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EXPONENTIALLY NASH VECTOR BUNDLES WITH GROUP ACTION

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1. Introduction.

Let G be an affine exponentially Nash group. In this note, we are concerned with fundamental properties of exponentially Nash G vector bundles. Our results in the present note are an exposition of [6].

Nash vector bundles (resp. Nash manifolds) are bundles (resp. manifolds) intermediate between real algebraic ones and C^ω ones. It is known that there are some useful categories between Nash one and C^∞ one (e.g. [3], [11], [12], [13], [14], [24], [25]). One of them is an exponentially Nash category.

Nash manifolds have been studies for a long time and there are many brilliant works (e.g. [1], [2], [10], [17], [18], [19], [20], [21]).

The semialgebraic subsets of $\mathbb{R}^n$ are just the subsets of $\mathbb{R}^n$ definable in the standard structure $\mathbf{R}_{stan} := (\mathbb{R}, <, +, \cdot, 0, 1)$ of the field $\mathbb{R}$ of real numbers [22]. It is known that there are only three useful collections of sets definable in $\mathbf{R}_{stan}$ [15]. These collections are the sets of semilinear sets, semibounded sets, and semialgebraic sets. However any non-polynomially bounded function is not definable in $\mathbf{R}_{stan}$, where a polynomially bounded function means a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ admitting an integer $N \in \mathbb{N}$ and a real number $x_0 \in \mathbb{R}$ with $|f(x)| \leq x^N$, $x > x_0$. C. Miller [16] proved that if there exists a non-polynomially bounded function definable in an o-minimal expansion $(\mathbb{R}, <, +, \cdot, 0, 1, ...)$ of $\mathbf{R}_{stan}$, then the exponential function $exp : \mathbb{R} \longrightarrow \mathbb{R}$ is definable in this structure. Hence $\mathbf{R}_{exp} := (\mathbb{R}, <, +, \cdot, exp, 0, 1)$ is a natural expansion of $\mathbf{R}_{stan}$. There are a number of results on $\mathbf{R}_{exp}$ (e.g. [11], [12], [13], [14], [25]), in particular $\mathbf{R}_{exp}$ is o-minimal. Since $\mathbf{R}_{exp}$ does not have elimination of quantifiers, in $\mathbf{R}_{exp}$ Tarski-Seidenberg Theorem does not hold true. Remark that there are another expansions of $\mathbf{R}_{stan}$ with similar properties of $\mathbf{R}_{exp}$ ([3], [4], [25]).

We say that a C^r manifold $(0 \leq r \leq \omega)$ is *an exponentially C^r Nash manifold* if it is definable in $\mathbf{R}_{exp}$ (See Definition 2.5). Equivariant such manifolds are defined in the similar way (See Definition 2.8). *Equivariant exponentially Nash vector bundles* are defined as well as Nash ones (See Definition 2.11).

In this note, all exponentially Nash groups, all exponentially Nash G manifolds and exponentially Nash G vector bundles are of class C^ω, and every manifold does not have boundary unless otherwise stated.

Theorem [6]. *Let G be a compact affine exponentially Nash group and let X be a compact affine exponentially Nash G manifold.*
(1) For every $C^\infty G$ vector bundle η over X, there exists a strongly exponentially Nash G vector bundle (See Definition 2.13) ζ which is $C^\infty G$ vector bundle isomorphic to η.
(2) For any two strongly exponentially Nash G vector bundles over X, they are exponentially Nash G vector bundle isomorphic if and only if they are $C^0 G$ vector bundle isomorphic.
(3) If $\dim X \geq 1$ and X has a 0-dimensional orbit, then for any $C^\infty G$ vector bundle η' of positive rank over X, there exists a non-strongly exponentially Nash G vector bundle ζ' which is $C^\infty G$ vector bundle isomorphic to η'.

In the equivariant Nash category, a stronger version of Theorem (3) holds true [9]. Remark that Nash structures of $C^\infty G$ manifolds and $C^\infty G$ vector bundles are studied in [8] and [5], respectively.

2. Exponentially Nash G manifolds and exponentially Nash G vector bundles.

Recall the definition of exponentially Nash G manifolds and exponentially Nash G vector bundles [7] and basic facts [7].

Definition 2.1. (1) An $\mathbf{R}_{exp}$-*term* is a finite string of symbols obtained by repeated applications of the following two rules:
[1] Constants and variables are $\mathbf{R}_{exp}$-terms.
[2] If f is an m-place function symbol of $\mathbf{R}_{exp}$ and $t_1, \dots, t_m$ are $\mathbf{R}_{exp}$-terms, then the concatenated string $f(t_1, \dots, t_m)$ is an $\mathbf{R}_{exp}$-term.
(2) An $\mathbf{R}_{exp}$-*formula* is a finite string of $\mathbf{R}_{exp}$-terms satisfying the following three rules:
[1] For any two $\mathbf{R}_{exp}$-terms t_1 and t_2, $t_1 = t_2$ and $t_1 > t_2$ are $\mathbf{R}_{exp}$-formulas.
[2] If ϕ and ψ are $\mathbf{R}_{exp}$-formulas, then the negation $\neg\phi$, the disjunction $\phi \vee \psi$, and the conjunction $\phi \wedge \psi$ are $\mathbf{R}_{exp}$-formulas.
[3] If ϕ is an $\mathbf{R}_{exp}$-formula and v is a variable, then $(\exists v)\phi$ and $(\forall v)\phi$ are $\mathbf{R}_{exp}$-formulas.
(3) *An exponentially definable set* $X \subset \mathbb{R}^n$ is the set defined by an $\mathbf{R}_{exp}$-formula (with parameters).
(4) Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be exponentially definable sets. A map $f : X \longrightarrow Y$ is called *exponentially definable* if the graph of $f \subset \mathbb{R}^n \times \mathbb{R}^m$ is exponentially definable.

On the other hand, using [12] any exponentially definable subset of $\mathbb{R}^n$ is the image of an $\mathfrak{R}_{n+m}$-semianalytic set by the natural projection $\mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ for some m. Here a subset X of $\mathbb{R}^n$ is called $\mathfrak{R}_n$-*semianalytic* if X is a finite union of sets of the following form:

$$\{x \in \mathbb{R}^n | f_i(x) = 0, g_j(x) > 0, 1 \leq i \leq k, 1 \leq j \leq l\},$$

where $f_i, g_j \in \mathbb{R}[x_1, \dots, x_n, exp(x_1), \dots, exp(x_n)]$.

The following is a collections of properties of exponentially definable sets (cf. [7]).

Proposition 2.2 (cf. [7]). *(1) Any exponentially definable set consists of only finitely many connected components.*

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be exponentially definable sets.
(2)The closure $Cl(X)$ and the interior $Int(X)$ of X in $\mathbb{R}^n$ are exponentially definable.
(3) The distance function $d(x, X)$ from x to X defined by $d(x, X) = \inf\{||x-y|| | y \in X\}$ is a continuous exponentially definable function, where $||\cdot||$ denotes the standard norm of $\mathbb{R}^n$.
(4) Let $f : X \longrightarrow Y$ be an exponentially definable map. If a subset A of X is exponentially definable then so is $f(A)$, and if $B \subset Y$ is exponentially definable then so is $f^{-1}(B)$.
(5) Let $Z \subset \mathbb{R}^l$ be an exponentially definable set and let $f : X \longrightarrow Y$ and $h : Y \longrightarrow Z$ be exponentially definable maps. Then the composition $h \circ f : X \longrightarrow Z$ is also exponentially definable. In particular for any two polynomial functions $f, g : \mathbb{R} \longrightarrow \mathbb{R}$, the function $h : \mathbb{R} - \{f = 0\} \longrightarrow \mathbb{R}$ defined by $h(x) = e^{g(x)/f(x)}$ is exponentially definable.
(6) The set of exponentially definable functions on X forms a ring.
(7) Any two disjoint closed exponentially definable sets X and $Y \subset \mathbb{R}^n$ can be separated by a continuous exponentially definable function. $\square$

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open exponentially definable sets. A C^r $(0 \le r \le \omega)$ map $f : U \longrightarrow V$ is called *an exponentially C^r Nash map* if it is exponentially definable. An exponentially C^r Nash map $g : U \longrightarrow V$ is called *an exponentially C^r Nash diffeomorphism* if there exists an exponentially C^r Nash map $h : V \longrightarrow U$ such that $g \circ h = id$ and $h \circ g = id$. Remark that the graph of an exponentially C^r Nash map may be defined by an $\mathbf{R}_{exp}$-formula with quantifiers.

Theorem 2.3 [14]. *Let $S_1, \ldots, S_k \subset \mathbb{R}^n$ be exponentially definable sets. Then there exists a finite family $\mathfrak{W} = \{\Gamma^d_\alpha\}$ of subsets of $\mathbb{R}^n$ satisfying the following four conditions:*
(1) Γ^d_α are disjoint, $\mathbb{R}^n = \cup_{\alpha,d}\Gamma^d_\alpha$ and $S_i = \cup\{\Gamma^d_\alpha | \Gamma^d_\alpha \cap S_i \neq \emptyset\}$ for $1 \le i \le k$.
(2) Each Γ^d_α is an analytic cell of dimension d.
(3) $\overline{\Gamma^d_\alpha} - \Gamma^d_\alpha$ is a union of some cells Γ^e_β with $e < d$.
(4) If $\Gamma^d_\alpha, \Gamma^e_\beta \in \mathfrak{W}, \Gamma^e_\beta \subset \overline{\Gamma^d_\alpha} - \Gamma^d_\alpha$ then $(\Gamma^d_\alpha, \Gamma^e_\beta)$ satisfies Whitney's conditions (a) and (b) at all points of Γ^e_β. $\square$

Theorem 2.3 allows us to define the *dimension* of an exponentially definable set E by

$$\dim E = \max\{\dim \Gamma | \Gamma \text{ is an analytic submanifold contained in } E\}.$$

Example 2.4. (1) The C^∞ function $\lambda : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\lambda(x) = \begin{cases} 0 & \text{if} \quad x \le 0 \\ e^{(-1/x)} & \text{if} \quad x > 0 \end{cases}$$

is an exponentially C^∞ Nash map. This example shows that an exponentially definable C^∞ map is not always analytic. This phenomenon does not occur in the usual Nash category. Notice that every C^∞ Nash map is a C^ω Nash map.

(2) The Zariski closure of the graph of the exponential function $exp : \mathbb{R} \longrightarrow \mathbb{R}$ in $\mathbb{R}^2$ is the whole space $\mathbb{R}^2$. Hence the dimension of the graph of exp is smaller than that of its Zariski closure.
(3) The continuous function $h : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$h(x) = \begin{cases} e^{x-n} & \text{if} \quad n \le x \le n+1 \\ e^{n+2-x} & \text{if} \quad n+1 \le x \le n+2 \end{cases}, \text{ for } n \in 2\mathbb{Z},$$

is not exponentially definable, but the restriction of h on any bounded exponentially definable set is exponentially definable. $\square$

Definition 2.5. Let r be a non-negative integer, ∞ or ω.
(1) *An exponentially C^r Nash manifold X of dimension d* is a C^r manifold admitting a finite system of charts $\{\phi_i : U_i \longrightarrow \mathbb{R}^d\}$ such that for each i and j $\phi_i(U_i \cap U_j)$ is an open exponentially definable subset of $\mathbb{R}^d$ and the map $\phi_j \circ \phi_i^{-1}|\phi_i(U_i \cap U_j) : \phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$ is an exponentially C^r Nash diffeomorphism (an exponentially Nash homeomorphism if $r = 0$). We call these atlas *exponentially C^r Nash*. Exponentially C^r Nash manifolds with compatible atlases are identified. A subset M of X is called *exponentially definable* if every $\phi_i(U_i \cap M)$ is exponentially definable.
(2) An exponentially definable subset X of $\mathbb{R}^n$ is called *a d-dimensional exponentially C^r Nash submanifold of $\mathbb{R}^n$* if for any $x \in X$ there exists an exponentially C^r Nash diffeomorphism ϕ from some open exponentially definable neighborhood U of the origin in $\mathbb{R}^n$ onto some open exponentially definable neighborhood V of x in $\mathbb{R}^n$ such that $\phi(0) = x, \phi(\mathbb{R}^d \cap U) = X \cap V$. Here $\mathbb{R}^d$ denotes the subset of $\mathbb{R}^n$ those which the last $(n-d)$ components are zero. An exponentially C^r $(r > 0)$ Nash submanifold is of course an exponentially C^r Nash manifold [7].
(3) Let X (resp. Y) be an exponentially C^r Nash manifold with exponentially C^r Nash atlas $\{\phi_i : U_i \longrightarrow \mathbb{R}^n\}_i$ (resp. $\{\psi_j : V_j \longrightarrow \mathbb{R}^m\}_j$). A C^r map $f : X \longrightarrow Y$ is said to be *an exponentially C^r Nash map* if for any i and j $\phi_i(f^{-1}(V_j) \cap U_i)$ is open and exponentially definable in $\mathbb{R}^n$, and that the map $\psi_j \circ f \circ \phi_i^{-1} : \phi_i(f^{-1}(V_j) \cap U_i) \longrightarrow \mathbb{R}^m$ is an exponentially C^r Nash map.
(4) Let X and Y be exponentially C^r Nash manifolds. We say that X is *exponentially C^r Nash diffeomorphic to Y* if one can find exponentially C^r Nash maps $f : X \longrightarrow Y$ and $h : Y \longrightarrow X$ such that $f \circ h = id$ and $h \circ f = id$.
(5) An exponentially C^r Nash manifold is said to be C^r *affine* if it is exponentially C^r Nash diffeomorphic to some exponentially C^r Nash submanifold of $\mathbb{R}^l$. We simply write *affine* instead of C^r affine if $r = \omega$.

Remark that any C^∞ Nash manifold is a C^ω Nash manifold, but there exists an exponentially C^∞ Nash manifold which is not an exponentially C^ω Nash manifold (See Example 2.4).

Definition 2.6. (1) A group G is called *an exponentially Nash group* (resp. *an affine exponentially Nash group*) if G is an exponentially Nash manifold (resp. an affine exponentially Nash manifold) and that the multiplication $G \times G \longrightarrow G$ and the inversion $G \longrightarrow G$ are exponentially Nash maps.
(2) Let G be an exponentially Nash group. A *representation* of G means a group homomorphism from G to some $GL(\mathbb{R}^n)$ which is an exponentially Nash map. We use a representation as a representation space.

Example 2.7. Subgroups

$$\{exp(t\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix})|t \in \mathbb{R}\} \text{ and } \{exp(t\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix})|t \in \mathbb{R}\}$$

of $GL_2(\mathbb{R})$ are exponentially Nash groups but not Nash ones.

Definition 2.8. Let G be an exponentially Nash group and let r be a non-negative integer, ∞ or ω.
(1) An exponentially C^r Nash submanifold in a representation of G is called *an exponentially C^r Nash G submanifold* if it is G invariant.
(2) *An exponentially C^r Nash G manifold* is a pair (X, θ) consisting of an exponentially C^r Nash manifold X and a group action θ of G on X, such that $\theta : G \times X \longrightarrow X$ is an exponentially C^r Nash map. For simplicity of notation, we write X instead of (X, θ).
(3) Let X and Y be exponentially C^r Nash G manifolds. An exponentially C^r Nash map $f : X \longrightarrow Y$ is called *an exponentially C^r Nash G map* if it is a G map. An exponentially C^r Nash G map $g : X \longrightarrow Y$ is said to be *an exponentially C^r Nash G diffeomorphism* if there exists an exponentially C^r Nash G map $h : Y \longrightarrow X$ such that $g \circ h = id$ and $h \circ g = id$.
(4) We say that an exponentially C^r Nash G manifold is C^r *affine* if it is exponentially C^r Nash G diffeomorphic to an exponentially C^r Nash G submanifold of some representation of G. If $r = \omega$, then we simply write *affine* instead of C^r affine.

We have the following implications on groups:

an algebraic group$\Longrightarrow$an affine Nash group$\Longrightarrow$an affine exponentially Nash group
$\Longrightarrow$ an exponentially Nash group $\Longrightarrow$ a Lie group.

Let G be an algebraic group. Then we obtain the following implications on G manifolds:

a nonsingular algebraic G set $\Longrightarrow$ an affine Nash G manifold
$\Longrightarrow$ an affine exponentially Nash G manifold $\Longrightarrow$ an exponentially Nash G manifold $\Longrightarrow$ a $C^\infty G$ manifold.

Moreover, notice that a Nash G manifold is not always an affine exponentially Nash G manifold.

In the equivariant exponentially Nash category, the equivariant tubular neighborhood result holds true [7].

Proposition 2.9 [7]. *Let G be a compact affine exponentially Nash group and let X be an affine exponentially Nash G submanifold possibly with boundary in a representation Ω of G. Then there exists an exponentially Nash G tubular neighborhood (U, p) of X in Ω, namely U is an affine exponentially Nash G submanifold in Ω and the orthogonal projection $p : U \longrightarrow X$ is an exponentially Nash G map.* □

The following lemma is useful to prove the existence of nonaffine exponentially Nash manifolds, which is a generalization of the usual Nash case (I.2.2.15 [21]).

Proposition 2.10 [7]. *Let M and N be exponentially Nash manifolds and let $h : M \longrightarrow N$ be a locally exponentially Nash map. If N is affine then h is an exponentially Nash map. Here we say that h is locally exponentially Nash if for any $x \in M$ and $f(x) \in N$, there exist open exponentially definable neighborhoods U of x in M and V of $f(x)$ in N such that $f(U) \subset V$ and $f|U : U \longrightarrow V$ is an exponentially Nash map.* $\square$

Definition 2.11. Let G be an exponentially Nash group and let r be a non-negative integer, ∞ or ω.
(1) A C^rG vector bundle (E, p, X) of rank k is said to be *an exponentially C^r Nash G vector bundle* if the following three conditions are satisfied:

(a) The total space E and the base space X are exponentially C^r Nash G manifolds.
(b) The projection p is an exponentially C^r Nash G map.
(c) There exists a family of finitely many local trivializations $\{U_i, \phi_i : U_i \times \mathbb{R}^k \longrightarrow p^{-1}(U_i)\}_i$ such that $\{U_i\}_i$ is an open exponentially definable covering of X and that for any i and j the map $\phi_i^{-1} \circ \phi_j|(U_i \cap U_j) \times \mathbb{R}^k : (U_i \cap U_j) \times \mathbb{R}^k \longrightarrow (U_i \cap U_j) \times \mathbb{R}^k$ is an exponentially C^r Nash map.

We call these local trivializations *exponentially C^r Nash*.
(2) Let $\eta = (E, p, X)$ (resp. $\zeta = (F, q, X)$) be an exponentially C^r Nash G vector bundle of rank n (resp. m). Let $\{U_i, \phi_i : U_i \times \mathbb{R}^n \longrightarrow p^{-1}(U_i)\}_i$ (resp. $\{V_j, \psi_j : V_j \times \mathbb{R}^m \longrightarrow q^{-1}(V_j)\}_j$) be exponentially C^r Nash local trivializations of η (resp. ζ). A C^rG vector bundle map $f : \eta \longrightarrow \zeta$ is said to be *an exponentially C^r Nash G vector bundle map* if for any i and j the map $(\psi_j)^{-1} \circ f \circ \phi_i|(U_i \cap V_j) \times \mathbb{R}^n : (U_i \cap V_j) \times \mathbb{R}^n \longrightarrow (U_i \cap V_j) \times \mathbb{R}^m$ is an exponentially C^r Nash map. A C^rG section s of η is called *exponentially C^r Nash* if each $\phi_i^{-1} \circ s|U_i : U_i \longrightarrow U_i \times \mathbb{R}^n$ is exponentially C^r Nash.
(3) Two exponentially C^r Nash G vector bundles η and ζ are said to be *exponentially C^r Nash G vector bundle isomorphic* if there exist exponentially C^r Nash G vector bundle maps $f : \eta \longrightarrow \zeta$ and $h : \zeta \longrightarrow \eta$ such that $f \circ h = id$ and $h \circ f = id$.

Recall universal G vector bundles (cf. [5]).

Definition 2.12. Let G be a compact exponentially Nash group. Let Ω be an n-dimensional representation of G and B the representation map $G \longrightarrow GL_n(\mathbb{R})$ of Ω. Suppose that $M(\Omega)$ denotes the vector space of $n \times n$-matrices with the action $(g, A) \in G \times M(\Omega) \longrightarrow B(g)^{-1}AB(g) \in M(\Omega)$. For any positive integer k, we define the vector bundle $\gamma(\Omega, k) = (E(\Omega, k), u, G(\Omega, k))$ as follows:

$$G(\Omega, k) = \{A \in M(\Omega) | A^2 = A, A = A', TrA = k\},$$

$$E(\Omega, k) = \{(A, v) \in G(\Omega, k) \times \Omega | Av = v\},$$

$$u : E(\Omega, k) \longrightarrow G(\Omega, k) : u((A, v)) = A,$$

where A' denotes the transposed matrix of A and TrA stands for the trace of A. Then $\gamma(\Omega, k)$ is an algebraic set. Since the action on $\gamma(\Omega, k)$ is algebraic, it is an algebraic G vector bundle. We call it *the universal G vector bundle associated with Ω and k*. Since $G(\Omega, k)$ and $E(\Omega, k)$ are nonsingular, $\gamma(\Omega, k)$ is a Nash G vector bundle, hence it is an exponentially Nash one.

Definition 2.13. Let G be a compact exponentially Nash group and let X be an exponentially C^r Nash G manifold. An exponentially C^r Nash G vector bundle $\eta = (E, p, X)$ of rank k is said to be *strongly exponentially C^r Nash* if the base space X is C^r affine and that there exist some representation Ω of G and an exponentially C^r Nash G map $f : X \longrightarrow G(\Omega, k)$ such that η is exponentially C^r Nash G vector bundle isomorphic to $f^*(\gamma(\Omega, k))$. If $r = \omega$, then strongly exponentially C^r Nash is abbreviated to strongly exponentially Nash.

Let G be a compact Nash group. Then we have the following implications on G vector bundles over an affine Nash G manifold:

a Nash G vector bundle $\Longrightarrow$ an exponentially Nash G vector bundle $\Longrightarrow$ a $C^\omega G$ vector bundle, and

a strongly Nash G vector bundle $\Longrightarrow$ a strongly exponentially Nash G vector bundle $\Longrightarrow$ an exponentially Nash G vector bundle.

3. Sketch of proof.

Sketch of proof of Theorem (1) and (2). We now give a sketch of proof of (1). Since G and X are compact, there exist a representation Ω of G and a $C^\infty G$ map $f : X \longrightarrow G(\Omega, k)$ such that η is $C^\infty G$ vector bundle isomorphic to $f^*(\gamma(\Omega, k))$, where k denotes the rank of η. Averaging a polynomial approximation of f and by Proposition 2.9, we have an exponentially Nash G map $h : X \longrightarrow \gamma(\Omega, k)$ which approximates f. By [24], $\zeta := h^*(\gamma(\Omega, k))$ is the required one.

We now sketch the proof of (2). Let ζ_1 and ζ_2 be two strongly exponentially Nash G vector bundles over X. Then $Hom(\zeta_1, \zeta_2)$ is a strongly exponentially Nash G vector bundle. By the assumption, there exists an element F in $Iso(\zeta_1, \zeta_2)$. Approximating F by an exponentially Nash G section of $Hom(\zeta_1, \zeta_2)$, we have the desired isomorphism because $Iso(\zeta_1, \zeta_2)$ is open in $Hom(\zeta_1, \zeta_2)$. $\square$

We prepare the following result to prove Theorem (3).

Proposition 3.1 [7]. *Let G be a compact affine exponentially Nash group and let $\eta = (E, p, Y)$ be an exponentially Nash G vector bundle of rank k over an affine exponentially Nash G manifold Y. Then η is strongly exponentially Nash if and only if E is affine.* $\square$

Sketch of proof of Theorem (3). By Theorem (1) we may assume that η' is exponentially Nash G vector bundle. Since X has a 0-dimensional orbit $G(x)$ and by Proposition 2.9, one can find an open G invariant exponentially definable neighborhood U of $G(x)$ such that $\eta|U$ is exponentially Nash G vector bundle isomorphic to $U \times \Xi$ for some representation Ξ. Using Proposition 2.9, we can construct three open G invariant exponentially definable subsets of U which cover U. We paste their overlaps with a collection of exponentially Nash G diffeomorphisms. By Proposition 2.10, we can show that the total space of the resulting exponentially Nash G vector bundle ζ' is nonaffine. Therefore we have (3) by Proposition 3.1. $\square$

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