, \beta_{13}, \beta_{23}\},\ \Sigma_{II}^+ := \{\beta_{12}, \beta_{32}, \beta_{13}\},\ \Sigma_{III}^+ := \{\beta_{12}, \beta_{32}, \beta_{31}\}.$$

and Ξ_J be the set of Harish-Chandra parameters correspond to positive root systems $\Sigma_J^+ (J = I, II, III)$ compatible with the positive compact root system $\Sigma_{c,+}$. We understand ρ_c the half-sum of the compact positive roots and ρ_n^J of the noncompact ones in Σ_J^+.

Proposition 5.2.1 ([Ya] Theorem 2.4) *Let π_Λ be a discrete series representation of G with Harish-Chandra parameter $\Lambda \in \Xi_J$, Blattner parameter $\lambda = \Lambda + \rho_J - 2\rho_c$, and η be the representation constructed in section 2. Assume Λ is far from walls, then the image of* $\mathrm{Hom}_{(\mathfrak{g}_C, K)}(\pi_\Lambda^*, \mathrm{Ind}_R^G \eta)$ *by the correspondence of subsection 4.2 in $C^\infty_{\eta,\tau_\lambda}(R\backslash G/K)$ is characterized by*

$$(D) \ : \qquad \mathcal{D}^{-\beta}_{\eta,\tau_\lambda}.F \ = \ 0 \qquad (\forall \beta \in \Sigma_J^+ \cap \Sigma_n).$$

In short

$$\mathrm{Hom}_K(\tau_\lambda^*, \mathrm{Ind}_R^G \eta) \ \cong \ \bigcap_{\beta \in \Sigma_J^+ \cap \Sigma_n} Ker\ \mathcal{D}^{-\beta}_{\eta,\tau_\lambda}.$$

6 Difference-differential equations for coefficients

6.1 Radial part of Schmid operators

For the representation $(\eta, L^2(\mathbb{R}))$ constructed in section 4 and for any finite dimensional K-module W, we denote the space of the smooth $\mathcal{S}(\mathbb{R}) \otimes_{\mathbb{C}} W$-valued functions on A by

$$C^\infty(A; \mathcal{S}(\mathbb{R}) \otimes_{\mathbb{C}} W) \ := \ \{\phi : A \to \mathcal{S}(\mathbb{R}) \otimes_{\mathbb{C}} W \mid C^\infty\text{-function}\}.$$

Let

$$\begin{array}{rcccl} \mathrm{res}_A & : & C^\infty_{\eta,\tau_\lambda}(R\backslash G/K) & \to & C^\infty(A; \mathcal{S}(\mathbb{R}) \otimes_{\mathbb{C}} V_\lambda), \\ \mathrm{res}_{A,\pm} & : & C^\infty_{\eta,\tau_\lambda \otimes Ad_{\mathfrak{p}_\pm}}(R\backslash G/K) & \to & C^\infty(A; \mathcal{S}(\mathbb{R}) \otimes_{\mathbb{C}} V_\lambda \otimes_{\mathbb{C}} \mathfrak{p}_\pm) \end{array}$$

be the restriction maps to A. Then we define *the radial part* $R(\nabla^\pm_{\eta,\tau_\lambda})$ of $\nabla^\pm_{\eta,\tau_\lambda}$ on the image of res_A by

$$R(\nabla^\pm_{\eta,\tau_\lambda}).(\mathrm{res}_A \varphi) \ = \ \mathrm{res}_{A,\pm}(\nabla^\pm_{\eta,\tau_\lambda}.\varphi).$$

Let us denote by ϕ and ∂ the restriction to A of $\varphi \in C^\infty_{\eta,\tau_\lambda}(R\backslash G/K)$ and the generator H of $\mathfrak{a}$, respectively, $\partial\phi = (H.\varphi)|_A$. We remark $\partial = r\frac{d}{dr}$: the Euler operator in variable r.

Proposition 6.1.1 ([K-O] Prop 4-1) *Let ϕ be the above element in $C^\infty(A; \mathcal{S}(\mathbb{R}) \otimes_{\mathbb{C}} V_\lambda)$. Then the radial part $R(\nabla^+_{\eta,\tau_\lambda})$ of $\nabla^+_{\eta,\tau_\lambda}$ is given by*

(i) $R(\nabla^+_{\eta,\tau_\lambda}).\phi$

$$= \frac{1}{2}\{\partial - \sqrt{-1}r^2\eta(E_1) - 4\}.(\phi \otimes X_{\beta_{13}}) + \frac{1}{2}(\tau_\lambda \otimes Ad_{\mathfrak{p}_+})(H'_{13}).(\phi \otimes X_{\beta_{13}})$$
$$-\frac{1}{2}r\{\eta(E_{2,+}) - \sqrt{-1}\eta(E_{2,-})\}.(\phi \otimes X_{\beta_{23}}) + (\tau_\lambda \otimes Ad_{\mathfrak{p}_+})(X_{\beta_{12}}).(\phi \otimes X_{\beta_{23}}).$$

Similarly for the radial part $R(\nabla^-_{\eta,\tau_\lambda})$ *of* $\nabla^-_{\eta,\tau_\lambda}$, [1] *we have*

(ii) $R(\nabla^-_{\eta,\tau_\lambda}).\phi$

$$= \frac{1}{2}\{\partial + \sqrt{-1}r^2\eta(E_1) - 4\}.(\phi \otimes X_{\beta_{31}}) - \frac{1}{2}(\tau_\lambda \otimes Ad_{\mathfrak{p}\_})(H'_{13}).(\phi \otimes X_{\beta_{31}})$$
$$-\frac{1}{2}r\{\eta(E_{2,+}) + \sqrt{-1}\eta(E_{2,-})\}.(\phi \otimes X_{\beta_{32}}) + (\tau_\lambda \otimes Ad_{\mathfrak{p}\_})(X_{\beta_{21}}).(\phi \otimes X_{\beta_{32}}).$$

□

6.2 Compatibility of S-type and K-type

If we write $\phi = \varphi|_A \in C^\infty(A; \mathcal{S}(\mathbb{R}) \otimes_{\mathbb{C}} V_\lambda)$ as

$$\phi(a) \;=\; \sum_{n=1}^{\infty}\sum_{k=0}^{d_\lambda} c_{nk}(a)(h_n \otimes v_k^\lambda)$$

in terms of basis $\{h_n | n \in \mathbb{N}_0\}$ and $\{v_k^\lambda | k = 0, \ldots, d_\lambda\}$ of $\mathcal{S}(\mathbb{R})$ and V_λ respectively, the compatibility of S-action and K-action implies the vanishing of many coefficients c_{nk}. Here is the precise statement.

Let $(\eta, \mathcal{S}(\mathbb{R}))$ be the representation constructed in section 2 as a tensor product $\widetilde{\chi}_\mu \otimes (\omega_\psi \times \rho_\psi)|_{\widetilde{R}}$. Here $\widetilde{\chi}_\mu$ is an odd character of $\widetilde{S} \cong \widetilde{SO}(2)$ parameterized by a half integer μ. Culculate $\phi(mam^{-1})$, $m \in S = M, a \in A$ in two different ways, first, $\phi(mam^{-1}) = \phi(a)$ since $M = Z_K(A)$, second, $\phi(mam^{-1}) = \eta(m)\tau_\lambda(m^{-1}).\phi(a)$ since $\phi = \varphi|_A$, $\varphi \in C^\infty_{\eta,\tau_\lambda}(R\backslash G/K)$, then we have next linear relation between n and k.

Lemma 6.2.1 (1) *The image of* res_A *in* $C^\infty(A; \mathcal{S}(\mathbb{R}) \otimes_{\mathbb{C}} V_\lambda)$ *is zero unless*

$$\frac{-\lambda_1 + 2\lambda_2}{3} \in \mathbb{Z} \qquad \text{and} \qquad \frac{-\lambda_1 + 2\lambda_2}{3} \geq \frac{1}{2} + \mu.$$

(2) *Assume* $\ell_\lambda \in \mathbb{Z}$, *then the* A*-radial part* ϕ *of* $\varphi \in C^\infty_{\eta,\tau_\lambda}(R\backslash G/K)$ *is written as*

$$\phi(a_r) \;=\; \sum_{k=0}^{d_\lambda} c_k(a_r)(h_n \otimes v_k^\lambda),$$

where $c_k(a_r)$*'s are* C^∞*-functions on* A *and the index* n *is given by*

$$n \;=\; -k + \frac{2\lambda_1 - \lambda_2}{3} - \frac{1}{2} - \mu.$$

□

6.3 Difference-differential equations

We first write down the A-radial part $R(\mathcal{D}^{-\beta}_{\eta,\tau_\lambda})$ of the β-shift operators $\mathcal{D}^{-\beta}_{\eta,\tau_\lambda}$ in terms of coefficient functions $c_k(a_r)$'s of ϕ.

[1] In [K-O], there is a misprint in the formula (ii). The symbol − after ∂ is + correctly.

Proposition 6.3.1 *Let ϕ be any function in $C^\infty(A; \mathcal{S}(\mathbb{R}) \otimes_{\mathbb{C}} V_\lambda)$ which is the A-radial part of $\varphi \in C^\infty_{\eta,\tau_\lambda}(R\backslash G/K)$. By using the lemma 6.2.1, we can express ϕ as*

$$\phi(a_r) = \sum_{k=0}^{d_\lambda} c_k(a_r)(h_{n_{\mu,\lambda}(k)} \otimes v_k^\lambda),$$

where we denote $-k + \frac{2\lambda_1-\lambda_2}{3} - \frac{1}{2} - \mu$ by $n_{\mu,\lambda}(k)$. Then for an arbitrary noncompact root β, the action of the A-radial part $R(\mathcal{D}^{-\beta}_{\eta,\tau_\lambda})$ of the β-shift operator is given in terms of c_k's as follows:

$$R(\mathcal{D}^{-\beta}_{\eta,\tau_\lambda})\phi(a_r) = \sum_{k=0}^{d_{\lambda-\beta}} c_k^{-\beta}(a_r)(h_{n_{\mu,\lambda-\beta}(k)} \otimes v_k^{\lambda-\beta})$$

with

$$\begin{aligned}
c_k^{-\beta_{23}}(a_r) &= \frac{1}{2}\{(d_\lambda - k + 1)(\partial + k - \lambda_2 - 2r^2 s).c_k(a_r) + k\frac{1+2s}{\sqrt{2}} r \cdot c_{k-1}(a_r)\}, \\
c_k^{-\beta_{13}}(a_r) &= \frac{1}{2}\{(\partial + k - 2d_\lambda - \lambda_2 - 1 - 2r^2 s).c_{k+1}(a_r) - \frac{1+2s}{\sqrt{2}} r \cdot c_k(a_r)\}, \\
c_k^{-\beta_{32}}(a_r) &= \frac{-1}{2}\{(\partial - k + \lambda_2 - 2 + 2r^2 s).c_k(a_r) \\
&\qquad -\sqrt{2}(1+2s)(n_{\mu,\lambda}(k)+1)r \cdot c_{k+1}(a_r)\}, \\
c_k^{-\beta_{31}}(a_r) &= \frac{1}{2}\{k(\partial - k + \lambda_2 + 2d_\lambda + 1 + 2r^2 s).c_{k-1}(a_r) \\
&\qquad +(d_\lambda - k + 1)\sqrt{2}(1+2s)(n_{\mu,\lambda}(k)+1)r \cdot c_k(a_r)\}.
\end{aligned}$$

□

Using the above proposition, we can write the differential equations (D) in Proposition 5.2.1 in terms of the coefficient functions c_k of the A-radial part ϕ of the algebraic generalized Whittaker function $F \in C^\infty_{\eta,\tau_\lambda}(R\backslash G/K)$, which comes from $l \in I_{\pi_\Lambda,\eta}$. As for the system of the difference-differential equations satisfied by c_k's, see [I] subsection 7.3.

7 An explicit formula and the multiplicity one theorem

7.1 An explicit formula for coefficients

Now we are in a position to formulate the generalized Whittaker functions with analytic condition. Let us define *the generalized Whittaker model for the representation π of G with K-type τ* as follow

$$\begin{aligned}
Wh_\eta^\tau(\pi) := \{F \in C^\infty_{\eta,\tau}(R\backslash G/K) \mid\ & F|_A(a_r) \text{ is of moderate growth when } r \to \infty, \\
& l(v^*) = \langle v^*, F(\cdot)\rangle_K,\ l \in I_{\pi,\eta},\ v^* \in V_\tau^*\}.
\end{aligned}$$

We call the elements in the space above *the generalized Whittaker functions associated to the representation π with K-type τ.*

Proposition 7.1.1 *Let $\phi = F|_A$ be a function in $C^\infty(A; \mathcal{S}(\mathbb{R}) \otimes_{\mathbb{C}} V_\lambda)$ which comes from $l \in \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi_\Lambda, \mathrm{Ind}_R^G \eta)$, $\Lambda \in \Sigma_{II}$. And c_k's are the coefficient functions of ϕ expanded with respect to the basis $\{h_n\}$ and $\{v_k^\lambda\}$ of $\mathcal{S}(\mathbb{R})$ and V_λ, respectively. Then each c_k $(0 \leq k \leq d_\lambda - 1)$ satisfies the following differential equation.*

$$\{\partial^2 - (2d_\lambda + 4)\partial + G_k(r)\}.c_k(a_r) \;=\; 0$$

where

$$G_k(r) = -4s^2r^4 - \{4(\lambda_2 - k + d_\lambda - 1)s + (n_k + 1)(1 + 2s)^2\}r^2 - (k - 2d_\lambda - \lambda_2 - 2)(k - \lambda_2 + 2).$$

Here we abbreviated $n_{\mu\lambda}(k) = -k + \frac{2\lambda_1 - \lambda_2}{3} - \mu - \frac{1}{2}$ as n_k. □

As a result, we obtain an explicit formula of the coefficient functions c_k's of the minimal K-type generalized Whittaker functions for large discrete series representations.

Theorem 7.1.2 *The coefficient functions c_k's of the A-radial part of the minimal K-type generalized Whittaker functions F's $\in Wh_\eta^{\tau_\Lambda}(\pi_\Lambda)$ for the large discrete series representations π_Λ's $(\Lambda \in \Sigma_{II})$ of $SU(2,1)$ are of the form*

$$c_k(a_r) \;=\; (\mathrm{const.}) \times r^{d_\lambda + 1} W_{\kappa,(k-\lambda)/2}(2|s|r^2)$$

with parameters

$$\kappa = \{-(\lambda_2 - k + d_\lambda - 1)s - (n_k + 1)(2s + 1)^2/4\}/2|s|,$$

$k = 0, \ldots, d_\lambda$. Here $\lambda = (\lambda_1, \lambda_2)$ is the Blattner parameter of π_Λ. Function $W_{\kappa,m}(x)$ is the classical Whittaker function. □

In the cases of the holomorphic and the antiholomorphic discrete series representations, the differential equations satisfied by the coefficient functions are of the first order. Consequently the solutions are essentially exponential functions.

Proposition 7.1.3 *Let $\phi = F|_A$ be a function in $C^\infty(A; \mathcal{S}(\mathbb{R}) \otimes_{\mathbb{C}} V_\lambda)$ which comes from $l \in \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi_\Lambda, \mathrm{Ind}_R^G \eta)$. And c_k's are the coefficient functions of ϕ expanded with respect to the basis $\{h_n\}$ and $\{v_k^\lambda\}$. Then each c_k $(0 \leq k \leq d_\lambda + 1)$ satisfies the following differential equation*

$$\{\partial + G_k(r)\}.c_k(a_r) \;=\; 0$$

where

$$G_k(r) = \begin{cases} -2sr^2 - (\lambda_2 + k) & \text{when } \Lambda \in \Sigma_I; \\ 2sr^2 + (\lambda_2 + k) & \text{when } \Lambda \in \Sigma_{III}. \end{cases}$$

□

Hence we have an explicit formula of c_k's.

Theorem 7.1.4 *The coefficient functions c_k's of the A-radial part of the minimal K-type generalized Whittaker functions F's $\in Wh_\eta^{\tau_\Lambda}(\pi_\Lambda)$ for the holomorphic (resp. antiholomorphic) discrete series representations π_Λ's $(\Lambda \in \Sigma_I(\text{resp.}\Sigma_{III})$ of $SU(2,1)$ are of the form*

$$c_k(a_r) \;=\; (\mathrm{const.}) \times r^{\lambda_2 + k} e^{sr^2},$$

$k = 0, \ldots, d_\lambda$ with $s < 0$, (resp.

$$c_k(a_r) \;=\; (\mathrm{const.}) \times r^{-\lambda_2 - k} e^{-sr^2},$$

$k = 0, \ldots, d_\lambda$ with $s > 0$). Here variable a_r is an element of A. □

Remark These explicit formulae of generalized Whittaker functions for the holomorphic and the antiholomolphic discrete series representations are compatible with the classical theory of Fourier-Jacobi expansion of holomolphic modular forms on $SU(2,1)$, or on the associated symmetric domain $SU(2,1)/K$. We just remark here that the conditions on parameter s of the central character of ρ_{ψ_s} in Theorem 7.1.4 is the consequence of the moderate growth condition on $Wh_\eta^{\tau_\lambda}(\pi_\Lambda)$.

7.2 The multiplicity one theorem for the discrete series

Assemble the parts prepared in previous sections, then we obtain simultaneously the multiplicity one theorem and an explicit form of elements in the minimal K-type generalized Whittaker model $Wh_\eta^{\tau_\lambda}(\pi)$ for the discrete series representations π's of $SU(2,1)$.

In order to formulate the multiplicity one theorem we have to introduce a $(\mathfrak{g}_{\mathbb{C}}, K)$-submodule $\mathcal{A}_\eta(R\backslash G)$ of $C_\eta^\infty(R\backslash G)$.

$$\mathcal{A}_\eta(R\backslash G) := \left\{ f \in C_\eta^\infty(R\backslash G) \;\middle|\; \begin{array}{l} c_{f,h} \text{ is right } K\text{-finite and} \\ c_{f,h}|_A(a_r) \text{ is of moderate growth} \\ \text{when } r \to \infty, \qquad \forall h \in (\eta, \mathcal{S}(\mathbb{R})) \end{array} \right\},$$

where $c_{f,h}$ is a $\mathbb{C}$-valued function on G defined as $c_{f,h}(g) := (f(g), h)_\eta$ and $c_{f,h}|_A$ is the A-radial part of $c_{f,h}$. It is easy to see that $\mathcal{A}_\eta(R\backslash G)$ is a $(\mathfrak{g}_{\mathbb{C}}, K)$-submodule of $C_\eta^\infty(R\backslash G)$.

Theorem 7.2.1 *The discrete series representation π_Λ of $SU(2,1)$ with Harish-Chandra parameter $\Lambda \in \Xi$ and Blattner parameter $\lambda = (\lambda_1, \lambda_2) \in L_T^+$ has multiplicity one property* i.e.

$$\dim_{\mathbb{C}} \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\mathcal{H}_{\pi_\Lambda}^*, \mathcal{A}_{\eta_{\mu,\psi}}(R\backslash G)) = 1$$

if and only if

$$\frac{2\lambda_1 - \lambda_2}{3} \in \mathbb{Z}, \qquad \frac{1}{2} + \mu \leq \frac{-\lambda_1 + 2\lambda_2}{3}.$$

Under this condition, the minimal K-type generalized Whittaker model $Wh_\eta^{\tau_\lambda}(\pi_\Lambda)$ of π_Λ has a basis $F_\eta^{\tau_\lambda}$ whose A-radial part is given as follows.

1) *When $\Lambda \in \Xi_{II}$* (i.e. *π_Λ is a large discrete series representation*),

$$F_\eta^{\tau_\lambda}(a_r) = \sum_{k=0}^{d_\lambda} r^{d_\lambda+1} W_{\kappa,\frac{k-\lambda}{2}}(2|s|r^2) \cdot \left(h_{n_{\mu\lambda}(k)} \otimes v_k^\lambda\right),$$

where

$$\kappa = \{-(\lambda_2 - k + d_\lambda - 1)s - (n_k+1)(2s+1)^2/4\}/2|s|.$$

2) *When $\Lambda \in \Xi_I$* (i.e. *π_Λ is a holomorphic discrete series represemtation*),

$$F_\eta^{\tau_\lambda}(a_r) = \sum_{k=0}^{d_\lambda} r^{\lambda_2+k} e^{sr^2} \cdot \left(h_{n_{\mu\lambda}(k)} \otimes v_k^\lambda\right),$$

where $s < 0$.

3) *When $\Lambda \in \Xi_{III}$* (i.e. *π_Λ is an antiholomorphic discrete series representation*),

$$F_\eta^{\tau_\lambda}(a_r) = \sum_{k=1}^{d_\lambda} r^{-\lambda_2-k} e^{-sr^2} \cdot \left(h_{n_{\mu\lambda}(k)} \otimes v_k^\lambda\right),$$

where $s > 0$.
Here $r \in \mathbb{R}_{>0}$, *and the index of each base* h_n *of* η *is*

$$n_{\mu\lambda}(k) = -k + \frac{2\lambda_1 - \lambda_2}{3} - \frac{1}{2} - \mu.$$

□

8 The case of principal series representations

The case of the principal series representation $\pi_{k,\nu} = \mathrm{Ind}_P^G(1_N \otimes e^\nu \otimes \chi_k)$, where

$$e^\nu \ : A \ni a \mapsto e^{(\nu+\rho)(\log a)} \in \mathbb{C}^*,$$
$$\chi_k \ : M \ni \mathrm{diag}(e^{i\theta}, e^{-2i\theta}, e^{i\theta}) \mapsto e^{ik\theta} \in U(1),$$

we can obtain an explicit formula for the corner K-type generalized Whittaker function by solving the differential equation arose from the Casimir action. For details and notation, see [I] §9, [K-O] §7.

8.1 Radial part of the Casimir operator

Proposition 8.1.1 *Let* $\phi \in C^\infty(A; \mathcal{S}(\mathbb{R}) \otimes_{\mathbb{C}} V_\tau)$ *be the A-radial part of* $\varphi \in C^\infty_{\eta,\tau}(R\backslash G/K)$. *Then the radial part* $R(\Omega)$ *of* Ω *is given by*

$$\begin{aligned} R(\Omega).\phi = \frac{1}{2}\{&\partial^2 - 4\partial + \frac{1}{3}k^2 - r^4\eta(E_1)^2 \\ &-2\sqrt{-1}r^2\eta(E_1)\tau(H'_{13}) + r^2(\eta(E_{2,+})^2 + \eta(E_{2,-})^2) \\ &+2r\eta(E_{2+})\tau(X_{\beta_{12}} + X_{\beta_{21}}) + 2\sqrt{-1}r\eta(E_{2-})\tau(X_{\beta_{12}} - X_{\beta_{21}})\}\phi. \end{aligned}$$

□

8.2 An explicit formula and the multiplicity one theorem

Theorem 8.2.1 *The irreducible principal series representation* $\pi_{k,\nu}$ *of* $SU(2,1)$ *has multiplicity one property* i.e.

$$\dim_{\mathbb{C}} \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi^*_{k,\nu}, \mathcal{A}_{\eta_{\mu,\psi}}(R\backslash G)) \ = \ 1$$

if and only if

$$\frac{k}{3} - \mu - \frac{1}{2} \in \mathbb{Z}_{\geq 0}.$$

Under this condition, the corner K-type generalized Whittaker model $Wh_\eta^{\tau(-k,-k)}(\pi_{k,\nu})$ *has a basis* $F_\eta^{\tau(-k,-k)}$ *whose A-radial part is given by*

$$F_\eta^{\tau(-k,-k)}(a_r) \ = \ rW_{\kappa,\frac{\nu}{2}}(2|s|r^2) \cdot \left(h_{n_0} \otimes v_0\right),$$

where

$$\kappa = \{-ks - (2n_0+1)(4s^2+1)/4\}/2|s|.$$

Here $r \in \mathbb{R}_{>0}$, *and the index of the base* h_{n_0} *of* η *is*

$$n_0 = \frac{k}{3} - \mu - \frac{1}{2}.$$

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