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LMI based control design: Solutions to nonconvex problems

Tetsuya Iwasaki[1] and Shinji Hara[2]

1 Introduction

In this article, a list of open computational problems arising in the field of control systems engineering is presented. The problems are all considered to be very important in the analysis/design of control systems. They are, however, also known to be extremely difficult, and only partial solutions are available till date. Some of the existing approaches to these problems are explained to motivate further development of "better" algorithms.

We use the following notation. For a matrix A, A' and A^* denote its transpose and its complex conjugate transpose, respectively. $A>0$ means that A is *symmetric* positive definite. The set of $n\times n$ real symmetric matrices is denoted by $\mathcal{S}_n$. A subset of $\mathcal{S}_n$ with a block diagonal structure is denoted by $\mathcal{DS}_n$. $\rho(A)$ and $\|A\|$ are the spectral radius and the spectral norm (the maximum singular value) of A, respectively.

2 Open problems in control engineering

In this section, we provide a list of some computational problems in the analysis and the synthesis of control systems. All the problems are characterized by feasibility problems of finding a matrix satisfying certain matrix inequalities. Many control problems can be reduced to one of these problems, and hence a complete solution to any of these problems can have a tremendous impact on the control theory. The significance of the problems can be found, for example, in [3, 9, 19].

The first problem arises in the robust stability/performance analysis of linear time-invariant uncertain systems [10, 13, 14].

Problem 1 *Let $M\in\mathbb{R}^{n\times m}$ and $\boldsymbol{\nabla}\subseteq\mathbb{C}^{m\times n}$ be given. Find $\Theta\in\mathbb{R}^{(n+m)\times(n+m)}$ such that*

$$\begin{pmatrix}\nabla & I\end{pmatrix}\Theta\begin{pmatrix}\nabla^*\\ I\end{pmatrix}\geq 0,\quad \forall\,\nabla\in\boldsymbol{\nabla} \tag{1}$$

$$\begin{pmatrix}I & M\end{pmatrix}\Theta\begin{pmatrix}I\\ M'\end{pmatrix}<0. \tag{2}$$

This problem is a convex problem; The set of feasible Θs is given by the intersection of (uncountably many) convex sets, each of which is defined by the LMIs for a fixed $\nabla\in\boldsymbol{\nabla}$.

However, it is difficult to solve this problem since the infinitely many constraints are not easily dealt with. In the context of control system analysis, the set $\boldsymbol{\nabla}$ has a particular structure, as illustrated by the following example.

Example 1 An important special case of Problem 1 is characterized by [4, 17]

$$\boldsymbol{\nabla} := \{\mathbf{diag}(\nabla_1 \cdots \nabla_m) : \ \nabla_i \in \mathbb{C}^{n_i \times n_i}, \ \|\nabla_i\| \leq 1\}.$$

If the number of the blocks in $\boldsymbol{\nabla}$ is three or less ($m \leq 3$), then a complete solution to Problem 1 is known [14]; In this case, it can be shown that if there exists Θ satisfying (1) and (2) then there exists another feasible Θ of the following structure:

$$\Theta = \mathbf{diag}(-\theta_1 I_{n_1}, \ldots, -\theta_m I_{n_m}, \ \theta_1 I_{n_1}, \ldots, \theta_m I_{n_m}) \tag{3}$$

for some real scalars $\theta_i > 0$. Note that for a Θ of this structure, condition (1) is automatically satisfied. Thus, Problem 1 reduces to the search for parameters $\theta_i > 0$ satisfying the LMI in (2), which can be solved efficiently. Unfortunately, this trick does not work for the case where $m > 3$; a matrix M can be constructed such that there is a feasible (unstructured) Θ but no feasible Θ of the structure in (3) exists. Nevertheless, the use of the structured Θ results in a sufficient condition for solvability of Problem 1, which is often of practical importance.

Another instance of specific $\boldsymbol{\nabla}$ is given by

$$\boldsymbol{\nabla} := \{\sum_{i=1}^{m} \lambda_i \nabla_i : \ \lambda_i \geq 0, \ \sum_{i=1}^{m} \lambda_i = 1\}$$

where ∇_i are given real matrices. Clearly,

$$\begin{pmatrix} \nabla_i & I \end{pmatrix} \Theta \begin{pmatrix} \nabla_i' \\ I \end{pmatrix} \geq 0, \quad \forall\, i = 1, \ldots, m \tag{4}$$

is necessary for (1). On the other hand, if

$$\Theta_{11} \leq 0, \quad \Theta := \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}' & \Theta_{22} \end{pmatrix} \tag{5}$$

then the vertex condition (4) becomes also sufficient for (1). This fact can readily be seen once we notice that the inequality in (1) is equivalent to

$$\begin{pmatrix} \nabla\Theta_{12} + \Theta_{12}'\nabla' + \Theta_{22} & \nabla\Theta_{11} \\ \Theta_{11}\nabla' & -\Theta_{11} \end{pmatrix} \geq 0$$

by the Schur complement formula, and thus the set of ∇ satisfying this inequality for a given Θ is convex. Hence, by searching for Θ satisfying (2), (4) and (5), a feasible Θ to Problem 1 may be found. Note, however, that this is only a sufficient condition due to (5) and this approach may fail even if Problem 1 admits a solution. See [1, 10, 13] for more details. ■

The following four problems arise in the synthesis of control systems. The first one probably covers the largest class of control problems among the four. The internal structure of each control problem has been lost by posing the problem in the general setting, and hence the first problem is perhaps the most difficult.

Problem 2 *Let real symmetric matrices Ω_0, F_i, G_j and H_{ij} be given, where $i = 1, \ldots, n_x$; $j = 1, \ldots, n_y$ and n_x and n_y are given positive integers. Find $x \in \mathbb{R}^{n_x}$ and $y \in \mathbb{R}^{n_y}$ such that*

$$\Omega(x,y) := \Omega_0 + \sum_{i=1}^{n_x} x_i F_i + \sum_{j=1}^{n_y} y_j G_j + \sum_{i=1}^{n_x}\sum_{j=1}^{n_y} x_i y_j H_{ij} > 0$$

Note that Ω is a bilinear function of variables x and y. We call $\Omega(x,y) > 0$ a bilinear matrix inequality (BMI). The feasible set is, in general, not convex. In particular, the smallest eigenvalue of $\Omega(x,y)$ is not necessarily a concave function of x and y. A particular example of Problem 2 is given below. See [9, 18] for more instances of control problems that can be reduced to Problem 2.

Example 2 Consider a linear time-invariant discrete-time system

$$x_{k+1} = Ax_k + Bu_k, \quad y_k = Cx_k \tag{6}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$ and $y_k \in \mathbb{R}^p$ are, respectively, the state, the control input, and the measured output; A, B and C are real matrices of appropriate dimensions; k is the time index. A fundamental problem in control design is the static output feedback stabilization problem: Find a feedback gain (constant matrix) K such that the control law

$$u_k = Ky_k \tag{7}$$

stabilizes the system. That is, find K such that the state of the closed loop system[3]

$$x_{k+1} = (A + BKC)x_k$$

approaches zero as the time goes to infinity for any given initial state x_0. It is well known that

$$\lim_{k\to\infty} x_k = 0, \quad \forall\, x_0$$

holds if and only if

$$\rho(A + BKC) < 1$$

holds. Note that

$$\begin{aligned} \rho(A+BKC) < 1 \;&\Leftrightarrow\; \exists\, D > 0 : \; \|D^{-1}(A+BKC)D\| < 1 \\ &\Leftrightarrow\; \exists\, P > 0 : \; P > (A+BKC)P(A+BKC)'. \end{aligned}$$

Hence, using the Schur complement formula, the stabilization problem is equivalent to finding matrices $P \in \mathcal{S}_n$ and K such that

$$\begin{pmatrix} P & (A+BKC)P \\ (A+BKC)'P & P \end{pmatrix} > 0. \tag{8}$$

Clearly, this is a BMI in variables P and K, and hence is a special case of Problem 2. ∎

[3] The difference equation for the closed loop system can be obtained by simply substituting (7) to (6).

The next problem is characterized by LMI constraints plus a rank constraint. This problem also covers a large class of control problems [9, 19].

Problem 3 *Let n and r be positive integers. Let an affine function $\Phi : \mathcal{S}_n \times \mathcal{S}_n \to \mathcal{S}_\ell$ be given. Find matrices $X \in \mathcal{DS}_n$ and $Y \in \mathcal{DS}_n$ such that*

$$\Phi(X,Y) > 0, \quad \begin{pmatrix} X & I \\ I & Y \end{pmatrix} \geq 0, \quad \mathbf{rank}\begin{pmatrix} X & I \\ I & Y \end{pmatrix} \leq r.$$

The first two constraints in Problem 3 are LMIs in terms of the variables X and Y, while the last one is not convex in general. Thus, the rank constraint complicates the problem. Note that if $r \geq 2n$, then the rank condition is automatically satisfied and Problem 3 becomes an LMI problem which can readily be solved. The following example reveals when this happens and gives a physical intuition to the rank condition.

Example 3 As in Example 2, we consider the stabilization problem for the system given in (6). This time, the controller to be designed is a dynamical system of the form

$$\begin{pmatrix} \hat{x}_{k+1} \\ u_k \end{pmatrix} = \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{pmatrix} \begin{pmatrix} \hat{x}_k \\ y_k \end{pmatrix}$$

where $\hat{x}_k \in \mathbb{R}^{\hat{n}}$ is the controller state. The number $\hat{n}$ is called the order of the controller. It can be shown [5, 11] that there exists a dynamic stabilizing controller of order $\hat{n}$ if and only if there exist matrices $X \in \mathcal{S}_n$ and $Y \in \mathcal{S}_n$ such that

$$\begin{array}{l} \mathcal{B}(X - AXA')\mathcal{B}' > 0 \\ \mathcal{C}(Y - A'YA)\mathcal{C}' > 0 \end{array}, \quad \begin{pmatrix} X & I \\ I & Y \end{pmatrix} \geq 0, \quad \mathbf{rank}\begin{pmatrix} X & I \\ I & Y \end{pmatrix} \leq n + \hat{n}$$

where $\mathcal{B}$ and $\mathcal{C}$ are given matrices defined by the system matrices B and C in (6), respectively. Clearly, the problem of finding such matrices X and Y is a special case of Problem 3. Note that $r := n + \hat{n}$ is greater than or equal to $2n$ when $\hat{n} \geq n$, i.e., the controller order is greater than or equal to the order of the system to be stabilized. Hence, if we allow for the controller order to be free in the design, then the stabilization problem becomes an LMI problem. In certain control applications, however, it is desired or required that the controller order be small, in order to allow practical implementations. This fact shows the significance of Problem 3. ∎

It can be shown that if $r = n$ then the last two conditions in Problem 3 are equivalent to $X = Y^{-1} > 0$. The following problem is concerned with this case.

Problem 4 *Let n and ℓ be a positive integer. Let an affine function $\Phi : \mathcal{S}_n \times \mathcal{S}_n \to \mathcal{S}_\ell$ be given. Find matrices $X \in \mathcal{DS}_n$ and $Y \in \mathcal{DS}_n$ such that*

$$\Phi(X,Y) > 0, \quad X > 0, \quad Y > 0, \quad XY = I.$$

As noted above, this is a special case of Problem 3 with $r = n$. In fact, *any* control problem known to be reducible to Problem 3 can also be cast as a special case of Problem 4 [9] with a different definition for the function Φ and an increased size of matrices X and Y. In this sense, Problems 3 and 4 are equivalent. However, algorithms that address these problems can be quite different due to the difference in the nature of the constraints.

Example 4 Let us consider the static output feedback stabilization problem defined in Example 2. As shown there, the problem has a solution if and only if there exist matrices $P = P'$ and K such that the BMI in (8) holds. It can be shown further [15] that the BMI admits a solution pair (P,K) if and only if there exist $X \in \mathcal{S}_n$ and $Y \in \mathcal{S}_n$ such that

$$\begin{array}{l} \mathcal{B}(X - AXA')\mathcal{B}' > 0 \\ \mathcal{C}(Y - A'YA)\mathcal{C}' > 0 \end{array}, \quad X > 0, \quad Y > 0, \quad XY = I$$

holds, where $\mathcal{B}$ and $\mathcal{C}$ are given matrices defined by the system matrices in (6). Note that the parameter K has been eliminated. This problem is a special case of Problem 4. The condition given above is precisely the necessary and sufficient condition for the existence of K satisfying (8) for a fixed P, where X and Y correspond to P and P^{-1}, respectively. ∎

The last problem presented below is characterized by LMI constraints plus a spectral radius condition, and is very similar to Problem 4. However, there is a difference that becomes significant when developing computational algorithms for these problems; namely, in the following problem, the feasible set can have a nonempty interior, while in Problem 4, there is no interior point in the feasible set due to the constraint $XY = I$.

Problem 5 *Let n and ℓ be a positive integer. Let an affine function $\Psi : \mathcal{S}_n \times \mathcal{S}_n \to \mathcal{S}_\ell$ be given. Find matrices $L \in \mathcal{DS}_n$ and $R \in \mathcal{DS}_n$ such that*

$$\Psi(L, R) > 0, \quad L > 0, \quad R > 0, \quad \rho(LR) \leq 1.$$

The troublesome factor of this problem is the last constraint given by the spectral radius inequality. This constraint is not convex, while all the other constraints are LMIs (and hence convex). In the context of control synthesis, Problems 4 and 5 are (different but) equivalent formulations of the same control problem. This point is clarified in the following example. Related results can be found in [16].

Example 5 Recall from Example 2 that the static output feedback stabilization problem is equivalent to finding matrices $P \in \mathcal{S}_n$ and K such that

$$P > 0, \quad P > (A + BKC)P(A + BKC)'.$$

It can readily be shown that, given K, there exists $P \in \mathcal{S}_n$ satisfying this condition if and only if there exist $R \in \mathcal{S}_n$ and $L \in \mathcal{S}_n$ such that

$$R^{-1} \geq L > 0, \quad L > (A + BKC)R^{-1}(A + BKC)'.$$

Note that matrices L and R correspond to the left and the right "scaling matrices" as in $\|L^{-1/2}(A + BKC)R^{-1/2}\| < 1$. Eliminating the parameter K from the second condition as explained in Example 4, we have

$$\mathcal{B}(L - AR^{-1}A')\mathcal{B}' > 0, \quad \mathcal{C}(R - A'L^{-1}A)\mathcal{C}' > 0$$

where $\mathcal{B}$ and $\mathcal{C}$ are computed from B and C, respectively. Noting that, for $L > 0$ and $R > 0$, condition $R^{-1} \geq L$ is equivalent to $\rho(LR) \leq 1$, and using the Schur complement formula, the problem reduces to: Find $L \in \mathcal{S}_n$ and $R \in \mathcal{S}_n$ such that

$$\begin{pmatrix} \mathcal{B}L\mathcal{B}' & \mathcal{B}A \\ A'\mathcal{B}' & R \end{pmatrix} > 0, \quad \begin{pmatrix} \mathcal{C}R\mathcal{C}' & \mathcal{C}A' \\ A\mathcal{C}' & L \end{pmatrix} > 0, \quad \rho(LR) \leq 1.$$

This is a special case of Problem 5. ∎

3 Computational Methods

In this section, we present several computational methods that address the problems listed in the previous section. As noted before, none of these problems have been solved completely. Our objective here is to give a tutorial overview of currently available computational techniques; The three methods given below often find solutions to Problems 3 and 4 without a guarantee for global convergence to the solution set.

3.1 Coordinate Descent Methods

This section presents an approach to Problem 4. The results given below are essentially from [12]. One way to address Problem 4 is to consider the following optimization problem:

$$\lambda_{\text{opt}} := \min_{(X,Y)\in\mathcal{C}} \lambda_{max}(XY) \tag{9}$$

$$\mathcal{C} := \{(X,Y) \in \mathcal{DS}_n \times \mathcal{DS}_n : \ \Phi(X,Y) > 0, \ \begin{pmatrix} X & I \\ I & Y \end{pmatrix} \geq 0\ \}$$

In order to motivate this formulation, note that, for $X > 0$ and $Y > 0$,

$$\begin{aligned} \begin{pmatrix} X & I \\ I & Y \end{pmatrix} \geq 0 \ &\Leftrightarrow \ X \geq Y^{-1} \\ &\Leftrightarrow \ Y^{1/2} X Y^{1/2} \geq I \\ &\Leftrightarrow \ \lambda_i(Y^{1/2} X Y^{1/2}) \geq 1 \\ &\Leftrightarrow \ \lambda_i(XY) \geq 1. \end{aligned}$$

Hence, the set $\mathcal{C}$ is nonempty whenever Problem 4 has a solution. Moreover, Problem 4 admits a solution iff $\lambda_{\text{opt}} = 1$ and the optimal value is attained at a point in $\mathcal{C}$. Unfortunately, the minimization problem is not convex since the objective function $\lambda_{max}(XY)$ is not, although the feasible set $\mathcal{C}$ is convex.

An approach to this nonconvex problem is to utilize the fact that, for each fixed $Y > 0$, the function $\phi(X) := \lambda_{max}(XY)$ is convex. Thus, we can successively minimize $\lambda_{max}(XY)$ over one variable while fixing the other by convex programming:

Coordinate Descent Algorithm

$$\text{initialization: } k = 1, \ \ (X_0, Y_1) \in \mathcal{C}$$

$$X_k := \textbf{argmin}\{\lambda_{max}(XY_k) : \ (X, Y_k) \in \mathcal{C}\ \}$$
$$Y_{k+1} := \textbf{argmin}\{\lambda_{max}(X_k Y) : \ (X_k, Y) \in \mathcal{C}\ \}$$

Of course, this algorithm may not give a globally optimal solution to the above minimization problem.

It is easily shown that the optimal value of each minimization problem is not attained, i.e. X_k and Y_{k+1} lie on the boundary of the open feasible domain $\mathcal{C}$. Hence, the set $\mathcal{C}$ must be replaced by a closed set inner approximation using a small scalar $\varepsilon > 0$ to guarantee $(X_k,Y_k) \in \mathcal{C}$ for all k. The initialization $(X_0, Y_1) \in \mathcal{C}$ guarantees that the feasible set of each minimization problem is nonempty at every iteration. Moreover, it can readily be shown that the value of the objective function is monotonically nonincreasing at every step:

$$\lambda_{max}(X_k Y_k) \geq \lambda_{max}(X_{k+1} Y_{k+1}).$$

Since $\lambda_{max}(X_k Y_k)$ is bounded below by one, the sequence is guaranteed to converge.

This simple algorithm works well for some cases but is suffered from slow convergence and a huge gap between the obtained solution and $\lambda_{\rm opt}$ for other cases. The following algorithm is a version of coordinate descent algorithms modified by replacing the minimization problems by the computation of the analytic center. Our numerical experience suggests that this algorithm tend to give a better solution with smaller amount of computations than the standard coordinate descent algorithm given above.

Successive Centering Algorithm (SCA)

1. Choose $\theta \in (0,1)$

2. Initialize $k = 1$, and, with $(\hat{X}, \hat{Y}) \in \mathcal{C}$,

$$Y_1 := \hat{Y}, \quad \beta_1 > \lambda_{max}(\hat{X}\hat{Y})$$

3. Compute the analytic center X_k and update α_k

$$X_k := \mathbf{ac}\{\ I < Y_k^{1/2} X Y_k^{1/2} < \beta_k I,\ (X, Y_k) \in \mathcal{C}\ \}$$

$$\alpha_k := (1-\theta)\lambda_{max}(X_k Y_k) + \theta\beta_k$$

4. Compute the analytic center Y_k and update β_k

$$Y_{k+1} := \mathbf{ac}\{\ I < X_k^{1/2} Y X_k^{1/2} < \alpha_k I,\ (X_k, Y) \in \mathcal{C}\ \}$$

$$\beta_k := (1-\theta)\lambda_{max}(X_k Y_{k+1}) + \theta\alpha_k$$

5. If $(X_k, X_k^{-1}) \in \mathcal{C}$ or $(Y_{k+1}^{-1}, Y_{k+1}) \in \mathcal{C}$, stop.
   Otherwise let $k \leftarrow k+1$ and go to Step 3.

Note that **ac** denotes the analytic center and is defined by (e.g. [2])

$$\mathbf{ac}\{\ \mho(X) > 0\ \} := \mathbf{argmin}\{\log\det \mho(X)^{-1} :\ \mho(X) > 0\ \}$$

where $\mho(X) > 0$ is an LMI.

In this algorithm, an initialization parameter $(\hat{X}, \hat{Y}) \in \mathcal{C}$ can be found by solving an LMI problem, or it can be determined that $\mathcal{C}$ is empty, in which case, Problem 4 has no solution. If $\mathcal{C}$ is nonempty, then the SCA is well-defined, i.e. each set defined by the argument of $\mathbf{ac}\cdot$ at Steps 3 and 4 is nonempty and bounded, and hence the analytic centers exist at any iteration $k \geq 1$. Moreover, upper bounds α_k and β_k on $\lambda_{max}(X_k Y_k)$ are strictly decreasing and bounded below by one:

$$\beta_k > \alpha_k > \beta_{k+1} > 1, \quad \forall\, k \geq 1$$

and hence the convergence is guaranteed.

Example 6 Consider

$$\min_{(x,y)\in\mathcal{C}} \; xy$$

$$\mathcal{C} := \{\ (x,y) :\ y \ge 5(1-x)+1,\ y \ge 0.5(1-x)+1\ \}$$

It can be checked that the global minimum value is one and is attained at $x = y = 1$. Applying the SCA, the following results are obtained.

Initial conditions: $y_0 = 2$, $\beta_0 = 5$, $\theta = 0.2$

Final values: $(x,y) = (1.0230, 0.9886)$, $xy = 1.0112$

The sequences x_k and y_k generated by the algorithm are plotted on the x-y plane as shown in Fig. 1.

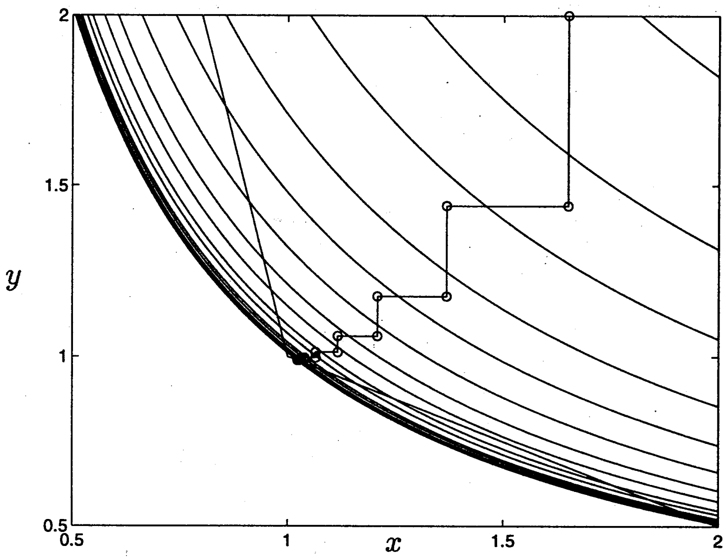

Figure 1. Behavior of the SCA on the x-y plane

In Fig. 1, the two straight lines intersecting at $(x,y) = (1,1)$ are the boundary of $\mathcal{C}$ which is the region above the lines. The curves are the hyperbolas $xy = \alpha_k$ and $xy = \beta_k$ for $k = 1, 2, \ldots$ For a given y_k, the analytic center x_k is determined to be the x-coordinate of the middle point of the segment defined by the intersection of the horizontal line $y = y_k$, the set $\mathcal{C}$, and the region between $xy = 1$ and $xy = \beta_k$.

Note that if we apply the coordinate descent algorithm, then the algorithm stops in one step, resulting in the final values $(x,y) = (0.8, 2)$, with $xy = 1.6$. ∎

A novel aspect of the SCA is that we do not require $\lambda_{max}(X_k Y_k) = 1$ to have a solution to Problem 4. Intuitively, the algorithm tries to make X_k and Y_k^{-1} (or equivalently, X_k^{-1} and Y_k) closer to each other while maintaining (X_k, Y_k) to be located "deep inside" the set $\mathcal{C}$. Thus, (X_k, X_k^{-1}) and (Y_k^{-1}, Y_k) are forced to move into $\mathcal{C}$. This point is illustrated by the following examples.

Example 7 Consider Problem 4 with

$$\Phi(X,Y) := \begin{pmatrix} \Phi_x(X) & 0 \\ 0 & \Phi_y(Y) \end{pmatrix}, \quad \begin{array}{r} \Phi_x(X) := -\mathcal{B}(AX + XA')\mathcal{B}' \\ \Phi_y(Y) := -\mathcal{C}(YA + A'Y)\mathcal{C}' \end{array}$$

$$A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{B} := \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \mathcal{C} := \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

The choice of Φ_x and Φ_y corresponds to the problem of static output feedback stabilization for continuous-time systems as opposed to the discrete-time setup given in Examples 2 and 4. The choice of $(A, \mathcal{B}, \mathcal{C})$ above corresponds to the fact that the underlying system is the double integrator $\ddot{y} = u$. It is known that there is no static output feedback stabilizing gain for this system, and thus Problem 4 is infeasible.

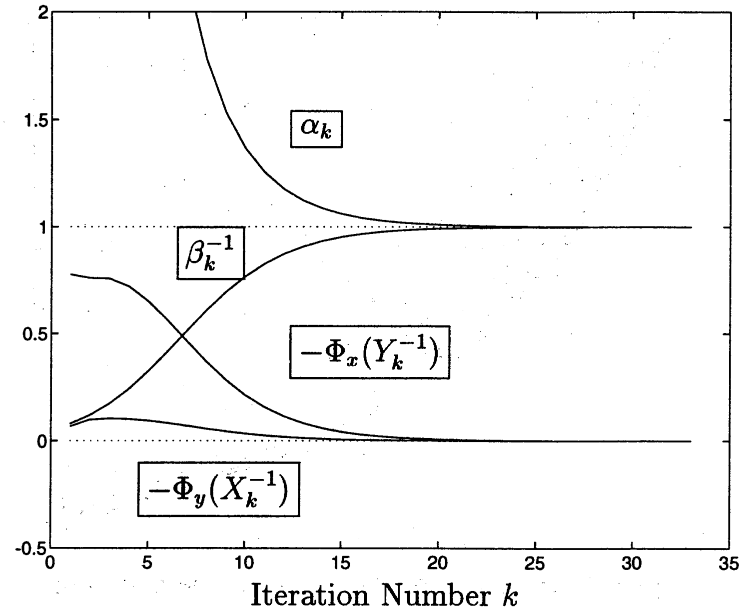

Figure 2. Behavior of the SCA (static output feedback)

The SCA is applied to this problem. The parameter θ is chosen to be $\theta = 0.2$. See [12] for details of the choice of an initial feasible point.

Fig. 2 shows the behavior of the SCA applied to this problem, where α_k and β_k are upper bounds on $\lambda_{max}(X_k Y_k)$. To visualize the convergence behavior, β_k^{-1} is plotted instead of β_k; in this way, the distance between the two curves corresponding to α_k and β_k^{-1} can be considered as a measure for the "distance" between X_k and Y_k^{-1}. Similarly, the curves $-\Phi_x(Y_k^{-1})$ and $-\Phi_y(X_k^{-1})$ indicate the "distance" between (Y_k^{-1}, Y_k) and $\mathcal{C}$, and (X_k, X_k^{-1}) and $\mathcal{C}$, respectively. For instance, $(Y_k^{-1}, Y_k) \in \mathcal{C}$ if and only if $-\Phi_x(Y_k^{-1}) < 0$.

Interestingly, the sequence $X_k Y_k$ approaches I (or equivalently, $\alpha_k \to 1$) even though Problem 4 is infeasible. What's happening here is that (Y_k^{-1}, Y_k) approaches $\mathcal{C}$, but never reaches (belongs to) $\mathcal{C}$, that is, $\Phi_x(Y_k^{-1}) \leq 0$ for all k. After 33 iterations, we have

$$X = \begin{pmatrix} 1.5470 & -4.4991 \times 10^{-5} \\ -4.4991 \times 10^{-5} & 4.7298 \end{pmatrix}, \quad \Phi_x(X) = 9.00\times 10^{-5}, \quad \Phi_y(X^{-1}) = -1.23\times 10^{-5},$$

$$Y = \begin{pmatrix} 0.64646 & -5.1534\times10^{-6} \\ -5.1534\times10^{-6} & 0.21144 \end{pmatrix}, \quad \Phi_x(Y^{-1}) = -7.54\times10^{-5}, \quad \Phi_y(Y) = 1.03\times10^{-5},$$
$$\lambda(XY) = 1.00008,\ \ 1.00002.$$

Thus, $X \cong Y^{-1}$ holds but neither (X, X^{-1}) nor (Y^{-1}, Y) belongs to $\mathcal{C}$. Note that, this example shows that $\alpha_k \to 1$ does not imply that Problem 4 is feasible. ■

Example 8 Consider the same problem as the one treated in Example 7 with different problem data:

$$A := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad C := \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}.$$

This problem corresponds to the stabilization of the double integrator system mentioned in Example 7 with a dynamic controller of order one (as opposed to the static output feedback controller in Example 7). In this case, it is known that a stabilizing controller exists, i.e., Problem 4 is feasible.

Again, the SCA is applied to the problem. The parameter θ is set to $\theta = 0.2$ as before. Fig. 3 shows the behavior of the algorithm. To design a stabilizing controller, four iterations suffice since $(X_k, X_k^{-1}) \in \mathcal{C}$ and $(Y_k^{-1}, Y_k) \in \mathcal{C}$ at $k = 4$. When the algorithm continues to run, (X_k, X_k^{-1}) and (Y_k^{-1}, Y_k) move "deep inside" the set $\mathcal{C}$. This is a typical behavior of the SCA. After 17 iterations, we have

$$Y^{-1} \cong X = \begin{pmatrix} 1.1458 & -0.6910 & -1.6325 \\ -0.6910 & 1.7966 & 2.0712 \\ -1.6325 & 2.0712 & 3.6253 \end{pmatrix}, \quad \lambda(XY) = \begin{cases} 1.000048 \\ 1.000048 \\ 1.000048 \end{cases}.$$

Thus we obtained a globally optimal solution to problem (9) within the optimality tolerance $|\lambda(XY) - \lambda_{\text{opt}}| \le 5\times10^{-5}$. ■

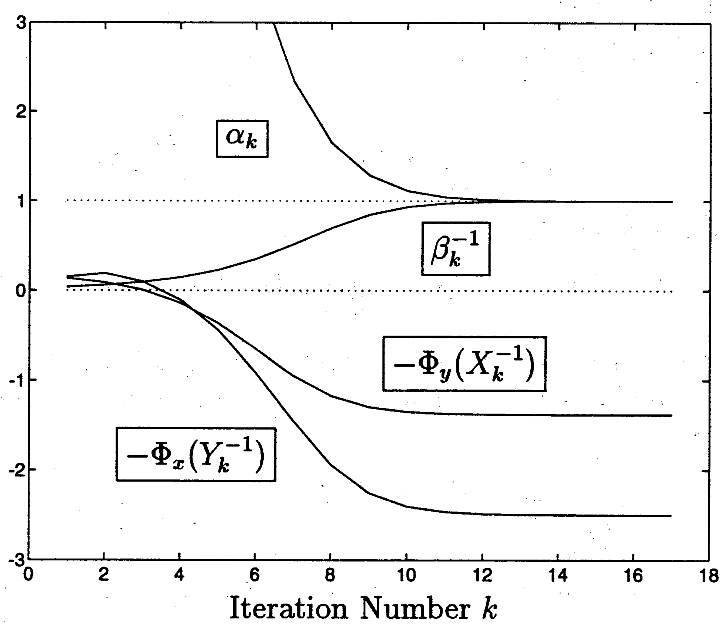

Figure 3. Behavior of the SCA (1*st* order dynamic output feedback)

3.2 Linearization methods

The computational method presented in this section addresses Problem 3 and is due to [6]. Here, we will try to minimize the bound r on the rank subject to the constraints in Problem 3. The idea is based on the following observation: The rank condition in Problem 3 is satisfied by pushing X and Y to the "boundary" of the second inequality in Problem 3. A natural way to do this is to consider the following minimization problem:

$$\phi_{\text{opt}} := \min_{(X,Y)\in\mathcal{C}} \phi(X,Y)$$

$$\mathcal{C} := \{(X,Y) : \Phi(X,Y) > 0, \begin{pmatrix} X & I \\ I & Y \end{pmatrix} \geq 0\}$$

where $\phi : \mathcal{C} \to \mathbb{R}$ is a function satisfying the properties:

(a) $\phi(X,Y) \geq \phi_{min}, \ \forall\,(X,Y) \in \mathcal{C}$

(b) $\phi(X,Y) = \phi_{min} \Leftrightarrow XY = I$

(c) ϕ is differentiable on $\mathcal{C}$

(d) ϕ is quasi-concave on $\mathcal{C}$

The function ϕ is called the "attracting function" due to properties (a) and (b). Recall from Section 3.1 that the second inequality in Problem 3 holds if and only if all the eigenvalues of XY is greater than or equal to one. If X and Y are "right" on the boundary, then $\lambda_i(XY) = 1$ for all i, in which case we have $XY = I$. A choice for ϕ is given by

$$\phi(X,Y) = 2\mathbf{tr}(XY)^{1/2}.$$

This function satisfies all the properties listed above. Then, the following algorithm can be applied to find a local solution to the above minimization problem.

Linearization Algorithm (LA)

1. Initialize $k = 0$ and $(X_0, Y_0) \in \mathcal{C}$.

2. Let
$$V_k := \frac{\partial\phi}{\partial X}(X_k, Y_k), \quad W_k := \frac{\partial\phi}{\partial Y}(X_k, Y_k)$$

3. Solve the LMI problem:
$$(X_{k+1}, Y_{k+1}) := \underset{(X,Y)\in\mathcal{C}}{\mathbf{argmin}}\ \mathbf{tr}(V_kX + W_kY)$$

Note that property (c) is necessary to compute the gradients V_k and W_k at step 2. In some cases, explicit formulas for the gradients exist. For instance, the gradients of $\phi(X,Y) = 2\mathbf{tr}(XY)^{1/2}$ are given by

$$\begin{matrix} V = H^{-1}(HYH)^{1/2}H^{-1} \\ W = V^{-1} \end{matrix}, \quad H := X^{1/2}.$$

It can be shown that if ϕ satisfies properties (a), (b) and (d), then the sequence $\phi(X_k, Y_k)$ is bounded below and decreasing. Thus, the LA converges. The idea for the proof is that a concave function is bounded above by its linear approximation.

Example 9 Consider Problem 3 with

$$\Phi(X,Y) := \begin{pmatrix} \Phi_x(X) & 0 \\ 0 & \Phi_y(Y) \end{pmatrix}, \quad \begin{array}{l} \Phi_x(X) := -\mathcal{B}(AX + XA' + 0.4X)\mathcal{B}' \\ \Phi_y(Y) := -\mathcal{C}(YA + A'Y + 0.4Y)\mathcal{C}' \end{array}$$

$$A := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}, \mathcal{B} := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}', \mathcal{C} := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}'.$$

This problem corresponds to the design of a controller that yields the closed loop poles in the region $\{ \lambda \in \mathbb{C} : Re(\lambda) < -0.2 \}$ where $Re(\cdot)$ is the real part of the complex number. This pole constraint imposes a requirement on the rate of convergence of the initial state response. The problem data $(A, \mathcal{B}, \mathcal{C})$ has a physical significance; the underlying system to be controlled is a mechanical system consisting of two masses connected by a spring, placed on a frictionless horizontal surface [20]. It is known that the system is stabilizable via $2nd$ order controller but no $1st$ order controller stabilizes the system. Therefore, we see that Problem 3 has no solution if $r \leq 5 (= 4 + 1)$.

Applying the LA to this problem with a particular choice of ϕ given above, we obtained a solution. The behavior of the LA is plotted in Fig. 4.

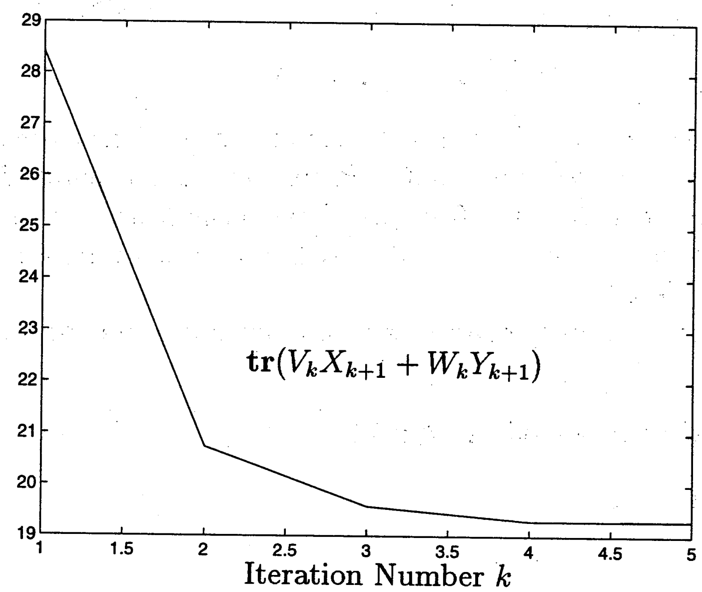

Figure 4. Behavior of the LA

After 5 iterations, the algorithm converged and we obtained the following result:

$$X = \begin{pmatrix} 2.8745 & 2.0917 & -0.5749 & -1.3931 \\ 2.0917 & 4.3544 & 0.5564 & -0.8709 \\ -0.5749 & 0.5564 & 1.4962 & -0.2256 \\ -1.3931 & -0.8709 & -0.2256 & 2.6111 \end{pmatrix}$$

$$Y = \begin{pmatrix} 2.6111 & -0.2256 & -0.8709 & -1.3931 \\ -0.2256 & 1.4962 & 0.5564 & -0.5749 \\ -0.8709 & 0.5564 & 4.3544 & 2.0917 \\ -1.3931 & -0.5749 & 2.0917 & 2.8745 \end{pmatrix}$$

$$\lambda(XY) = 1.0000, \ 1.0000, \ 8.9284, \ 21.7285$$

Note that the rank in Problem 3 is reduced from the full rank by 2 (the number of the eigenvalues of XY that are equal to one). Thus we have found a solution to Problem 3 with $r = 6$. Since there is no solution for $r \leq 5$, we conclude that the global minimum rank is attained. ∎

3.3 Projection Method

We consider Problem 3 again. The projection method presented here is from [7] and its extensions.

In this method, Problem 3 is formulated as follows: Find Z_{feas} such that

$$Z_{\text{feas}} \in \mathcal{C} \cap \mathcal{R}$$

where

$$\mathcal{C} := \{Z = \begin{pmatrix} X & I \\ I & Y \end{pmatrix} : \ \Phi(X,Y) \geq 0, \ Z \geq 0, \ X \in \mathcal{DS}_n, \ Y \in \mathcal{DS}_n \}$$

$$\mathcal{R} := \{Z : \ \mathbf{rank}(Z) \leq r \}$$

Here, $\mathcal{C}$ is a closed convex set and $\mathcal{R}$ is closed but not convex. The closedness of these sets is crucial for the projection method. Note that in Problem 3, the inequality for $\Phi(X,Y)$ is strict, while it is weakened to be $\Phi(X,Y) \geq 0$ in the above formulation. If this is not acceptable, one may use a fixed small parameter $\varepsilon > 0$ to define a closed inner-approximation for the feasible domain of Problem 3.

The following algorithm locally converges to a point in the intersection of the two sets $\mathcal{C}$ and $\mathcal{R}$.

Alternating Projection Algorithm (APA)

1. Initialize $k = 0$ and Z_0.

2. $\hat{Z}_{k+1} := \mathcal{P}_{\mathcal{C}} Z_k$

3. $Z_{k+1} := \mathcal{P}_{\mathcal{R}} \hat{Z}_{k+1}$

where $\mathcal{P}_{\mathcal{S}}$ is the projection onto the set $\mathcal{S}$ ($= \mathcal{C}$ or $\mathcal{R}$):

$$\mathcal{P}_{\mathcal{S}} Z_0 := \mathbf{argmin} \ \{\|Z - Z_0\|_F : \ Z \in \mathcal{S} \} \tag{10}$$

with $\|\cdot\|_F$ being the Frobenius norm.

This algorithm generates a sequence Z_k by alternatingly projecting the current point in one set onto the other set (see Fig. 5 for the conceptual sketch). Note that the algorithm does not require any property for the initial point; it can be chosen arbitrarily.

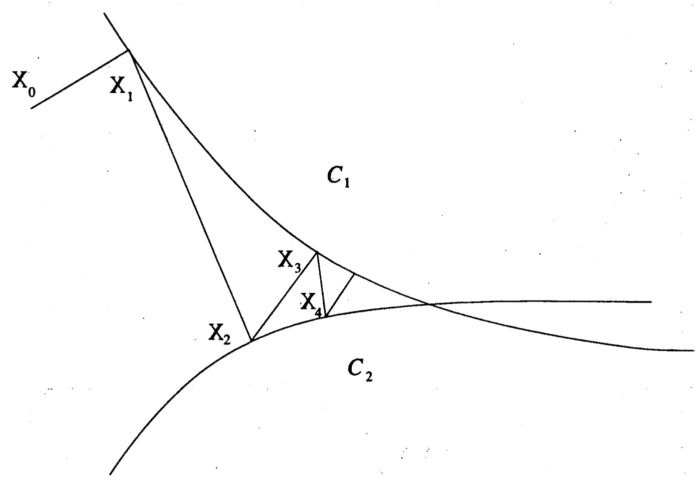

Figure 5. Alternating projection algorithm

If the set $\mathcal{S}$ is convex, then the projection $\mathcal{P}_{\mathcal{S}}Z_0$ is unique, i.e. the minimization problem in (10) has a unique minimizer. In fact, if both $\mathcal{C}$ and $\mathcal{R}$ were convex, then the APA is guaranteed to find a Z_{feas} whenever $\mathcal{C} \cap \mathcal{R}$ is nonempty, regardless of the initial point Z_0. If $\mathcal{C} \cap \mathcal{R}$ is empty, then the algorithm generates, in the steady state, a sequence that alternates between the two points giving the minimum distance of $\mathcal{C}$ and $\mathcal{R}$. Thus, under the convexity assumption, the APA is globally convergent. However, the set $\mathcal{R}$ is not convex and the APA is only locally convergent when applied to Problem 3.

The projection $\mathcal{P}_{\mathcal{C}}Z_k$ can be computed by solving an LMI problem. A projection $\mathcal{P}_{\mathcal{R}}\hat{Z}_{k+1}$ can be obtained by replacing the $2n-r$ small singular values of $\hat{Z}_{k+1}$ by zeros [8]. This procedure gives a minimizer to problem (10) with $\mathcal{S}$ and Z_0 replaced by $\mathcal{R}$ and $\hat{Z}_{k+1}$, respectively. Note that such minimizers may not be unique.

Example 10 We consider the problem treated in Example 9. Fig. 6 shows the behavior of the APA, where $\lambda_1(XY)$ and $\lambda_2(XY)$ are the smallest two eigenvalues of XY. The curves are plotted using the data $\hat{Z}_k \in \mathcal{C}$. Note that the smallest eigenvalue $\lambda_1(XY)$ is always one. This is because the projection $\mathcal{P}_{\mathcal{C}}Z_k$ always lies on the boundary of $\mathcal{C}$ when $Z_k \notin C$ (in particular the "boundary" of $Z \geq 0$). For this particular problem, the APA took more iterations than the LA. It is possible to (heuristically) improve the rate of convergence by taking the directional information into account [19].

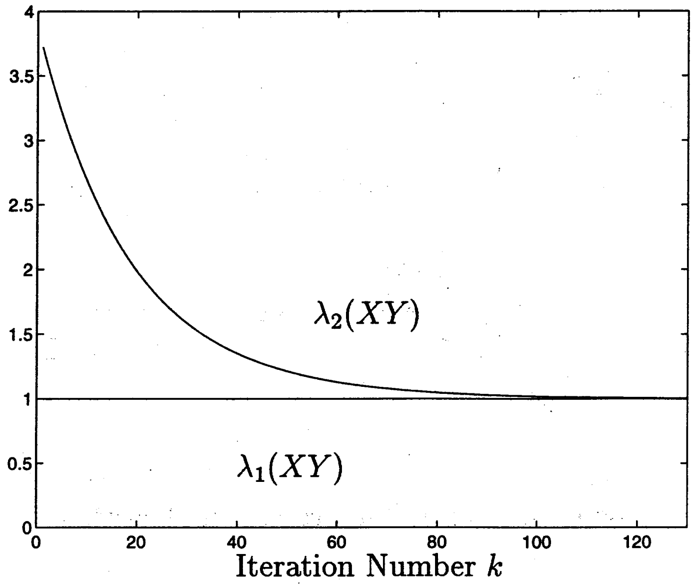

Figure 6. Behavior of the APA

After 130 iterations, the APA converged to

$$X = \begin{pmatrix} 2.8643 & 1.8096 & -0.7312 & -1.5714 \\ 1.8096 & 4.5226 & 0.8893 & -0.9077 \\ -0.7312 & 0.8893 & 1.8242 & -0.2438 \\ -1.5714 & -0.9077 & -0.2438 & 3.0529 \end{pmatrix}$$

$$Y = \begin{pmatrix} 3.0529 & -0.2438 & -0.9077 & -1.5714 \\ -0.2438 & 1.8242 & 0.8893 & -0.7312 \\ -0.9077 & 0.8893 & 4.5226 & 1.8096 \\ -1.5714 & -0.7312 & 1.8096 & 2.8643 \end{pmatrix}$$

$$\lambda(XY) = 0.9990, \;\; 1.0024, \;\; 13.7234, \;\; 25.6750.$$

Thus, we have found a feasible solution to Problem 3 with $r = 6$ which is the minimum attainable rank. ∎

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