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Conserved Quantities of "Random-Time Toda Equation"

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ABSTRACT: "Random-time Toda equation" is obtained by replacing the time-interval of the discrete-time Toda equation by random variables. The random-time Toda equation has higher-order conserved quantities in spite of the randomness introduced to the equation. Also obtained are the higher-order conserved quantities of a class of "Random-time soliton equations" which are related to the random-time Toda equation via Miura transformations.

KEYWORDS: Toda equation,randomness,conserved quantities,Miura transformation

In this letter we present "Random-time Toda equation" where the time-interval of the discrete-time Toda equation are replaced by random variables, and show a Lax pair of the random-time Toda equation which gives higher-order conserved quantities in spite of the randomness introduced to the equation.

We have the Toda equation of the form

$$\frac{d}{dt}J_n = V_{n-1} - V_n, \tag{1}$$

$$\frac{d}{dt}\log V_n = J_n - J_{n+1}, \tag{2}$$

which we discretized in a previous paper [1)] in the following form

$$J_n^{m+1} - \delta V_{n-1}^{m+1} = J_n^m - \delta V_n^m, \tag{3}$$

$$V_n^{m+1}(1 - \delta J_n^{m+1}) = V_n^m(1 - \delta J_{n+1}^m), \tag{4}$$

where δ is the time-interval and $t = m\delta$ for integers m. We called a couple of equations (3) and (4) "Discrete-time Toda equation".

Now we replace the time-interval δ in Eqs.(3) and (4) by random variables δ^m in the following way

$$J_n^{m+1} - \delta^{m+1} V_{n-1}^{m+1} = J_n^m - \delta^m V_n^m, \tag{5}$$

$$V_n^{m+1}(1 - \delta^{m+1} J_n^{m+1}) = V_n^m(1 - \delta^m J_{n+1}^m), \tag{6}$$

which we call "Random-time Toda Equation".

Let us introduce new dependent variables $x_n, \hat{x}_n, y_n, \hat{y}_n$, by the following relations:

$$x_n^m = J_n^m - \delta^m V_{n-1}^m, \tag{7}$$

$$\hat{x}_n^m = J_n^m - \delta^m V_n^m, \tag{8}$$

$$y_n^m = V_n^m(1 - \delta^m J_n^m), \tag{9}$$

$$\hat{y}_n^m = V_n^m(1 - \delta^m J_{n+1}^m). \tag{10}$$

Then the random-time Toda equation is written in a simpe form:

$$x_n^{m+1} = \hat{x}_n^m, \tag{11}$$

$$y_n^{m+1} = \hat{y}_n^m. \tag{12}$$

Then, a Lax pair $\{L, A\}$ of the random-time Toda equation under the periodic boundary conditions:

$$V_{N+1}^m = V_1^m, \tag{13}$$

$$I_{N+1}^m = I_1^m, \tag{14}$$

is expressed as follows.

$$L^m = \begin{pmatrix} 1 - c^m x_1^m & c^m & 0 & \cdots & 0 & c^m y_N^m \\ c^m y_1^m & 1 - c^m x_2^m & c^m & \cdots & 0 & 0 \\ 0 & c^m y_2^m & 1 - c^m x_3^m & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & 1 - c^m x_{N-1}^m & c^m \\ c^m & 0 & 0 & \cdots & c^m y_{N-1}^m & 1 - c^m x_N^m \end{pmatrix},$$

$$A^m = \begin{pmatrix} 1 & 0 & 0 & \cdots & \delta^m V_N^n \\ \delta^m V_1^m & 1 & 0 & \cdots & 0 \\ 0 & \delta^m V_2^m & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \delta^m V_{n-1}^m & 1 \end{pmatrix}, \tag{15}$$

where c^m are arbitrary constants.

It is easy to see that a commutation relation:

$$A^m L^{m+1} - L^m A^m = 0 \tag{16}$$

gives the random-time Toda equation under the periodic boundary condition.

Eq.(16) gives higher conserved quantities $H_n (= \mathrm{Trace}[L^m]^n,\ n = 1, 2, 3, \cdots)$ of the random-time Toda equation because that

$$\mathrm{Trace}[L^{m+1}]^n = \mathrm{Trace}[L^m]^n. \tag{17}$$

We have shown in a previous paper [2)] that Miura transformations generate higher-order conserved quantities of a class of discrete soliton equations which are related to the discrete-time Toda equation. Similarly we obtain in the present paper higher-order conserved quantities of a class of "Random-time soliton equations" which are related to the random-time Toda equation via Miura transformations.

We have the Random-time Toda equation

$$J_n^{m+1} - \delta^{m+1} V_{n-1}^{m+1} = J_n^m - \delta^m V_n^m, \tag{18}$$

$$V_n^{m+1}(1 - \delta^{m+1} J_n^{m+1}) = V_n^m(1 - \delta^m J_{n+1}^m), \tag{19}$$

which is related to "Random-time Lotka-Volterra equation of type I"

$$v_n^{m+1}(1 - \delta^{m+1} v_{n-1}^{m+1}) = v_n^m(1 - \delta^m v_{n+1}^m) \tag{20}$$

via the Miura transformation:

$$V_n^m = v_{2n}^m v_{2n+1}^m, \tag{21}$$

$$J_n^m = v_{2n-1}^m + v_{2n}^m - \delta^m v_{2n-1}^m v_{2n}^m. \tag{22}$$

The random-time Lotka-Volterra equation of type I" is related to "Random-time Lotka-Volterra equation of type II"

$$\frac{w_n^{m+1}}{(1 + \delta^{m+1} w_{n-1}^{m+1})(1 + \delta^{m+1} w_n^{m+1})} = \frac{w_n^m}{(1 + \delta^m w_n^m)(1 + \delta^m w_{n+1}^m)} \tag{23}$$

via the Miura transformation:

$$v_n^m = \frac{w_n^m}{1 + \delta_n^m w_n^m}. \tag{24}$$

The random-time Lotka-Volterra equation of type II" is related to "Random-time KdV equation "

$$\frac{1}{u_n^{m+1}} + \delta^{m+1} \frac{1}{u_{n-1}^{m+1}} = \frac{1}{u_n^m} + \delta^m \frac{1}{u_{n+1}^m} \tag{25}$$

via the Miura transformation:

$$w_n^m = u_n^m u_{n+1}^m. \tag{26}$$

Following the same procedure as one developed in the previous paper [2], higher order conserved quantities of these equations are expressed by using the higher order conserved quantities of the random-time Toda equation H_n.

References

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