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Cuntz の Canonical Endomorphism の エントロピーについて

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Cuntz 環 $\mathcal{O}_n$ の生成元 を $\{S_i\}_{i=1}^n$ とした時、所謂 Cuntz の canonical endomorphism Φ は、

$$\Phi(x)=\sum_{i=1}^n S_i x S_i^*, \quad (x\in\mathcal{O}_n)$$

によって、定義される。

Voiculescu の topological entropy は、nuclear C^*-環の automorphism に対して、定義された概念であるが、その定義と、ここで用いる性質は、彼の結果をそのまま、適用する事により、*-endomorphism に対しても、有効である。

ここでは、Cuntz の canonical endomorphism Φ の $\mathcal{O}_n$ の unique $\log n$-KMS state ϕ に対する Connes-Narnhofer-Thirring の entropy $h_\phi(\Phi)$ と、Voiculescu の topological entropy $ht(\Phi)$ との間の、次の関係を報告する：

$$h_\phi(\Phi)=ht(\Phi)=\log n.$$

この証明と基本的に同じやり方で、$n\times n$ 行列環 の無限テンソル積 A の shift automorphism α_n に対して、$A\rtimes_{\alpha_n}\mathbb{Z}$ の implimenting unitary を u とした時、

$$h_\tau(Ad_u)=ht(Ad_u)=\log n$$

も導きだす事ができる。但し、τ は $A\rtimes_{\alpha_n}\mathbb{Z}$ の tracial state である。

2. Preliminaries

2.1. Let H_0 be a Hilbert space of dimension $n<\infty$. Put $H_i=H_0,\ i\in\mathbb{Z}$. For two integers i and j with $i<j$, we put

$$H_{[i,j]}=H_i\otimes H_{i+1}\otimes\cdots\otimes H_j.$$

Let $\{\delta(i):i=1,...,n\}$ be an orthonormal basis of H_0. The emmbedding $H_{[i,j]}\hookrightarrow H_{[i-1,j+1]}$ is given by $\xi\in H_{[i,j]}\to\delta(1)\otimes\xi\otimes\delta(1)\in H_{[i-1,j+1]}$. We denote by $\mathcal{H}_i$ the inductive limit of $\{H_{[i,i+j]}:j=0,1,...\}$ and by $\mathcal{H}$ the inductive limit of the incleasing sequence $\{\mathcal{H}_i:i=0,-1,...\}$.

Given $k,l\in\mathbb{Z}\ \ k<l$, let

$$W^n_{[k,l]}=\{\mu=(\mu_k,\cdots,\mu_l):\mu_i\in\{1,\cdots,n\},\ (k\le i\le l)\}.$$

Let $\mu \in W^n_{[k,l]}$ and $\nu \in W^n_{[l+1,m]}$. We put

$$\mu \cdot \nu = (\mu_k, \cdots, \mu_l, \nu_{l+1}, \cdots, \nu_m).$$

Further, let

$$W^n_0 = \{0\}, \quad W^n_{[0,\infty]} = \cup_{k=0}^{\infty} W^n_{[0,k]} \quad \text{and} \quad W^n_{\infty} = \cup_{k=0}^{\infty} W^n_{[-k,k]}.$$

The shift $\alpha : i \in \mathbb{Z} \to i+1$ induces the mapping on W^n_{∞}, which we denote by the same notation α.

For $\mu \in W^n_{[k,l]}$, we put

$$\delta(\mu) = \delta(\mu_k) \otimes \cdots \otimes \delta(\mu_l) \in H_{[k,l]}.$$

Then $\{\delta(\mu) : \mu \in W^n_{[k,l]}\}$ is an orthonormal basis in $H_{[k,l]}$.

Let $A_0 = B(H_0)$ and $\{e(i,j) : i,j = 1,...,n\}$ be the matrix unit of A_0 with respect to the orthonormal basis $\{\delta(i) : i = 1,...,n\}$. We denote the trace (1/n)Tr of A_0 by τ_0. Put $A_i = A_0, (i \in \mathbb{Z})$ and $\tau_i = \tau_0$. For two integers $i < j$, let

$$A_{[i,j]} = A_i \otimes A_{i+1} \otimes \cdots \otimes A_j.$$

For $\mu, \nu \in W^n_{[k,l]}$, we put

$$e(\mu,\nu) = e(\mu_k, \nu_k) \otimes \cdots \otimes e(\mu_l, \nu_l) \in A_{[k,l]}.$$

Then $\{e(\mu,\nu) : \mu, \nu \in W^n_{[k,l]}\}$ is a matrix units of $A_{[k,l]}$.

2.2. We apply the entropy of Connes-Narnhofer-Thirring and Voiculescu's topological entropy to both of automorphisms and unital *-endomorphisms on C^*-algebras. To fix notations, we recall the definition of the topological entropy. Let B be a nuclear C^*-algebra with unity. Let $CAP(B)$ be triples (ρ, η, C), where C is a finite dimensional C^*-algebra, and $\rho : B \to C$ and $\eta : C \to B$ are unital completely positive maps. Let Ω be the set of finite subsets of B. For an $\omega \in \Omega$, put

$$rcp(\omega;\delta) = \inf\{\text{rank } C : (\rho,\eta,C) \in CAP(B), \|\eta \cdot \rho(a) - a\| < \delta, a \in B\},$$

where rank C means the dimension of a maximal abelian self-adjoint subalgebra of C. For a unital *-endomorphism β of B, put

$$ht(\beta, \omega\ ;\delta) = \overline{\lim_{\mathrm{N}\to\infty}} \frac{1}{\mathrm{N}} \log rcp(\omega \cup \beta(\omega) \cup \cdots \cup \beta^{\mathrm{N}-1}(\omega);\delta)$$

and

$$ht(\beta, \omega) = \sup_{\delta > 0} ht(\beta, \omega; \delta).$$

Then the topological entropy $ht(\beta)$ of β is defined by

$$ht(\beta) = \sup_{\omega \in \Omega} ht(\beta, \omega).$$

Assume that there exists an increasing sequence $(\omega_j)_{j \in \mathbb{N}}$ of finite subsets of B such that the linear span of $\cup_{j \in \mathbb{N}} \omega_j$ is dense in B. Even in the case of *-endomorphisms which are not automorphisms, by the obvious analogoues of [V : 4.3 Proposition], $ht(\cdot)$ is obtained as the following form which we use later:

$$ht(\beta) = \sup_{j \in \mathbb{N}} ht(\beta, \omega_j).$$

Let ϕ be a state of B with $\phi \cdot \beta = \phi$. The essential relation between $ht(\beta)$ and Connes-Narnhofer-Thirring entropy $h_\phi(\beta)$ is by [V: 4.6 Proposition]

$$h_\phi(\beta) \leq ht(\beta).$$

3. Entropy of Cuntz's canonical endomorphism

3.1. To compute the entropy of Φ, we recall some of the representation for the Cuntz algebra $\mathcal{O}_n$ as a crossed product in [Cu1], (cf., [Ch2, I2, P, R]). We use the same notations as in §2.1.

For a $j \in \mathbb{Z}$, $\mathcal{A}_j$ is given as the infinite tensor product:

$$\mathcal{A}_j = \bigotimes_{i=j}^{\infty} A_i.$$

Define embeddings

$$\mathcal{A}_j \hookrightarrow \mathcal{A}_{j-1} \hookrightarrow \mathcal{A}_{j-2} \hookrightarrow \cdots$$

by $x \in \mathcal{A}_j \to e(1,1) \otimes x \in \mathcal{A}_{j-1}$. The inductive limit of this sequence is denoted by $\mathcal{A}$. Since two embeddings $\mathcal{A}_j \hookrightarrow \mathcal{A}_{j-1}$ and $\mathcal{H}_i \hookrightarrow \mathcal{H}_{i-1}$ are compatible, we can consider $\mathcal{A}$ acting faithfully on $\mathcal{H}$.

The automorphism σ of $\mathcal{A}$ is induced by the shift $\alpha : i (\in \mathbb{Z}) \to i+1$. Then the crossed product $\mathcal{A} \rtimes_\sigma \mathbb{Z}$ acts faithfully on the Hilbert space

$$K = \sum_{i \in \mathbb{Z}} \bigoplus u^i \mathcal{H},$$

where u is the implimenting unitary in $\mathcal{A} \rtimes_\sigma \mathbb{Z}$ for the automorphism σ of $\mathcal{A}$. Let p be the unit of $\mathcal{A}_0 \subset \mathcal{A} \rtimes_\sigma \mathbb{Z}$ and put

$$w = up.$$

We remark $u^j p = w^j$.

Then Cuntz algebra $\mathcal{O}_n$ is reresented as $p(\mathcal{A} \rtimes_\sigma \mathbb{Z})p$, which is the C^* subalgebra $C^*(\mathcal{A}_0, w)$ of $(\mathcal{A} \rtimes_\sigma \mathbb{Z})$ generated by $\{\mathcal{A}_0,\ w\}$. There exists a conditional expectation E of $C^*(\mathcal{A}_0, w)$ onto $\mathcal{A}_0$ with $E(w^j) = 0$ for all $j = 1, 2, \cdots$. The unique tracial state τ of $\mathcal{A}_0$ is extended to the state ϕ of $C^*(\mathcal{A}_0, w)$ by $\phi = \tau \cdot E$. Then ϕ is the unique $\log n$-KMS state of $C^*(\mathcal{A}_0, w)$ ([DP]).

3.2. Since

$$\sigma^j(p)(\mathcal{H}) = \mathcal{H}_j, \quad j \in \mathbb{Z},$$

the algebra $p(\mathcal{A} \rtimes_\sigma \mathbb{Z})p$ is acting faithfully on

$$pK = \sum_{i \in \mathbb{Z}} \bigoplus u^i \mathcal{H}_{-i}.$$

The restriction $\sigma|_{\mathcal{A}_0}$ of σ to $\mathcal{A}_0$ is the one sided non commutative Bernoulli shift. Cuntz's canonical inner endomorphism Φ of $\mathcal{O}_n$ is nothing but the extension of $\sigma|_{\mathcal{A}_0}$ to the Cuntz algebra $C^*(\mathcal{A}_0, w)$ which maps

$$a \to \sigma(a), \ (a \in \mathcal{A}_0), \quad \text{and} \quad w \to vw,$$

where

$$v = \sum_{j=1}^{n} e((j,1),(1,j)) \in A_{[0,1]},$$

([Cu2], cf. [Ch2]).

3.3. Let $k, m \in \mathbb{N}$. We define

$$K(k,m) = \sum_{l=-k}^{k} \bigoplus u^l H_{[-l,-l+m]}$$

and we denote the orthogonal projection of K onto $K(k,m)$ by $Q(k,m)$. The set $\{u^j \delta(\mu) : -k \leq j \leq k,\ \mu \in W^n_{[-j,-j+m]}\}$ is an orthonomal basis of $K(k,m)$. We denote by $E((j,\mu),(l,\nu))$ the partial isometry in $B(K(k,m))$ such that

$$E((j,\mu),(l,\nu)) : u^l \delta(\nu) \to u^j \delta(\mu), \quad (\mu \in W^n_{[-j,-j+m]},\ \nu \in W^n_{[-l,-l+m]}).$$

Then the set

$$\mathcal{E}(k,m) = \{E((j,\mu),(l,\nu)) : -k \leq j, l \leq k,\ \mu \in W^n_{[-j,-j+m]},\ \nu \in W^n_{[-l,-l+m]}\}$$

is a matrix units of $B(K(k,m))$.

3.4. Let $k, m \in \mathbb{N}$. We define the completely positive unital linear map

$$\varphi_{k,m} : p(\mathcal{A} \rtimes_\sigma \mathbb{Z})p \to B(K(k,m))$$

by

$$\varphi_{k,m}(x) = Q(k,m)xQ(k,m)|_{K(k,m)}.$$

For two integers a and b with $a < b$, we let

$$\omega_{a,b} = \{e(\mu,\nu)w^j : 0 \leq j \leq a \quad \text{and} \quad \mu, \nu \in W^n_{[0,b]}\}.$$

3.5. We define the linear map

$$\psi_{k,m} : B(K(k,m)) \to p(\mathcal{A} \rtimes_\sigma \mathbb{Z})p$$

by

$$\psi_{k,m}(E((j,\mu),(l,\nu)) = \frac{1}{2k+1} pu^j e(\mu,\nu)u^{*l}p,$$

for $E((j,\mu),(l,\nu)) \in \mathcal{E}(k,m)$. Then we have that $\psi_{k,m}$ is a unital completely positive map. Since $u^j p = w^j$, we have

$$\psi_{k,m} \cdot \varphi_{k,m}(e(\mu,\nu)w^j) = \frac{2k-j+1}{2k+1} e(\mu,\nu)w^j,$$

for all $e(\mu,\nu)w^j \in \omega_{a,b}$, $a \leq k$ and $b \leq m$.

Applying Voiculescu's definition of topological entropy to these completely positive maps and the above increasing sequence of finite sets $(\omega_{a,b})_{a,b}$, we have the following Theorem.

3.6. Theorem. *Let Φ be Cuntz's canonical inner endomorphism of $\mathcal{O}_n$. Then*

$$ht(\Phi) = \log n.$$

3.7. Corollary. *Let ϕ be the unique* $\log n$*-KMS state of $\mathcal{O}_n$. Then*

$$h_\phi(\Phi) = \log n.$$

Proof. Let τ be the unique tracial state of $\mathcal{A}_0$ and E be the conditional expectation of $p(\mathcal{A} \rtimes_\sigma \mathbb{Z})p$ onto $\mathcal{A}_0$, then $\phi = \tau \cdot E$. Hence $\phi \cdot \Phi = \phi$. This relation implies, by the endomorphism version of [V: 4.6 Proppsition],

$$\log n = h_\tau(\sigma|\mathcal{A}_0) \leq h_\phi(\Phi) \leq ht(\Phi) = \log n.$$

Therefore $h_\phi(\Phi) = \log n$. □

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