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RELATIONS BETWEEN EQUIVALENCE RELATIONS OF MAPS AND FUNCTIONS

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There are many kinds of equivalence relations of maps and functions, e.g. C^∞ $\mathcal{R}$-$\mathcal{L}$, C^∞ $\mathcal{R}$, C^0 $\mathcal{R}$-$\mathcal{L}$ and C^0 $\mathcal{R}$ equivalence relations. Some relations between them are clear. For example, C^∞ $\mathcal{R}$-$\mathcal{L}$ ($\mathcal{R}$) equivalence implies C^0 $\mathcal{R}$-$\mathcal{L}$ ($\mathcal{R}$, respectively,) equivalence, and the converse does not necessarily hold. But we do not know all the relations. The present paper is a list of relations, which is far from complete.

We treat map germs in §1, function germs in §2, global maps in §3 and global functions in §4.

A *Nash manifold* is a semialgebraic C^ω submanifold of a Euclidean space. A *semialgebraic* (*subanalytic*) *map* between semialgebraic (subanalytic) sets is a C^0 map with semialgebraic (subanalytic) graph. A *Nash map* between Nash manifolds is a semialgebraic C^ω map. For a point a and a set, a map or a sheaf A, let A_a denote the germ of A at a or the stalk of A over a. A map or function germ means a germ at a point unless otherwise specified.

§1. MAP GERMS

Let $f,\ g\colon \mathbf{R}^n_0 \to \mathbf{R}^m_0$ be C^∞ map germs. We call f and g *formally* $\mathcal{R}$-$\mathcal{L}$ *equivalent* if there exist C^∞ diffeomorphism germs π of $\mathbf{R}^m_0$ and τ of $\mathbf{R}^n_0$ such that $\pi\circ f - g\circ\tau$ is flat at 0. We define naturally *formal* $\mathcal{R}$ and $\mathcal{L}$ *equivalences*. Let $\mathbf{R}[[\cdots]]$ and $\mathbf{R}\{\cdots\}$ denote the formal and convergent power series rings, respectively. In these sets, we always assume the Krull topology (the $\mathfrak{m}$-adic topology, where $\mathfrak{m}$ denotes the maximal ideals). Let t-dim and dim denote dimension as a topological set and as an analytic set or a ring, respectively. An *analytic closure* of a set germ is the smallest analytic set germ including it.

Fact 1.1. *Formal $\mathcal{R}$ equivalence of C^ω map germs implies C^ω $\mathcal{R}$ equivalence. C^ω $\mathcal{R}$ equivalence of Nash map germs implies Nash $\mathcal{R}$ equivalence.*

Proof. The former and latter statements are trivial by Artin First Approximation Theorem [A$_1$] and by the following small generalization of the Second [A$_2$] (which also we call Artin Approximation Theorem), respectively.

Let $F\colon \mathbf{R}^n_0 \times \mathbf{R}^m_0 \to \mathbf{R}^k_0$ be a Nash map germ, and let $f\colon \mathbf{R}^n_0 \to \mathbf{R}^m_0$ be C^ω map germ such that $F(x, f(x)) = 0$. Then f is approximated by a Nash solution.

The proof is the following. Let X denote the Nash closure of graph f (the smallest Nash set germ including graph f). We can assume $F^{-1}(0) = X$. If X is algebraic, the statement is clear by [A$_2$]. So suppose X is not algebraic, and let $X \cup X_1 \cup \cdots$

be the Nash irreducible decomposition of the Zariski closure of X. By [A_2] we can approximate f by a Nash germ whose graph is contained in $X \cup X_1 \cup \cdots$. Since each graph is Nash irreducible, it is contained in X or some X_1. If it is always in X, the statement holds. Hence assume there is a sequence of Nash maps $f_l\colon \mathbf{R}^n_0 \to \mathbf{R}^m_0$ converging to f with graph in X_1. Let ϕ be a Nash function on $\mathbf{R}^{n+m}_0$ with zero set $= X_1$. Then $\phi(\mathrm{id}, f_l) = 0$. Hence $\phi(\mathrm{id}, f) = 0$, i.d. graph $f \subset X_1$, which contradicts the assumption that X is the Nash closure of graph f. □

Conjecture 1.2. *Formal $\mathcal{R}$-$\mathcal{L}$ equivalence of Nash map germs implies Nash $\mathcal{R}$-$\mathcal{L}$ equivalence.*

The following fact was suggested by S. Izumi.

Fact 1.3. *Formal $\mathcal{L}$ equivalence of Nash map germs implies Nash $\mathcal{L}$ equivalence.*

Proof. Let f, g be formally $\mathcal{L}$ equivalent Nash map germs from $\mathbf{R}^n_0$ to $\mathbf{R}^m_0$, and let $\pi = (\pi_1, ..., \pi_m)$ be an invertible element of $\mathbf{R}[[y_1, ..., y_m]]^m$ such that $\pi \circ f = g$ in $\mathbf{R}[[x_1, ..., x_n]]^m$. Set $\phi = (f, g) = \mathbf{R}^n_0 \to \mathbf{R}^m_0 \times \mathbf{R}^m_0$. Let

$$\phi_1^*\colon \mathbf{R}[[y_1, ..., y_m, z_1, ..., z_m]] \to \mathbf{R}[[x_1, ..., x_n]],$$
$$\phi_2^*\colon \mathbf{R}\{y_1, ..., y_m, z_1, ..., z_m\} \to \mathbf{R}\{x_1, ..., x_n\}$$

denote the homomorphisms induced by ϕ. Clearly $\operatorname{Ker}\phi_1^*$ and $\operatorname{Ker}\phi_2^*$ are prime ideals and

$$\operatorname{Ker}\phi_1^* \supset \operatorname{Ker}\phi_2^*\mathbf{R}[[y_1, ..., y_m, z_1, ..., z_m]].$$

Moreover, we have

$$(*) \qquad \operatorname{Ker}\phi_1^* = \operatorname{Ker}\phi_2^*\mathbf{R}[[y_1, ..., y_m, z_1, ..., z_m]]$$

for the following reason.

It suffices to see

$$\dim \mathbf{R}[[y_1, \ldots, z_m]]/\operatorname{Ker}\phi_1^* \geq \dim \mathbf{R}[[y_1, \ldots, z_m]]/\operatorname{Ker}\phi_2^*\mathbf{R}[[y_1, \ldots, z_m]].$$

Recall (see [M]) that the completion of $\mathbf{R}\{y_1, \ldots, z_m\}/\operatorname{Ker}\phi_2^*$ is $\mathbf{R}[[y_1, \ldots, z_m]]$ $/\operatorname{Ker}\phi_2^*\mathbf{R}[[y_1, \ldots, z_m]]$ and the dimension of a local ring is invariable after its completion. Hence

$$\dim \mathbf{R}\{y_1, \ldots, z_m\}/\operatorname{Ker}\phi_2^* = \dim \mathbf{R}[[y_1, \ldots, z_m]]/\operatorname{Ker}\phi_2^*\mathbf{R}[[y_1, \ldots, z_m]],$$

and what we prove is

$$\dim \mathbf{R}[[y_1, \ldots, z_m]]/\operatorname{Ker}\phi_1^* \geq \dim \mathbf{R}\{y_1, \ldots, z_m\}/\operatorname{Ker}\phi_2^*.$$

Let $\phi^{\mathbf{C}}\colon \mathbf{C}^n_0 \to \mathbf{C}^m_0 \times \mathbf{C}^m_0$ denote the complexification of ϕ. It is easy to show that $\dim \mathbf{R}\{y_1, \ldots, z_m\}/\operatorname{Ker}\phi_2^*$ is equal to the dimension of the complex analytic closure of (i.e. the smallest complex analytic set including) $\phi^{\mathbf{C}}(U)$ for a sufficiently small neighborhood U of 0 in $\mathbf{C}^n$. The dimension of the complex analytic closure equals the half of its topological dimension because the image is a subset of a complex algebraic set of the same topological dimension. On the other hand, t-dim Im ϕ = t-dim Im $\phi^{\mathbf{C}}/2$. So we need only prove

$$\dim \mathbf{R}[[y_1,\dots,z_m]]/\operatorname{Ker}\phi_1^* \geq \text{t-dim}\operatorname{Im}\phi.$$

Set $k = \text{t-dim}\operatorname{Im}\phi$. Choose a linear map $p\colon \mathbf{R}^m\times\mathbf{R}^m \to \mathbf{R}^k$ so that t-dim Im $p\circ\phi = k$. Then

$$\dim \mathbf{R}[[y_1,\dots,z_m]]/\operatorname{Ker}\phi_1^* \geq \dim \mathbf{R}[[u_1,\dots,u_k]]/\operatorname{Ker}(p\circ\phi)_1^*$$

because a formal power series ring is a Cohen-Macaulay ring. Hence what we need to show is $\dim \mathbf{R}[[u_1,\dots,u_k]]/\operatorname{Ker}(p\circ\phi)_1^* = k$, which is equivalent to that $(p\circ\phi)_1^*$ is injective. Assume $(p\circ\phi)_1^*$ is not so. Let $\alpha \in \mathbf{R}[[u_1,\dots,u_k]]$ be in $\operatorname{Ker}(p\circ\phi)_1^*$ and of the minimal order. We have an equality of matrices:

$$0 = \begin{pmatrix} \frac{\partial \alpha\circ p\circ\phi}{\partial x_1} \\ \vdots \\ \frac{\partial \alpha\circ p\circ\phi}{\partial x_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial p_1\circ\phi}{\partial x_1} & \cdots & \frac{\partial p_k\circ\phi}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial p_1\circ\phi}{\partial x_n} & \cdots & \frac{\partial p_k\circ\phi}{\partial x_n} \end{pmatrix} \begin{pmatrix} \frac{\partial \alpha}{\partial u_1}(p\circ\phi) \\ \vdots \\ \frac{\partial \alpha}{\partial u_k}(p\circ\phi) \end{pmatrix},$$

where $p = (p_1,\dots,p_k)$. Now the Jacobian matrix of $p\circ\phi$ is of rank k, if we regard its elements as in the quotient field of $\mathbf{R}\{x_1,\dots,x_n\}$, because t-dim Im$(p\circ\phi) = k$. Hence $\frac{\partial\alpha}{\partial u_j}(p\circ\phi) = 0$ for all j. But some of $\frac{\partial\alpha}{\partial u_j}$ is nonzero and of order $<$ order α, which is a contradiction. Therefore, (*) holds.

We have Nash generators $\alpha_1,\dots,\alpha_l$ of $\operatorname{Ker}\phi_2^*$ because Im $\phi^{\mathbf{C}}$ is a constructible set. Set $z_i - \pi_i(y) = \beta_i(y,z)$, $i = 1,\dots,m$. Then $\beta_i \in \operatorname{Ker}\phi_1^*$ and $\cap_i\beta_i^{-1}(0) = \operatorname{graph}\pi$. Hence $\begin{pmatrix}\beta_1\\ \vdots\\ \beta_m\end{pmatrix} = \begin{pmatrix}\gamma_{1,1} & \cdots & \gamma_{1,l}\\ \vdots & & \vdots\\ \gamma_{m,1} & \cdots & \gamma_{m,l}\end{pmatrix}\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_l\end{pmatrix}$ for some $\gamma_{i,j} \in \mathbf{R}[[y_1,\dots,z_m]]$. Let $\gamma'_{i,j}$ be Nash germ approximations of $\gamma_{i,j}$ and define Nash germ approximations β'_i of β_i by $\begin{pmatrix}\beta'_1\\ \vdots\\ \beta'_m\end{pmatrix} = \begin{pmatrix}\gamma'_{1,1} & \cdots & \gamma'_{1,l}\\ \vdots & & \vdots\\ \gamma'_{m,1} & \cdots & \gamma'_{m,l}\end{pmatrix}\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_l\end{pmatrix}$ so that $\beta'_i = z_i - \pi_i(y)+$ formal power series of order > 1. The former equality implies $\beta'_i(f,g) = 0$, $i = 1,\dots,m$. On the other hand, by the latter and the implicit function theorem, there exist uniquely Nash function germs $\beta''_i(y,z)$ of the form $z_i - \pi''_i(y)$, $i = 1,\dots,m$, such that $\cap_i\beta_i'^{-1}(0) = \cap_i\beta''^{-1}(0)$. Set $\pi'' = (\pi''_1,\dots,\pi''_m)$. Then π'' is a Nash diffeomorphism germ and $\pi''\circ f = g$. □

Fact 1.4. *C^∞ $\mathcal{L}$ ($\mathcal{R}$-$\mathcal{L}$) equivalence of C^ω map germs does not necessarily imply C^ω $\mathcal{L}$ ($\mathcal{R}$-$\mathcal{L}$, respectively,) equivalence.*

Proof (cf. [G1]). Let f, $g\colon \mathbf{R}_0^2 \to \mathbf{R}_0^4$ be the analytic map germs defined by

$$f(x_1,x_2) = (x_1, x_1x_2, x_1x_2e^{x_2}, 0),$$

$$g(x_1,x_2) = (x_1, x_1x_2, x_1x_2e^{x_2}, \sum_{k=1}^{\infty}\sum_{i=0}^{\infty}\frac{k!}{(k+i)!}x_1^k x_2^{k+i+1}).$$

Then f and g are C^∞ $\mathcal{L}$ equivalent but not C^ω $\mathcal{R}$-$\mathcal{L}$ equivalent.

Proof of C^∞ $\mathcal{L}$ equivalence. Define $\pi \in \mathbf{R}[[y_1,..,y_4]]^4$ by

$$\pi(y) = (y_1, y_2, y_3, \pi_4(y)) \quad \text{for } y = (y_1, .., y_4),$$
$$\pi_4(y) = y_4 + \sum_{k=1}^{\infty}(k!y_1^{k-1}y_3 - \sum_{i=1}^{k}\frac{k!}{(i-1)!}y_1^{k-i}y_2^i).$$

It is easy to calculate $\pi \circ f = g$ in $\mathbf{R}[[x_1,x_2]]^4$.

We will find a C^∞ diffeomorphism germ $\tilde{\pi}$ of $\mathbf{R}_0^4$ so that its Taylor expansion equals π and $\tilde{\pi} \circ f = g$. For each $n \in \mathbf{N}$, let π_{4n} denote the homogeneous part of π_4 of degree n. Let ϕ be a C^∞ function on $\mathbf{R}^4$ which equals 0 outside a small neighborhood of 0 and 1 on a smaller one. Let $N\colon \mathbf{N} \to \mathbf{N}$ be a sufficiently large map. Set

$$\tilde{\pi}_4(y) = \sum_{n\in\mathbf{N}} \pi_{4n}(y)\phi(N(n)y), \quad \tilde{\pi}(y) = (y_1, y_2, y_3, \tilde{\pi}_4(y)) \quad \text{for } y = (y_1, .., y_4) \in \mathbf{R}^4.$$

Then it is easy to see that $\tilde{\pi}$ is a C^∞ map between $\mathbf{R}^4$, its germ at 0 is a C^∞ diffeomorphism germ of $\mathbf{R}_0^4$, and its Taylor expansion at 0 equals π. But we can not expect $\tilde{\pi} \circ f = g$ as germs at 0. We need to modify $\tilde{\pi}$ so that this equality holds.

Since g is convergent on $\{|x_1x_2| < 1\}$, we regard f and g as C^ω maps defined on the domain. Then $f - \tilde{\pi}^{-1} \circ g$ is flat on $\{x_1 = 0\}$ for the following reason. Clearly, it is so at 0. Let $l \in \mathbf{N}$. By the form of f, the C^ω map:

$$(y_1, y_2, y_3, \sum_{n<l} \pi_{4n}(y)\phi(N(n)y)) \circ f - (y_1, y_2, y_3, \sum_{n<l} \pi_{4n}(y)) \circ f$$

vanishes on a neighborhood of $\{x_1 = 0\} - 0$ in $\mathbf{R}^2$ and converges to $\tilde{\pi} \circ f - g$ as $l \to \infty$ in the C^∞ compact-open topology. Therefore, $\tilde{\pi} \circ f - g$ and hence $f - \tilde{\pi}^{-1} \circ g$ are flat on $\{x_1 = 0\} - 0$. By the definition of $\tilde{\pi}$, $\tilde{\pi}^{-1} \circ g$ is of the form

$$(x_1, x_1x_2, x_1x_2e^{x_2}, h(x_1, x_2)).$$

Then h is a C^∞ function on $\{|x_1x_2| < 1\}$ and flat on $\{x_1 = 0\}$.

We want to find a C^∞ diffeomorphism germ τ of $\mathbf{R}_0^4$ of the form $(y_1, y_2, y_3, y_4 + \tau_4(y_1, y_2))$ such that $\tau \circ f = \tilde{\pi}^{-1} \circ g$. For that, it suffices to construct a C^∞ function $\tilde{\tau}_4(x_1, x_2)$ on $\{|x_2| < 1\}$ such that $\tilde{\tau}_4(x_1, x_1x_2) = h(x_1, x_2)$ on $\{|x_1x_2| < 1\}$. Define $\tilde{\tau}_4(x_1, x_2)$ to be 0 on $\{x_1 = 0,\ |x_2| < 1\}$ and $h(x_1, x_2/x_1)$ on $\{x_1 \neq 0,\ |x_2| < 1\}$. Then $\tilde{\tau}_4(x_1, x_1x_2) = h(x_1, x_2)$, and it follows from the above flatness that $\tilde{\tau}_4$ is of class C^∞.

Proof of non C^ω $\mathcal{R}$-$\mathcal{L}$ equivalence. If they are C^ω $\mathcal{R}$-$\mathcal{L}$ equivalent, there is a C^ω diffeomorphism germ π of $\mathbf{R}_0^4$ such that $\pi(\operatorname{Im} f) = \operatorname{Im} g$. But there exist a non-zero C^ω function germ on $\mathbf{R}_0^4$ which vanishes on $\operatorname{Im} f$ and there does not for $\operatorname{Im} g$ as shown in [G₁]. That is a contradiction. □

Conjecture 1.5. *Formal $\mathcal{L}$ ($\mathcal{R}$-$\mathcal{L}$) equivalence of C^ω map germs implies C^∞ $\mathcal{L}$ ($\mathcal{R}$-$\mathcal{L}$, respectively,) equivalence.*

Another natural question is under what conditions formal $\mathcal{L}$ ($\mathcal{R}$-$\mathcal{L}$) equivalence of C^ω map germs implies C^ω $\mathcal{L}$ ($\mathcal{R}$-$\mathcal{L}$, resp.) equivalence. A partial answer is the following.

The next fact also was suggested by S. Izumi.

Fact 1.6. *Let $f, g\colon \mathbf{R}^n_0 \to \mathbf{R}^m_0$ be formally $\mathcal{L}$ equivalent C^ω map germs. If $\operatorname{Im} f$ and its analytic closure are of the same topological dimension, then f and g are C^ω $\mathcal{L}$ equivalent.*

The assumption is satisfied if the topological dimension of $\operatorname{Im} f$ equals 1, m or the height of the ideal of $\mathbf{R}\{x_1, \dots, x_n\}$ generated by $f_1 \dots, f_n$, where $f = (f_1, \dots, f_n)$. The last condition is equivalent to that $2n = \text{t-dim} \operatorname{Im} f^{\mathbf{C}} + \text{t-dim} f^{\mathbf{C}-1}(0)$, where $f^{\mathbf{C}}$ denotes the complexification of f.

Proof. Define ϕ, ϕ_1^*, ϕ_2^*, etc., as in the proof of 1.4. We set

$$\text{f-rank} f^{\mathbf{C}} = \dim \mathbf{C}[[y_1, \dots, y_m]]/\operatorname{Ker} f_1^{\mathbf{C}*},$$

and for a set germ $A \subset \mathbf{R}^n (\subset \mathbf{C}^n)$ at 0, a-dim A denotes the dimension of the (complex, resp.) analytic closure of A as an (complex, resp.) analytic set germ. By the proof of 1.4 it suffices to prove

$$\text{(0)} \qquad \text{t-dim} \operatorname{Im} \phi^{\mathbf{C}}/2 = \text{a-dim} \operatorname{Im} \phi^{\mathbf{C}}.$$

By assumption,

$$\text{t-dim} \operatorname{Im} f = \text{a-dim} \operatorname{Im} f.$$

Complexification of this equality holds, namely,

$$\text{(1)} \qquad \text{t-dim} \operatorname{Im} f^{\mathbf{C}}/2 = \text{a-dim} \operatorname{Im} f^{\mathbf{C}},$$

because

$$\text{t-dim} \operatorname{Im} f = \text{t-dim} \operatorname{Im} f^{\mathbf{C}}/2, \qquad \text{a-dim} \operatorname{Im} f = \text{a-dim} \operatorname{Im} f^{\mathbf{C}},$$

which we see easily. It is also clear that

$$\text{(2)} \qquad \text{f-rank} f^{\mathbf{C}} \le \text{a-dim} \operatorname{Im} f^{\mathbf{C}},$$

$$\text{(3)} \qquad \text{f-rank} \phi^{\mathbf{C}} = \text{f-rank} f^{\mathbf{C}},$$

$$\text{(4)} \qquad \text{t-dim} \operatorname{Im} f^{\mathbf{C}} \le \text{t-dim} \operatorname{Im} \phi^{\mathbf{C}},$$

and we know (Lemma 1.5 in [I])

$$\text{(5)} \qquad \text{t-dim} \operatorname{Im} f^{\mathbf{C}}/2 \le \text{f-rank} f^{\mathbf{C}}, \qquad \text{t-dim} \operatorname{Im} \phi^{\mathbf{C}}/2 \le \text{f-rank} \phi^{\mathbf{C}}.$$

Therefore,

$$\text{t-dim} \operatorname{Im} \phi^{\mathbf{C}}/2 = \text{f-rank} \phi^{\mathbf{C}},$$

which implies (0) by Theorem 4.8 in [G_2]. □

Let $f = (f_1, \dots, f_n)\colon \mathbf{R}^n_0 \to \mathbf{R}^m_0$ be a C^ω map germ. We say that f is of *finite singularity type* if $\dim_{\mathbf{R}} \mathbf{R}\{x_1, \dots, x_n\}/(f_1, \dots, f_m, J_f)$ is finite, where J_f denotes the *Jacobian ideal*, i.e. the ideal of $\mathbf{R}\{x_1, \dots, x_n\}$ generated by the minors of the Jacobian matrix of f of degree m ($J_f = \{0\}$ if $m > n$). (We say also that f defines an *isolated complete intersection singularity* in the case of $m \le n$ [L] or f is *finitely C^ω $\mathcal{K}$ determined* [W].) If $m \ge n$, this condition is equivalent to that f is finite, i.e., $\dim_{\mathbf{R}} \mathbf{R}\{x_1, \dots, x_n\}/(f_1, \dots, f_m)$ is finite.

Let f be of finite singularity type. Let $U \subset \mathbf{C}^n$ and $V \subset \mathbf{C}^m$ be open neighborhoods of 0, and let $\tilde{f}^{\mathbf{C}}\colon U \to V$ be a complex analytic map whose germ at 0 is the complexification of f. Let $\Sigma_{\tilde{f}^{\mathbf{C}}}$ denote the singular point set of $\tilde{f}^{\mathbf{C}}$. Then we know the following facts.

(1) In the case of $m \le n$, J_f is reduced if and only if the following subset of $\Sigma_{\tilde{f}^{\mathbf{C}}}$ is dense around 0 (Proposition 4.5 in [L]):

$$\{x \in \Sigma_{\tilde{f}^{\mathbf{C}}} \colon \tilde{f}^{\mathbf{C}}_x \text{ is } C^\omega\ \mathcal{R}\text{-}\mathcal{L} \text{ equivalent to } (x_1, \dots, x_n) \to (x_1, \dots, x_{m-1}, \sum_{i=m}^{n} x_i^2)\}.$$

(2) $\mathbf{R}\{x_1, \dots, x_n\}/J_f$ is normal if the jet section $j^1\tilde{f}^{\mathbf{C}}\colon U - 0 \to J^1(U - 0, V)$ is transversal to the canonical stratification of the jet space $J^1(U, V)$ by the rank of the Jacobian matrix (cf. the proof of Theorem 4.7 in [L]).

(3) This condition is satisfied when f is finitely C^∞ $\mathcal{R}$-$\mathcal{L}$ determined.

Fact 1.7. *Two formally $\mathcal{R}$-$\mathcal{L}$ equivalent C^ω (Nash) map germs $f, g\colon \mathbf{R}^n_0 \to \mathbf{R}^m_0$ are C^ω (Nash, resp.) $\mathcal{R}$-$\mathcal{L}$ equivalent if f is of finite singularity type, if $\mathbf{R}\{x_1, \dots, x_n\}/J_f$ is normal in the case of $2 < m < n$ and if J_f is reduced in the case of $m = 2 < n$.*

Later we shall globalize this (Fact 3.7).

Proof. First we prove that if f and g are C^ω $\mathcal{R}$-$\mathcal{L}$ equivalent Nash map germs then they are Nash $\mathcal{R}$-$\mathcal{L}$ equivalent. Replacing g with $\tau \circ g \circ \pi$ for some Nash diffeomorphism germs π and τ, we can assume g and the C^ω diffeomorphism germs of equivalence are arbitrarily close to f and id, respectively. If $m = 1$, f is C^∞ $\mathcal{R}$ finitely determined (e.g. Proposition 2.3 in[W]) and hence Nash $\mathcal{R}$ finitely determined. Therefore, we assume $m > 1$. Let us consider the case of $2 < m < n$, and postpone the other cases. For simplicity of notation, we assume f is the germ of a Nash map $\tilde{f}\colon \mathbf{R}^n \to \mathbf{R}^m$. Let $\Sigma_{\tilde{f}}$ denote the singular point set of $\tilde{f}$, let $\phi_1(\tilde{f}), \dots, \phi_k(\tilde{f})$ denote the minors of degree m of the Jacobian matrix of $\tilde{f}$, and let $J_{\tilde{f}}$ denote the sheaf of $\mathcal{O}$-ideals generated by $\phi_1(\tilde{f}), \dots, \phi_k(\tilde{f})$, where $\mathcal{O}$ is the sheaf of analytic function germs on $\mathbf{R}^n$. Write $\phi_i(\tilde{f})_0 = \phi_i(f)$, $J_{\tilde{f},0} = J_f$ and $\Sigma_{\tilde{f},0} = \Sigma_f$. We assume also a complexification $\tilde{f}^{\mathbf{C}}$ is defined on $\mathbf{C}^n$ for simplicity of notation, and define $J_{\tilde{f}^{\mathbf{C}}}$, $\mathcal{O}^{\mathbf{C}}$, $\Sigma_{\tilde{f}^{\mathbf{C}}}$ in the same way.

We need the following known facts.

(1) There exist small open neighborhoods U of 0 in $\mathbf{C}^n$ and V of 0 in $\mathbf{C}^m$ such that $U \cap \Sigma_{\tilde{f}^{\mathbf{C}}}$ is everywhere of dimension $m - 1$ and $\tilde{f}^{\mathbf{C}}|_{U \cap \Sigma_{\tilde{f}^{\mathbf{C}}}} \colon U \cap \Sigma_{\tilde{f}^{\mathbf{C}}} \to V$ is a finite-to-one proper map. (See Theorem 2.6 in [W] and Theorem 2.8 in [L].)

First we prove:

(2) For every point x of $\mathbf{C}^n$ near 0, the ring $\mathcal{O}_x^{\mathbf{C}}/J_{\bar{f}^{\mathbf{C}},x}$ is normal.

It suffices to consider the case of $x = 0$ by a theorem of Oka (see Remark in p. 126 of [H]). The ideal $J_{f^{\mathbf{C}}}$ is prime for the following reason. Assume it is not so. For a prime ideal $\mathfrak{p}$ of $\mathbf{C}\{x_1, \dots, x_n\}$, $\overline{\mathfrak{p}} = \{\overline{g}\colon g \in \mathfrak{p}\}$ also is prime, and

$$\mathfrak{p} \cap \overline{\mathfrak{p}} = (\mathfrak{p} \cap \mathbf{R}\{x_1, \dots, x_n\})\mathbf{C}\{x_1, \dots, x_n\},$$

where $\overline{g}$ is defined by $\overline{g}(x) = \overline{g(\overline{x})}$ and $^-$ stands for the conjugate operator. Hence it is easy to see $J_{f^{\mathbf{C}}} = \mathfrak{p} \cap \overline{\mathfrak{p}}$ for some prime ideal $\mathfrak{p} \neq J_{f^{\mathbf{C}}}$ of $\mathbf{C}\{x_1, \dots, x_n\}$. Then there exists $g = g_1 + ig_2 \in \mathfrak{p} - J_{f^{\mathbf{C}}}$, $g_i \in \mathbf{R}\{x_1, \dots, x_n\}$, such that $g\overline{g} \in J_{f^{\mathbf{C}}}$. It follows

$$g_1^2 + g_2^2 \in J_f, \qquad g_1,\ g_2 \notin J_f.$$

Hence

$$\left(\frac{g_1}{g_2}\right)^2 + 1 = 0,\ \frac{g_1}{g_2} \neq 0 \quad \text{in the quotient field of } \mathbf{R}\{x_1, \dots, x_n\}/J_f,$$

which contradicts the assumption that $\mathbf{R}\{x_1, \dots, x_n\}/J_f$ is normal.

Let A denote the integral closure of $\mathbf{C}\{x_1, \dots, x_n\}/J_{f^{\mathbf{C}}}$ in its quotient field. Then $A = \overline{A}$. Hence A is generated by elements defined by real analytic function germs. Since $\mathbf{R}\{x_1, \dots, x_n\}/J_f$ is normal, it follows that $A = \mathbf{C}\{x_1, \dots, x_n\}/J_{f^{\mathbf{C}}}$, i.e., the ring $\mathbf{C}\{x_1, \dots, x_n\}/J_{f^{\mathbf{C}}}$ is normal.

We have C^ω diffeomorphism germs π of $\mathbf{R}_0^n$ and τ of $\mathbf{R}_0^m$ close to id such that $f \circ \pi = \tau \circ g$. As usual we assume $g^{\mathbf{C}}, \pi^{\mathbf{C}}$ and $\tau^{\mathbf{C}}$ are the germs of a C^ω function $\tilde{g}^{\mathbf{C}}\colon \mathbf{C}^n \to \mathbf{C}^m$, C^ω diffeomorphisms $\tilde{\pi}^{\mathbf{C}}$ of $\mathbf{C}^n$ and $\tilde{\tau}^{\mathbf{C}}$ of $\mathbf{C}^m$, respectively, for simplicity of notation. Clearly we have

$$\pi(\Sigma_g) = \Sigma_f, \qquad \pi^{\mathbf{C}}(\Sigma_{g^{\mathbf{C}}}) = \Sigma_{f^{\mathbf{C}}},$$
$$f(\Sigma_f) = \tau \circ g(\Sigma_g), \qquad f^{\mathbf{C}}(\Sigma_{f^{\mathbf{C}}}) = \tau^{\mathbf{C}} \circ g^{\mathbf{C}}(\Sigma_{g^{\mathbf{C}}}).$$

Here $f^{\mathbf{C}}, g^{\mathbf{C}}, ..$ are the complexifications of $f, g, ..$.

We construct a Nash germ approximation π' of π such that

(3) $$\pi'(\Sigma_g) = \Sigma_f, \qquad \pi'^{\mathbf{C}}(\Sigma_{g^{\mathbf{C}}}) = \Sigma_{f^{\mathbf{C}}}.$$

Since $\phi_1(f) \circ \pi, \dots, \phi_k(f) \circ \pi$ are generators of J_g, there exist convergent power series $\psi_{i,j}$, $i, j = 1, \dots, k$, such that

$$\phi_i(g) = \sum_{j=1}^{k} \psi_{i,j} \cdot (\phi_j(f) \circ \pi), \quad i = 1, \dots, k.$$

By Artin Approximation Theorem we have Nash germ approximations $\psi'_{i,j}$ of $\psi_{i,j}$ and π' of π such that

$$\phi_i(g) = \sum_{j=1}^{k} \psi'_{i,j} \cdot (\phi_j(f) \circ \pi'), \quad i = 1, \dots, k.$$

Then (3) is satisfied.

We want to approximate also τ by a Nash diffeomorphism germ τ' so that

(4) $$f(\Sigma_f) = \tau' \circ g(\Sigma_g), \qquad f^{\mathbf{C}}(\Sigma_{f^{\mathbf{C}}}) = \tau'^{\mathbf{C}} \circ g^{\mathbf{C}}(\Sigma_{g^{\mathbf{C}}}).$$

By (1), $f^{\mathbf{C}}(\Sigma_{f^{\mathbf{C}}})$ and $g^{\mathbf{C}}(\Sigma_{g^{\mathbf{C}}})$ are complex Nash set germs everywhere of dimension $m-1$. We have also

$$f(\Sigma_f) = \tau \circ g(\Sigma_g), \qquad f^{\mathbf{C}}(\Sigma_{f^{\mathbf{C}}}) \cap \mathbf{R}^m_0 = \tau(g^{\mathbf{C}}(\Sigma_{g^{\mathbf{C}}}) \cap \mathbf{R}^m_0),$$
$$f^{\mathbf{C}}(\Sigma_{f^{\mathbf{C}}}) = \tau^{\mathbf{C}} \circ g^{\mathbf{C}}(\Sigma_{g^{\mathbf{C}}}),$$

because $f^{\mathbf{C}} \circ \pi^{\mathbf{C}} = \tau^{\mathbf{C}} \circ g^{\mathbf{C}}$. Hence by the same arguments as above we obtain a Nash germ approximation τ' of τ such that

$$f(\Sigma_f) = \tau' \circ g(\Sigma_g), \qquad f^{\mathbf{C}}(\Sigma_{f^{\mathbf{C}}}) \cap \mathbf{R}^m_0 = \tau'(g^{\mathbf{C}}(\Sigma_{g^{\mathbf{C}}}) \cap \mathbf{R}^m_0),$$
$$f^{\mathbf{C}}(\Sigma_{f^{\mathbf{C}}}) = \tau'^{\mathbf{C}} \circ g^{\mathbf{C}}(\Sigma_{g^{\mathbf{C}}}).$$

Replace g with $\tau' \circ g \circ \pi'^{-1}$. Then g is close to f, and by (3) and (4) we have

$$\Sigma_g = \Sigma_f, \qquad \Sigma_{g^{\mathbf{C}}} = \Sigma_{f^{\mathbf{C}}}, \qquad f(\Sigma_f) = g(\Sigma_g), \qquad f^{\mathbf{C}}(\Sigma_{f^{\mathbf{C}}}) = g^{\mathbf{C}}(\Sigma_{g^{\mathbf{C}}}).$$

Next we want to reduce the problem to the case where

(5) $$f^{\mathbf{C}} = g^{\mathbf{C}} \quad \text{on } \Sigma_{f^{\mathbf{C}}}.$$

Clearly by (1), the sets

$$\Sigma_{f^{\mathbf{C}}} \cap f^{\mathbf{C}-1}(\operatorname{Sing} f^{\mathbf{C}}(\Sigma_{f^{\mathbf{C}}})) \quad \text{and} \quad \Sigma_{f^{\mathbf{C}}} \cap g^{\mathbf{C}-1}(\operatorname{Sing} f^{\mathbf{C}}(\Sigma_{f^{\mathbf{C}}}))$$

are complex Nash set germs and everywhere of dimension $< m-1$, where Sing means the singular point set germ. Set

$$X = \{x \in \Sigma_{\tilde{f}^{\mathbf{C}}} : \mathcal{O}^{\mathbf{C}}_x / J_{\tilde{f}^{\mathbf{C}},x} \text{ is not regular}\}.$$

Then X is of dimension $< m-1$. By the same arguments as above we have a Nash diffeomorphism germ π'' of $\mathbf{R}^n_0$ such that π'' is close to id, and

$$\pi''(\Sigma_f) = \Sigma_f, \qquad \pi''^{\mathbf{C}}(\Sigma_{f^{\mathbf{C}}}) = \Sigma_{f^{\mathbf{C}}},$$
$$\pi''(\Sigma_f \cap f^{-1}(\operatorname{Sing} f(\Sigma_f))) = \Sigma_f \cap g^{-1}(\operatorname{Sing} f(\Sigma_f)),$$
$$\pi''^{\mathbf{C}}(X_0 \cup (\Sigma_{f^{\mathbf{C}}} \cap f^{\mathbf{C}-1}(\operatorname{Sing} f^{\mathbf{C}}(\Sigma_{f^{\mathbf{C}}})))) = X_0 \cup (\Sigma_{f^{\mathbf{C}}} \cap g^{\mathbf{C}-1}(\operatorname{Sing} f^{\mathbf{C}}(\Sigma_{f^{\mathbf{C}}}))).$$

Replace g with $g \circ \pi''^{-1}$. Then we have a complex Nash subset germ S of $\Sigma_{f^{\mathbf{C}}}$ of dimension $< m-1$, which does not depend on g and is defined by polynomial functions with real coefficients, such that

$$S \supset X_0, \qquad f^{\mathbf{C}}(S) = g^{\mathbf{C}}(S), \qquad \Sigma_{f^{\mathbf{C}}} \cap f^{\mathbf{C}-1}(f^{\mathbf{C}}(S)) = \Sigma_{f^{\mathbf{C}}} \cap g^{\mathbf{C}-1}(f^{\mathbf{C}}(S)) = S,$$

and the map germs $f^{\mathbf{C}}|_{\Sigma_{f^{\mathbf{C}}}-S}$ and $g^{\mathbf{C}}|_{\Sigma_{f^{\mathbf{C}}}-S}$ are complex Nash covering germs onto $f^{\mathbf{C}}(\Sigma_{f^{\mathbf{C}}}-S)$. Hence there exists a complex Nash diffeomorphism germ $\rho = (\rho_1, \dots, \rho_n)$ of $\Sigma_{f^{\mathbf{C}}} - S$ such that

$$f^{\mathbf{C}} \circ \rho = g^{\mathbf{C}} \quad \text{on } \Sigma_{f^{\mathbf{C}}} - S$$

and ρ is close to id in the sense that for a large integer l_1

$$|x - \rho(x)| \le |x|^{l_1} \quad \text{for } x \in \Sigma_{f^{\mathbf{C}}} - S,$$

which is possible by the Łojasiewicz inequality. Such a ρ is unique because for some positive integer l_2 and a complex algebraic subset germ S_1 of $\Sigma_{f^{\mathbf{C}}}$ of dimension 1 which does not intersect with $S - 0$,

$$\mathrm{dist}(x, \Sigma_{f^{\mathbf{C}}} \cap f^{\mathbf{C}-1}(f^{\mathbf{C}}(x)) - x) \ge |x|^{l_2} \quad \text{for } x \in S_1.$$

Clearly ρ is bounded. Hence, since $\mathcal{O}_0^{\mathbf{C}}/J_{f^{\mathbf{C}}}$ is normal (2), we can extend ρ to a complex analytic map germ $P = (P_1, \dots, P_n)$ of $\mathbf{C}_0^n$. Here we can choose P so that
(6) P is close to id,
(7) $P(\mathbf{R}_0^n) = \mathbf{R}_0^n$, and
(8) P is semialgebraic for the following reason.

For such P, replace g with $g \circ (P^{-1}|_{\mathbf{R}_0^n})$. Then we have (5).

***Proof of* (6).** We assume $\tilde{f}^{\mathbf{C}}|_{\Sigma_{\tilde{f}^{\mathbf{C}}}-\tilde{S}} : \Sigma_{\tilde{f}^{\mathbf{C}}} - \tilde{S} \to \tilde{f}^{\mathbf{C}}(\Sigma_{\tilde{f}^{\mathbf{C}}} - \tilde{S})$ is a covering for some complex Nash set $\tilde{S} \subset \Sigma_{\tilde{f}^{\mathbf{C}}}$ of dimension $< m-1$, and P is the germ of a C^ω diffeomorphism $\tilde{P}$ of $\mathbf{C}^n$. By the equality $\tilde{f}^{\mathbf{C}} \circ \tilde{P} = \tilde{\tau}^{\mathbf{C}-1} \circ \tilde{f}^{\mathbf{C}} \circ \tilde{\pi}^{\mathbf{C}}$ on $\Sigma_{\tilde{f}^{\mathbf{C}}}$, for any large integer l, if g is sufficiently close to f, we have

$$|\tilde{f}^{\mathbf{C}} \circ \tilde{P}(x) - \tilde{f}^{\mathbf{C}}(x)| \le |x|^l \quad \text{for } x \in \Sigma_{\tilde{f}^{\mathbf{C}}} \text{ around } 0.$$

Now there exists a number $l' > 0$ such that

$$\sup_{x \in U \cap \Sigma_{\tilde{f}^{\mathbf{C}}} \cap \tilde{f}^{\mathbf{C}-1}(y)} \mathrm{dist}(x, U \cap \Sigma_{\tilde{f}^{\mathbf{C}}} \cap \tilde{f}^{\mathbf{C}-1}(y')) \le |y - y'|^{l'}$$
$$\text{for } (y, y') \in (V \cap \tilde{f}^{\mathbf{C}}(\Sigma_{\tilde{f}^{\mathbf{C}}} - \tilde{S}))^2,$$

where U and V are the neighborhoods of 0 given in (1), which follows from the Łojasiewicz inequality because the both sides are semialgebraic functions on $(V \cap \tilde{f}^{\mathbf{C}}(\Sigma_{\tilde{f}^{\mathbf{C}}} - \tilde{S}))^2$ vanishing on the diagonal. Hence we have

$$\mathrm{dist}(x, U \cap \Sigma_{\tilde{f}^{\mathbf{C}}} \cap \tilde{f}^{\mathbf{C}-1}(\tilde{g}^{\mathbf{C}}(x))) \le |f^{\mathbf{C}}(x) - g^{\mathbf{C}}(x)|^{l'}$$
$$= |\tilde{f}^{\mathbf{C}} \circ \tilde{P}(x) - \tilde{f}^{\mathbf{C}}(x)|^{l'} \quad \text{for } x \in U \cap \Sigma_{\tilde{f}^{\mathbf{C}}} - \tilde{S}.$$

Consequently,

$$\mathrm{dist}(x, U \cap \Sigma_{\tilde{f}^{\mathbf{C}}} \cap \tilde{f}^{\mathbf{C}-1}(\tilde{g}^{\mathbf{C}}(x))) \le |x|^{ll'} \quad \text{for } x \in \Sigma_{\tilde{f}^{\mathbf{C}}} - \tilde{S} \text{ around } 0.$$

Choose large l. Then, since ρ is unique and close to id,

$$\mathrm{dist}(x, U \cap \Sigma_{\tilde{f}^{\mathbf{C}}} \cap \tilde{f}^{\mathbf{C}-1}(\tilde{g}^{\mathbf{C}}(x))) = |\tilde{P}(x) - x| \quad \text{for } x \in \Sigma_{\tilde{f}^{\mathbf{C}}} - \tilde{S} \text{ around } 0.$$

In conclusion,

$$|\tilde{P}(x) - x| \leq |x|^{ll'} \quad \text{for } x \in \Sigma_{\tilde{f}^{\mathbf{C}}} - \tilde{S} \text{ around } 0.$$

Moreover, this holds for $x \in \tilde{S}$ around 0 also because $\tilde{P}$ is continuous.

The maximum of l'' such that $|P(x) - x| \leq |x|^{l''}$ for $x \in \Sigma_{f^{\mathbf{C}}}$ is called the geometrical order of $P|_{\Sigma_{f^{\mathbf{C}}}} - \mathrm{id}$, and its algebraic order is by definition the maximal number of $l^{(3)}$ such that

$$P_i(x) - x_i \in (\mathfrak{m}^{\mathbf{C}})^{l^{(3)}} + J_{f^{\mathbf{C}}}, \quad i = 1, \dots, n,$$

where $x = (x_1, \dots, x_n) \in \mathbf{C}^n$ and $\mathfrak{m}^{\mathbf{C}}$ means the maximal ideal of $\mathcal{O}_0^{\mathbf{C}}$. By the theorems of [R] and [L-T] on relations between geometrical order and algebraic order, we have

$$l^{(3)} \longrightarrow \infty \quad \text{as } l'' \longrightarrow \infty.$$

Therefore, replacing each P_i with the sum of P_i and an element of $J_{f^{\mathbf{C}}}$, we can assume

$$P_i(x) - x_i \in (\mathfrak{m}^{\mathbf{C}})^{l^{(3)}}, \quad i = 1, \dots, n,$$

and hence P is close to id.

***Proof of* (7).** We have

$$(P_i + \bar{P}_i)(x)/2 - x_i \in (\mathfrak{m}^{\mathbf{C}})^{l^{(3)}}, \quad i = 1, \dots, n,$$

and, since $P = \bar{P}$ on $\Sigma_{f^{\mathbf{C}}}$,

$$f^{\mathbf{C}} \circ ((P + \bar{P})/2) = g^{\mathbf{C}} \quad \text{on } \Sigma_{f^{\mathbf{C}}}.$$

Hence we can replace P with $(P + \bar{P})/2$, which is real-valued on $\mathbf{R}_0^n$.

***Proof of* (8).** Since $f^{\mathbf{C}} \circ P = g^{\mathbf{C}}$ on $\Sigma_{f^{\mathbf{C}}}$ and $J_{f^{\mathbf{C}}}$ is a prime ideal by (2), we have real convergent power series $\alpha_{i,j}$, $i = 1, \dots, k$, $j = 1, \dots, m$, in n-variables such that

$$f \circ P - g = \Big(\sum_{i=1}^{k} \alpha_{i,1} \phi_i(f), \dots, \sum_{i=1}^{k} \alpha_{i,m} \phi_i(f)\Big).$$

Hence by Artin Approximation Theorem we can assume P is of semialgebraic.

From (2) and (5) it follows that

$$f_i - g_i \in J_f, \quad i = 1, \dots, m,$$

because $J_{f^{\mathbf{C}}}$ is reduced. On the other hand, as we have chosen g to be close to f, for a large integer r, each $f_i - g_i$ is contained in $\mathfrak{m}^r$. Therefore, by Artin-Rees Theorem

(9) $$f_i - g_i \in \mathfrak{m} J_f, \quad i = 1, \dots, m,$$

where $\mathfrak{m}$ denotes the maximal ideal of $\mathcal{O}_0$. From (9) it shall follows that f and g are C^∞ $\mathcal{R}$ equivalent. Hence by Artin Approximation Theorem they are Nash $\mathcal{R}$ equivalent, which completes the proof in the case of $2 < m < n$.

It remains to prove C^∞ $\mathcal{R}$ equivalence of f and g. Recall the following fact. Its function case is Lemma 1.1 in [S$_1$], and the map case is proved in the same way. We omit the proof.

Assume C^∞ function germs $a_i(x,t)$, $i = 1, \dots, n$, at $0 \times [0,1]$ in $\mathbf{R}^n \times \mathbf{R}$ such that

(10)
$$g(x) - f(x) = \sum_{i=1}^n a_i(x,t)\left(\frac{\partial g(x)}{\partial x_i}t + \frac{\partial f(x)}{\partial x_i}(1-t)\right) \quad \text{as germs at } 0\times[0,1],$$
$$a_i(0,t) = 0, \quad i = 1, \dots, n.$$

Then f and g are C^∞ $\mathcal{R}$ equivalent. (Here we do not need the hypothesis that f and g are of class Nash. The one of class C^∞ is sufficient.)

Using (9) we will construct such a_i's. First we show

(11)
$$\frac{\partial(f_j - g_j)}{\partial x_i} \in \mathfrak{m}J_f, \quad i = 1, \dots, n, \; j = 1, \dots, m.$$

By (1) there exists a complex analytic Nash subset $X \subset \Sigma_{\tilde{f}^{\mathbf{C}}}$ of dimension $< m-1$ such that for each $x_0 \in \Sigma_{\tilde{f}^{\mathbf{C}}} - X$ near 0, if we choose suitable local coordinate systems $u = (u_1, \dots, u_n)$ around x_0 in $\mathbf{C}^n$ and $v = (v_1, \dots, v_m)$ around $\tilde{f}^{\mathbf{C}}(x_0) = \tilde{g}^{\mathbf{C}}(x_0)$ in $\mathbf{C}^m$ with $u = 0$ at x_0 and $v = 0$ at $\tilde{f}^{\mathbf{C}}(x_0)$, then the germs $\tilde{f}^{\mathbf{C}}_{x_0}$ and $\tilde{g}^{\mathbf{C}}_{x_0}$ are of the form:

$$(u_1, \dots, u_{m-1}, \alpha(u)) \quad \text{and} \quad (u_1, \dots, u_{m-1}, \beta(u)),$$

and

$$\Sigma_{\tilde{f}^{\mathbf{C}}, x_0} = \mathbf{C}_0^{m-1} \times 0^{n-m+1},$$

where α and β vanish and are singular on $\mathbf{C}_0^{m-1} \times 0^{n-m+1}$. Hence all $\frac{\partial(\tilde{f}^{\mathbf{C}} - \tilde{g}^{\mathbf{C}})}{\partial x_i}$ vanish on $\Sigma_{\tilde{f}^{\mathbf{C}}} - X$ near 0, which implies (11).

Set

$$B = \begin{pmatrix} \frac{\partial g}{\partial x_1}t + \frac{\partial f}{\partial x_1}(1-t) \\ \vdots \\ \frac{\partial g}{\partial x_n}t + \frac{\partial f}{\partial x_n}(1-t) \end{pmatrix}.$$

It follows from (11) and Nakayama's Lemma that if we fix $t \in [0,1]$, the minors of B of degree m are generators of J_f. Hence the minors $\phi_1(gt + f(1-t)), \dots \phi_k(gt + f(1-t))$ of B are generators of the ideal of the ring $\mathcal{O}_{0\times[0,1]}$ of C^ω function germs at $0 \times [0,1]$ in $\mathbf{R}^n \times \mathbf{R}$ generated by $\phi_1(f), \dots, \phi_k(f)$, which are regarded as function germs at $0 \times [0,1]$ in $\mathbf{R}^n \times \mathbf{R}$. Therefore, by (9) there exists a (k,m) matrix C with elements in $\mathcal{O}_{0\times[0,1]}$ such that

$$g - f = (\phi_1(gt + f(1-t)), \dots \phi_k(gt + f(1-t)))C,$$
$$C = 0 \quad \text{on } 0 \times [0,1].$$

Now we can restate (10) as follows:

$$(10') \qquad (\phi_1(gt+f(1-t)),\dots,\phi_k(gt+f(1-t)))C = AB,$$

where $A=(a_1,\dots,a_n)$. To solve (10′) we need only consider

$$(0,\dots,0,\phi_i(gt+f(1-t)),0,\dots,0)C = A_iB, \quad i=1,\dots,k.$$

Hence we can reduce the problem as follows.

Let ϕ_1 be the minor of the upper m rows. Let $C_1 \in (\mathcal{O}_{0\times[0,1]})^m$ with $C_1=0$ on $0\times[0,1]$. Then there exists $A_1 \in (\mathcal{O}_{0\times[0,1]})^n$ such that

$$\phi_1(gt+f(1-t))C_1 = A_1B, \qquad A_1 = 0 \quad \text{on } 0\times[0,1].$$

This is clear if we ask only for $A_1 \in (\mathcal{O}_{0\times[0,1]})^m \times 0^{n-m}$.

Case of $m=2<n$. We proceed as above. The fact (1) holds true. But (2) does not hold, namely, $\mathcal{O}_0^{\mathbf{C}}/J_{f^{\mathbf{C}}}$ is not always normal. Hence we need to modify the above arguments on extension of ρ. As in the first case, let g be close to f. Assume $\Sigma_{f^{\mathbf{C}}}$ has singularities. Then, since $\dim \Sigma_{f^{\mathbf{C}}}=1$, 0 is the isolated singularity of $\Sigma_{f^{\mathbf{C}}}$. By the same reason as in the first case we assume

$$\Sigma_g=\Sigma_f, \qquad \Sigma_{g^{\mathbf{C}}}=\Sigma_{f^{\mathbf{C}}}, \qquad f(\Sigma_f)=g(\Sigma_g), \qquad f^{\mathbf{C}}(\Sigma_{f^{\mathbf{C}}})=g^{\mathbf{C}}(\Sigma_{g^{\mathbf{C}}}).$$

Then define a complex analytic diffeomorphism germ $\rho=(\rho_1,\dots,\rho_n)$ of $\Sigma_{f^{\mathbf{C}}}-0$ so that

$$f^{\mathbf{C}}\circ\rho = g^{\mathbf{C}} \quad \text{on } \Sigma_{f^{\mathbf{C}}}-0.$$

We extend ρ to $\Sigma_{f^{\mathbf{C}}}$ by setting $\rho(0)=0$. In general, ρ is not extensible to a complex analytic map germ : $\mathbf{C}_0^n \to \mathbf{C}_0^n$, but:

If g is sufficiently close to f then ρ is extensible.

Proof of extendability. Without loss of generality we assume any irreducible component of $\Sigma_{f^{\mathbf{C}}}$ is not contained in any hyperplane $\{x_i=0\}$. By a theorem of Oka (Theorem IV.14 in [H]) there exists a positive integer s such that $x_1^s(\rho_i(x)-x_i)$, $i=1,\dots,n$, and x_i^s/x_1, $i=2,\dots,n$, on $\Sigma_{f^{\mathbf{C}}}$ are extensible to complex analytic function germs on $\mathbf{C}_0^n$. Let α_i,β_i be respective extensions. Then by the same arguments as in the first case each α_i is chosen to be arbitrarily close to 0. Hence each α_i is of the form

$$(12) \qquad \sum_{\substack{\gamma=(\gamma_1,\dots,\gamma_n)\in\mathbf{N}^n\\ |\gamma|=s'}} a_{i,\gamma}x^\gamma, \quad a_{i,\gamma}\in\mathcal{O}_0^{\mathbf{C}},$$

where

$$x^\gamma = x_1^{\gamma_1}\cdots x_n^{\gamma_n}, \qquad |\gamma|=\gamma_1+\cdots+\gamma_n,$$

and s' is some large integer. Hence if g is close to f then we can choose arbitrarily large s'. Note

$$x_i^s = \beta_i x_1 \quad \text{on } \Sigma_{f^{\mathbf{C}}}, \quad i=2,\dots n.$$

In (12) replace x^γ with $\beta^\delta x_1^{|\delta|}x^\gamma/x^{s\delta}$, where $\delta=(0,\delta_2,\dots,\delta_n)\in\mathbf{N}^n$ is the maximum such that $\gamma\ge s\delta$, i.e. $\gamma_i\ge s\delta_i$, $i=2,\dots;n$. Then each α_i becomes divisible by $x_1^{s''}$, where s'' is an integer such that $s''\to\infty$ as $s'\to\infty$. Let g be so close to f that $s''>s$. Then α_i/x_1^s is a complex analytic extension of $\rho_i(x)-x_i$. Hence ρ is extensible.

The above proof shows, moreover, that we can choose an extension P of ρ to be close to id. The rest of proof is the same as in the first case except that $J_{f^{\mathbf{C}}}$ is prime. In the present case it is only reduced but sufficient for the rest of proof.

Case of $m\ge n$. We modify the proof in the first case. The proof becomes easier. In this case $f^{\mathbf{C}}|_U\colon U\to V$ is proper and finite-to-one for small open neighborhoods U of 0 in $\mathbf{C}^n$ and V of 0 in $\mathbf{C}^m$. Let g be close to f. Then as in the first case we obtain Nash diffeomorphism germs π of $\mathbf{R}^n_0$ and τ of $\mathbf{R}^m_0$ such that they are close to id, and

$$\begin{aligned}\pi(\Sigma_g)&=\Sigma_f, & \pi^{\mathbf{C}}(\Sigma_{g^{\mathbf{C}}})&=\Sigma_{f^{\mathbf{C}}},\\ f(\Sigma_f)&=\tau\circ g(\Sigma_g), & f^{\mathbf{C}}(\Sigma_{f^{\mathbf{C}}})&=\tau^{\mathbf{C}}\circ g^{\mathbf{C}}(\Sigma_{g^{\mathbf{C}}}),\\ f(\mathbf{R}^n_0)&=\tau\circ g(\mathbf{R}^n_0), & f^{\mathbf{C}}(\mathbf{C}^n_0)&=\tau^{\mathbf{C}}\circ g^{\mathbf{C}}(\mathbf{C}^n_0).\end{aligned}$$

Replace g with $\tau\circ g\circ\pi^{-1}$. Then

$$\Sigma_g=\Sigma_f,\quad \Sigma_{g^{\mathbf{C}}}=\Sigma_{f^{\mathbf{C}}},\quad f(\Sigma_f)=g(\Sigma_g),\quad f^{\mathbf{C}}(\Sigma_{f^{\mathbf{C}}})=g^{\mathbf{C}}(\Sigma_{g^{\mathbf{C}}}),$$
$$f(\mathbf{R}^n_0)=g(\mathbf{R}^n_0),\qquad f^{\mathbf{C}}(\mathbf{C}^n_0)=g^{\mathbf{C}}(\mathbf{C}^n_0).$$

Next by the same reason as in the first case we can assume a complex Nash subset germ S of $\mathbf{C}^n_0$ of dimension $<n$, defined by polynomial functions with real coefficients, such that

$$f^{\mathbf{C}}(S)=g^{\mathbf{C}}(S),\qquad f^{\mathbf{C}-1}(f^{\mathbf{C}}(S))=g^{\mathbf{C}-1}(g^{\mathbf{C}}(S))=S,$$

and the map germ $f^{\mathbf{C}}|_{\mathbf{C}^n_0-S}$ and $g^{\mathbf{C}}|_{\mathbf{C}^n_0-S}$ are complex analytic covering germs onto $f^{\mathbf{C}}(\mathbf{C}^n_0-S)$. Define a complex analytic diffeomorphism germ ρ of $\mathbf{C}^n_0-S$ so that

$$f^{\mathbf{C}}\circ\rho=g^{\mathbf{C}}\quad\text{on }\mathbf{C}^n_0-S.$$

Then ρ is semialgebraic and extensible to a complex Nash diffeomorphism germ P of $\mathbf{C}^n_0$ because $\mathcal{O}^{\mathbf{C}}_0$ is normal. Clearly $P|_{\mathbf{R}^n_0}$ is a Nash diffeomorphism germ of $\mathbf{R}^n_0$, and we have

$$f\circ P=g\quad\text{on }\mathbf{R}^n_0.$$

Proof of C^ω $\mathcal{R}$-$\mathcal{L}$ equivalence of C^ω map germs. Let $f,\,g\colon\mathbf{R}^n_0\to\mathbf{R}^m_0$ be formally $\mathcal{R}$-$\mathcal{L}$ equivalent C^ω map germs of finite singularity type. We can prove that they are C^ω $\mathcal{R}$-$\mathcal{L}$ equivalent in the same way as above. The difference is that for an invertible element $\pi\in\mathbf{R}[[x_1,\dots,x_n]]^n$ the equality $\pi(\Sigma_g)=\Sigma_f$ is meaningless. We replace it with

$$J_g=\mathbf{R}\{x_1,\dots,x_n\}\cap J_f\circ\pi,$$

where

$$J_f\circ\pi=\{\psi\circ\pi\colon\psi\in J_f\}.$$

Then the above arguments work. We omit the detail. □

Conjecture 1.8. *We can remove the assumptions of normality and reducedness in* 1.7.

Fact 1.9. 1.7 *is not correct for* C^∞ *map germs. Namely, there exist two formally* $\mathcal{R}$-$\mathcal{L}$ *equivalent* C^∞ *map germs which are of finite singularity type but not* C^∞ $\mathcal{R}$-$\mathcal{L}$ *equivalent. For example, define* $f\colon \mathbf{R}_0^2 \to \mathbf{R}_0^2$ *by* $f(x,y) = (x, y^3)$ *and choose* $g = f + (a$ C^∞ *function germ flat at* 0) *with an isolated singularity at* 0. *Then* f *and* g *are formally the same each other and of finite singularity type but not* C^∞ $\mathcal{R}$-$\mathcal{L}$ *equivalent.*

This is one reason why I expect a better theory of C^ω and Nash singularities than C^∞ ones.

§2. Function germs

Fact 2.1 (Theorem II.7.1 in [S3]). *Let* $X \subset Y \subset \mathbf{R}^n$ *be semialgebraic (subanalytic) sets, and let* $f, g\colon Y \to \mathbf{R}$ *be semialgebraic (subanalytic) functions with* $f^{-1}(0) = g^{-1}(0) = X$. *Then the germs of* f *and* g *at* X *are semialgebraically (subanalytically)* $\mathcal{R}$ *equivalent up to sign, i.e., the germs of* $|f|$ *and* $|g|$ *at* X *are semialgebraically (subanalytically)* $\mathcal{R}$ *equivalent. Here we can choose the semialgebraic (subanalytic) homeomorphisms of equivalence to be the identity map on* X. *Consequently, if the germs of* f *and* g *at* X *are semialgebraically (subanalytically)* $\mathcal{R}$-$\mathcal{L}$ *equivalent, then the germs of* f *and* g *are semialgebraically (subanalytically)* $\mathcal{R}$ *equivalent or the germs of* f *and* $-g$ *are so.*

This fact is one of typical properties of semialgebraic (subanalytic) function germs and semialgebraic (subanalytic) equivalence relation. Clearly there exist two non-negative C^∞ function germs on $\mathbf{R}_0$ whose zero sets are both 0 and which are not C^0 $\mathcal{R}$-$\mathcal{L}$ equivalent. It is also clear that the function germs $x \to x^2$ and $x \to x^4$ are semialgebraically $\mathcal{R}$ equivalent but not C^1 $\mathcal{R}$-$\mathcal{L}$ equivalent. Moreover, C^ω $\mathcal{R}$-$\mathcal{L}$ equivalence of non-negative C^ω function germs does not imply C^1 $\mathcal{R}$ equivalence as follows.

Fact 2.2. *Define polynomial function germs* f *and* g *on* $\mathbf{R}_0^2$ *by*

$$f(x,y) = y^2(y-x^2)^2(y-x^4)^2, \qquad g = 4f.$$

Then f *and* g *are linearly* $\mathcal{L}$ *equivalent but not* C^1 $\mathcal{R}$ *equivalent.*

Proof. Set

$$A = \{y=0\}_0, \qquad B = \{y = x^2\}_0, \qquad C = \{y = x^4\}_0.$$

Assume there exists a C^1 diffeomorphism $\pi = (\pi_1, \pi_2)$ of $\mathbf{R}_0^2$ such that $f \circ \pi = g$, and let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ denote its Jacobian matrix. There are two cases : $\pi(A) = A$ and $\pi(B) = B$, or $\pi(A) = C$ and $\pi(B) = B$. Let us consider the first case. Then we can change the definition of f and g by $f = y(y-x^2)(y-x^4)$, $g = 2f$.

We have

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \begin{cases} (0, x^6) & \text{on } A \\ (-2x^5 + 2x^7, x^4 - x^6) & \text{on } B, \end{cases}$$

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 2\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right).$$

Hence

$$c(x,0)\pi_1^6(x,0) = 0, \qquad d(x,0)\pi_1^6(x,0) = 2x^6,$$

$$b(x,x^2)(-2\pi_1^5(x,x^2) + 2\pi_1^7(x,x^2)) + d(x,x^2)(\pi_1^4(x,x^2) - \pi_1^6(x,x^2)) = 2x^4 - 2x^6.$$

Since

$$\frac{\pi_1(x,y)}{x} \longrightarrow a(0) \quad \text{as } (x,y) \longrightarrow 0 \quad \text{on } A \cup B,$$

it follows that

$$c(0) = 0, \qquad d(0)a^6(0) = 2, \qquad d(0)a^4(0) = 2.$$

Therefore,

$$a^2(0) = 1, \qquad d(0) = 2.$$

We can assume $a(0) = 1$ because $f(-\pi_1, \pi_2) = g$. Set

$$h_1(y) = \pi_1(0,y) - b(0)y, \qquad h_2(y) = \pi_2(0,y) - 2y,$$

which are C^1 function germs on $\mathbf{R}_0$ such that $h_1'(0) = h_2'(0) = 0$. Consider $f \circ \pi = 2f$ on $\{x = 0\}_0$. Then we have

$$(2y + h_2)(2y + h_2 - (b(0)y + h_1)^2)(2y + h_2 - (b(0)y + h_1)^4) = 2y^3 \quad \text{on } \mathbf{R}_0.$$

Divide the both sides by y^3, and take the limits as $y \to 0$. Then $8 = 2$, which is impossible.

In the case of $\pi(A) = C$ we arrive at a contradiction in the same way but more easily. We omit the detail. □

Fact 2.3 (Example II.7.9 in [S$_3$]). *There exist two homogeneous polynomial functions on $\mathbf{R}^7$ with an isolated singularity at 0 which are C^0 $\mathcal{R}$ equivalent but not subanalytically $\mathcal{R}$ equivalent and whose germs at 0 are also C^0 $\mathcal{R}$ equivalent but not subanalytically $\mathcal{R}$ equivalent.*

Fact 2.4. *Two formally $\mathcal{R}$-$\mathcal{L}$ equivalent C^ω (Nash) function germs are C^ω (Nash, resp.) $\mathcal{R}$-$\mathcal{L}$ equivalent.*

Proof. Let f and g be formally $\mathcal{R}$-$\mathcal{L}$ equivalent C^ω function germs on $\mathbf{R}_0^n$ with $f(0) = g(0) = 0$ which are singular at 0, and let $\pi \in \mathbf{R}[[x_1, \dots, x_n]]^n$ and $\tau \in \mathbf{R}[[y]]$ be invertible elements such that $f \circ \pi = \tau \circ g$. By 1.1 it suffices to find an invertible polynomial element $\tau_1 \in \mathbf{R}[[y]]$ such that $\tau_1 \circ g$ and $\tau \circ g$ are formally $\mathcal{R}$ equivalent. Let $\tau_1 \in \mathbf{R}[[y]]$ be a polynomial and sufficiently close to τ, and set $\tau_2 = \tau_1^{-1} \circ \tau$. Then τ_2 is close to id, and hence $g - \tau_2 \circ g$ is contained in the ideal of $\mathbf{R}[[x_1, \dots, x_n]]$ generated by g^p for a large integer p. Complexify g and apply Hilbert zero point theorem to $g^{\mathbf{C}}$ and the Jacobian ideal $J_{g^{\mathbf{C}}} = (\frac{\partial g^{\mathbf{C}}}{\partial x_1}, \dots, \frac{\partial g^{\mathbf{C}}}{\partial x_n})$. Then we have $g^q \in J_g$ for some integer q. Hence we can assume $g - \tau_2 \circ g \in \mathfrak{m} J_g^2$ where $\mathfrak{m}$ denotes the maximal ideal of $\mathbf{R}[[x_1, \dots, x_n]]$. Then it is known that g and $\tau_2 \circ g$ are formally $\mathcal{R}$ equivalent. (The C^∞ germ case is Proposition II.2 in [T]. The formal case is proved in the same way.) □

§3. Global maps

Let $\mathcal{O}(M)$ ($\mathcal{N}(M)$) denote the ring of C^ω (Nash) functions on a C^ω (Nash, respectively) manifold M, and let $\mathcal{O}^M$ denote the sheaf of C^ω function germs on M. We write $\mathcal{O}$ for $\mathcal{O}^M$ if no confusion happens.

For an analytic set $A \subset M$, the *sheaf of ideals* $\mathcal{I}(A) \subset \mathcal{O}$ *defined by* A is such that $\mathcal{I}(A)_x$ consists of germs vanishing at A_x. We call A *coherent* if $\mathcal{I}(A)$ is coherent, which is equivalent to the following statement by the fundamental theorem A on Stein manifolds. There exist C^ω functions f_i on M vanishing on A such that for each $x \in M$, f_{ix} generate $\mathcal{I}(A)_x$.

For a subset A of a C^ω manifold M, the *analytic closure* of A is the intersection of the zero sets of C^ω functions vanishing on A, and the *Nash closure* of A is defined in the same way if M is a Nash manifold. Note that the analytic (Nash) closure is the zero set of one C^ω (Nash, respectively) function, which is proved as follows. This is clear if M is compact or in the Nash case. So assume M is a noncompact C^ω manifold. Let A^a denote the analytic closure, and let ϕ_i, $i \in \mathbf{N}$, be C^ω functions on M whose common zero set is A^a. Since the ring $C^\omega(K)$ of C^ω function germs at a compact semianalytic set K in M is Noetherian, there exist a finite number of ϕ_i such that another ϕ_i is their linear combination with coefficients in $C^\omega(K)$ as germs at K. Hence we have a complexification $A^{a\mathbf{C}} \subset M^{\mathbf{C}}$ of the pair $A^a \subset M$ such that $M^{\mathbf{C}}$ is a Stein manifold and $A^{a\mathbf{C}}$ is an analytic set in $M^{\mathbf{C}}$. Using the fundamental theorem A we can assume the complexifications of ϕ_i are defined on $M^{\mathbf{C}}$. Then $\sum a_i\phi_i^2$ is the required function, where a_i are small positive numbers.

From now, for a C^ω manifold M, let $M^{\mathbf{C}}$ denote the germ at M of a complexification of M. Let $f\colon M_1 \to M_2$ be a C^ω map between C^ω manifolds. Let $f^{\mathbf{C}}\colon M_1^{\mathbf{C}} \to M_2^{\mathbf{C}}$ always denote the germ at M_1 of a complexification of f, let Σ_f denote the singular point set of f, and let $J_f \subset \mathcal{O}^{M_1}$ denote the sheaf of the Jacobian ideals of f defined so that for each $x \in M_1$, $J_{f,x}$ is the Jacobian ideal of f_x. Remember $J_f = \{0\}$ if $\dim M_1 < \dim M_2$. We say f is of *finite singularity type* if the germ f_x at each point x is of finite singularity type. We define naturally $\mathcal{O}^{M_i^{\mathbf{C}}}$, $\mathcal{I}^{\mathbf{C}}(A)$, $\Sigma_{f^{\mathbf{C}}}$ and $J_{f^{\mathbf{C}}}$, and write $\mathcal{O}^{\mathbf{C}}$.

We give to C^∞ and C^ω map and function spaces the Whitney C^∞ topology and to Nash ones the Nash topology (see [$S_{2,3}$]) unless otherwise specified.

Fact 3.1. *C^ω $\mathcal{L}$, $\mathcal{R}$-$\mathcal{L}$, or $\mathcal{R}$ equivalence of Nash maps does not necessarily imply Nash $\mathcal{L}$, $\mathcal{R}$-$\mathcal{L}$, or $\mathcal{R}$ equivalence, respectively, if the Nash manifolds are noncompact.*

Proof. See Fact 4.1 for a counter-example of $\mathcal{R}$-$\mathcal{L}$ and $\mathcal{R}$ equivalences.

Let us construct a counter-example of $\mathcal{L}$ equivalence. By Remark VI.2.6 in [S_2] there exist Nash manifolds $M_1 \subset M_2$ and a C^ω diffeomorphism $\phi\colon (M_1 \times]-1,1[, M_1 \times 0) \to (M_2, M_1)$ such that M_1 is compact, $M_1 \times]-1,0]$ and $\phi(M_1 \times]-1,0])$ are Nash diffeomorphic, and $M_1 \times [0,1[$ and $\phi(M_1 \times [0,1[)$ are not so. Set $M_3 = M_1 \times S^1$. Let p and q denote the projections of $M_1 \times]-1,1[$ onto the first and second factors, respectively. Let $\phi'\colon M_1 \times]-1,1[\to M_2$ be a Nash imbedding such that $\phi'|_{M_1\times 0} = \mathrm{id}$. Let $h\colon S^1 \to]-1,1[$ be a Nash map such that h and $-h$ are not Nash $\mathcal{L}$ equivalent by any orientation preserving Nash diffeomorphism of

$]-1,1[$. Define Nash maps $f,\ g\colon M_3 \to M_2$ so that

$$\operatorname{Im} f, \operatorname{Im} g \subset \operatorname{Im} \phi', \qquad p \circ \phi'^{-1} \circ f = p \circ \phi'^{-1} \circ g = \text{proj},$$
$$q \circ \phi'^{-1} \circ f = h, \qquad q \circ \phi'^{-1} \circ g = -h.$$

Then f and g are C^ω $\mathcal{L}$ equivalent but not Nash $\mathcal{L}$ equivalent. □

Conjecture and Examples 3.2. *A conjecture is that C^∞ $\mathcal{R}$ equivalence of Nash (or C^ω) maps implies C^ω $\mathcal{R}$ equivalence.* Some special cases are proved in [S₁].

If the following globalization of Artin Approximation Theorem holds then the Conjecture is clear.

Let $F_i\colon M \times N \to \mathbf{R}$ be a finite number of C^ω functions for C^ω manifolds M and N. Let $y = y(x)\colon M \to N$ be a C^∞ solution of $F_i(x, y(x)) = 0$. Then we can approximate it by a C^ω solution.

But this globalization is not always true. For example, let $M = \mathbf{R}$, $N = \mathbf{R}^2$ and $F(x, y_1, y_2) = y_1 y_2$, and let $y = y(x)$ be defined so that its image is contained in $\{y_1 y_2 = 0\}$ and of the form $\llcorner$. Clearly we can not approximate the solution by a C^ω solution.

A counter-example in the case where $M = N$ and $y = y(x)$ is a C^∞ diffeomorphism is the following. Set $M = N = \mathbf{R}^2$ and $F = (x_1 - y_1)(x_2 - y_2)$. Let $f = (f_1, f_2)\colon \mathbf{R}^2 \to \mathbf{R}^2$ be a small C^∞ perturbation of id such that

$$f_1(x_1, x_2)\begin{cases} = x_1 & \text{if } x_1 \le 0 \\ > x_1 & \text{if } x_1 > 0, \end{cases}$$
$$f_2(x_1, x_2)\begin{cases} > x_2 & \text{if } x_1 < 0 \\ = x_2 & \text{if } x_1 \ge 0. \end{cases}$$

Then $y = f(x)$ is a C^∞ solution of $F = 0$, and there does not exist its C^ω approximation because any strong C^∞ approximation solution is a solution of $x_1 = y_1$ on $\{x_1 \le -1\}$ and a solution of $x_2 = y_2$ on $\{x_1 \ge 1\}$ and hence a C^ω approximation is uniquely id.

Fact 3.3. *C^ω $\mathcal{R}$ equivalence of Nash maps implies Nash $\mathcal{R}$ equivalence if the source Nash manifold is compact.*

Proof. This follows from the global approximation theorem (Theorem 0.0 in [C-R-S]). □

Conjecture 3.4. *C^ω $\mathcal{R}$-$\mathcal{L}$ equivalence of Nash maps implies Nash $\mathcal{R}$-$\mathcal{L}$ equivalence if the manifolds are compact.*

Fact 3.5. *C^∞ $\mathcal{L}$ equivalence of Nash maps does not imply C^ω $\mathcal{R}$-$\mathcal{L}$ equivalent even if the manifolds are compact. But C^ω $\mathcal{L}$ equivalence implies Nash $\mathcal{L}$ equivalence if the manifolds are compact.*

Proof. See the proof of Fact 3.8 for the first statement.

The proof of the second is similar to it of 1.3. Let $f,\ g\colon M_1 \to M_2$ be C^ω $\mathcal{L}$ equivalent Nash maps, and let π be a C^ω diffeomorphism of M_2 such that $\pi \circ$

$f = g$. Assume M_2 is contained in $\mathbf{R}^m$. Set $\phi = (f, g)\colon M_1 \to M_2 \times M_2$ and $\pi = (\pi_1, \dots, \pi_m)$. Let

$$\phi_1^*\colon \mathcal{O}(M_2 \times M_2) \to \mathcal{O}(M_1), \qquad \phi_2^*\colon \mathcal{N}(M_2 \times M_2) \to \mathcal{N}(M_1)$$

denote the homomorphism induced by ϕ. We want to see

$$\operatorname{Ker} \phi_1^* = \operatorname{Ker} \phi_2^* \mathcal{O}(M_2 \times M_2).$$

The inclusion $\supset$ is clear. We show the reverse inclusion. We have

$$\operatorname{Ker} \phi_1^* = \{h \in \mathcal{O}(M_2 \times M_2)\colon h = 0 \text{ on } \operatorname{Im} \phi\},$$
$$\operatorname{Ker} \phi_2^* = \{h \in \mathcal{N}(M_2 \times M_2)\colon h = 0 \text{ on } \operatorname{Im} \phi\}.$$

Hence it suffices to prove the following two statements. (1) For a semialgebraic set $A \subset M_2$, the Nash closure of A is the analytic closure of A. (2) For a Nash set $A \subset M_2$, set

$$I = \{h \in \mathcal{O}(M_2)\colon h = 0 \text{ on } A\}, \qquad J = \{h \in \mathcal{N}(M_2)\colon h = 0 \text{ on } A\}.$$

Then $I \subset J\mathcal{O}(M_2)$.

(1) is an immediate consequence of Proposition 0.4 in [C-R-S]. To show (2) we can assume A is irreducible as a Nash set. Then J is prime, and so is $J\mathcal{O}(M_2)$ by Proposition 0.5 in [C-R-S]. On the other hand, it is easy to see that the coheights of I and J equal t-dim A. Hence (2) follows.

Since the Nash function ring on a Nash manifold is Noetherian, $\operatorname{Ker} \phi_2^*$ is finitely generated. Let $\alpha_1, \dots, \alpha_l$ be generators. Set

$$z_i - \pi_i(y) = \beta_i(y, z) \quad \text{for } (y, z) = (y, z_1, \dots, z_m) \in M_2 \times \mathbf{R}^m, \ i = 1, \dots, m.$$

Then $\beta_i|_{M_2 \times M_2} \in \operatorname{Ker} \phi_1^*$ and $\cap_i \beta_i^{-1}(0) = \operatorname{graph} \pi$. Hence there exist $\gamma_{i,j} \in \mathcal{O}(M_2 \times M_2)$ such that $\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix} = \begin{pmatrix} \gamma_{1,1} & \cdots & \gamma_{1,l} \\ \vdots & & \vdots \\ \gamma_{m,1} & \cdots & \gamma_{m,l} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_l \end{pmatrix}$ on $M_2 \times M_2$. Let $\gamma'_{i,j}$ be Nash function approximations of $\gamma_{i,j}$ and define Nash function approximations β'_i of the restrictions of β_i to $M_2 \times M_2$ by $\begin{pmatrix} \beta'_1 \\ \vdots \\ \beta'_m \end{pmatrix} = \begin{pmatrix} \gamma'_{1,1} & \cdots & \gamma'_{1,l} \\ \vdots & & \vdots \\ \gamma'_{m,1} & \cdots & \gamma'_{m,l} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_l \end{pmatrix}$.

Extend the small C^ω functions $\beta'_i - \beta_i$ to small C^ω functions on $M_2 \times \mathbf{R}^m$, and approximate β_i+the extensions by Nash functions $\tilde{\beta}'_i$ in the compact-open C^∞ topology fixing them on $M_2 \times M_2$. Then $\tilde{\beta}'_i$ are close to β_i on a neighborhood of $M_2 \times M_2$ in $M_2 \times \mathbf{R}^m$, and $\begin{pmatrix} \tilde{\beta}'_1 \\ \vdots \\ \tilde{\beta}'_m \end{pmatrix} = \begin{pmatrix} \gamma'_{1,1} & \cdots & \gamma'_{1,l} \\ \vdots & & \vdots \\ \gamma'_{m,1} & \cdots & \gamma'_{m,l} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_l \end{pmatrix}$ on $M_2 \times M_2$.

By the implicit function theorem, there exist uniquely Nash functions $\beta''_i(y, z)$ on a neighborhood of $M_2 \times M_2$ in $M_2 \times \mathbf{R}^m$ of the form $z_i - \pi''_i(y)$, $i = 1, \dots, m$, such that $\cap_i \tilde{\beta}'^{-1}_i(0) = \cap_i \beta''^{-1}(0)$ on the neighborhood. Set $\pi'' = (\pi''_1, \dots, \pi''_m)$. Then π'' is a Nash imbedding of M_2 into $\mathbf{R}^m$ and $\pi'' \circ f = g$. Let p be the orthogonal projection of a tubular neighborhood of M_2 in $\mathbf{R}^m$. Then $p \circ \pi''$ is the required Nash diffeomorphism of M_2 of $\mathcal{L}$ equivalence. □

Fact 3.6. *Let f, g: $M_1 \to M_2$ be C^∞ $\mathcal{L}$ equivalent C^ω maps between C^ω manifolds. Assume* $\operatorname{Im} f$ *is a coherent analytic set in M_2. Then f and g are C^ω $\mathcal{L}$ equivalent.*

Proof. Let $\pi = (\pi_1, \dots, \pi_m)$, ϕ, ϕ_1^* and β_i be defined as in the proof of 3.5. Then by the proof it suffices to prove that each β_i can be approximated by an element of $\operatorname{Ker} \phi_1^*$. Assume
(*) $\operatorname{Im} \phi$ is a coherent analytic set.
Now $\operatorname{Ker} \phi_1^*$ is the global cross-sections of the sheaf of ideals $\mathcal{I}(\operatorname{Im} \phi) \subset \mathcal{O}^{M_2 \times M_2}$ defined by $\operatorname{Im} \phi$. Hence by (*) and Theorem VI.3.10 in [Mal],

$$\beta_i \in \operatorname{Ker} \phi_1^* C^\infty(M_2 \times M_2).$$

Such β_i can be approximated by an element of $\operatorname{Ker} \phi_1^*$ by the following assertion.

(**) Let $U \subset \mathbf{C}^n$ be a Stein open set containing $\mathbf{R}^n$, let $X \subset U$ be a complex analytic set such that $X \cap \mathbf{R}^n$ is coherent and X is a complexification of $X \cap \mathbf{R}^n$, and let γ be a C^∞ function on $\mathbf{R}^n$ which vanishes on $X \cap \mathbf{R}^n$. Then γ is approximated by a C^ω function on $\mathbf{R}^n$ with the same property.

Proof of (*). By 1.6, for each $(y, z) \in \operatorname{Im} \phi$, $(\operatorname{Im} \phi)_{y,z}$ is the graph of a C^ω map germ defined on $(\operatorname{Im} f)_y$. Hence, by coherence of $\operatorname{Im} f$, $\mathcal{I}(\operatorname{Im} \phi)$ is coherent.

Proof of (**). This is a small generalization of Lemma 6.2.3 in [S$_1$]. Set

$$K_c = \{x \in \mathbf{R}^n \colon |x| \le c\}, \quad K_c^{\mathbf{C}} = \{x \in \mathbf{C}^n \colon |x| \le c\} \quad \text{for } c > 0.$$

Let h be a C^∞ function on $\mathbf{R}^n$ with $h = 0$ on $K_{1/2}$ and $h = 1$ outside K_1. Shrink U and let $\psi_1, \psi_2, \dots$ be generators of $H^0(U, \mathcal{I}^{\mathbf{C}}(X))$ and real-valued on $\mathbf{R}^n$. Then for each $c > 0$, $H^0(U \cap K_c^{\mathbf{C}}, \mathcal{I}^{\mathbf{C}}(X \cap K_c^{\mathbf{C}}))$ is generated by $\psi_1, \dots, \psi_{l_c}$ for some l_c. Hence $\gamma|_{K_2}$ is of the form $(\xi_1\psi_1 + \dots + \xi_{l_2}\psi_{l_2})|_{K_2}$ for some C^∞ functions $\xi_1, \dots, \xi_{l_2}$ on $\mathbf{R}^n$. Approximate each ξ_i by a polynomial function in the compact-open C^∞ topology. Then we have $\Gamma_1 \in H^0(U, \mathcal{I}^{\mathbf{C}}(X))$ such that $\Gamma_1|_{K_2}$ is an approximation of $\gamma|_{K_2}$. We choose Γ_1 so that $(\gamma - \Gamma_1)h_1$ is also sufficiently small on K_2, where $h_1(x) = h(x/2)$. Repeating this construction we can obtain a sequence of complex C^ω functions $\Gamma_1, \Gamma_2, \dots$ so that $(\Gamma_1 + \Gamma_2 + \cdots)|_{\mathbf{R}^n}$ is a C^∞ approximation of γ which vanishes on $X \cap \mathbf{R}^n$.

For analyticity of $\Gamma_1 + \Gamma_2 + \cdots$, we define Γ_2 precisely as follows. Let $\Gamma_2' \in H^0(U, \mathcal{I}^{\mathbf{C}}(X))$ be such that $\gamma - \Gamma_1 - \Gamma_2'$ and $(\gamma - \Gamma_1 - \Gamma_2')h_2$ on K_3 are small, where $h_2(x) = h(x/3)$. Set

$$\Gamma_2''(x) = dk^n \int_{\mathbf{R}^n} h_1(y) e^{-k^2|x-y|^2} dy \quad \text{for } x \in \mathbf{C}^n,$$

here $d = 1/\int_{\mathbf{R}^n} e^{-y^2} dy$ and k is a large number. Then Γ_2'' is analytic on $\mathbf{C}^n$, and by Lemma 5 in [Wh], $\Gamma_2'' - h_1$ on K_3 and Γ_2'' on $K_{1/2}^{\mathbf{C}}$ are small. Hence $\Gamma_2 = \Gamma_2'\Gamma_2''$ on K_3 is an approximation of $(\gamma - \Gamma_1)h_1|_{K_3}$, and $\gamma - \Gamma_1 - \Gamma_2$ on K_3 is small because $\gamma - \Gamma_1$ and $(\gamma - \Gamma_1)h_1$ on K_2 are small and because h_1 is close to 1 on $K_3 - K_2$.

Define $\Gamma_3, \Gamma_4, \dots$ in the same way and set $\Gamma = \Gamma_1 + \Gamma_2 + \cdots$. Then Γ is convergent on U because Γ_2'' on $K_{1/2}^{\mathbf{C}}$, Γ_3'' on $K_{3/4}^{\mathbf{C}}, \dots$ are small, and $\Gamma|_{\mathbf{R}^n}$ is an approximation of γ and vanishes on $X \cap \mathbf{R}^n$. □

Fact 3.7. *Two C^ω $\mathcal{R}$-$\mathcal{L}$ equivalent Nash maps $f, g\colon M_1 \to M_2$ are Nash $\mathcal{R}$-$\mathcal{L}$ equivalent if M_1 and M_2 are compact, if f is of finite singularity type, if $\mathcal{O}/J_f$ is normal in the case of $2 < \dim M_2 < \dim M_1$ and if J_f is reduced in the case of* $\dim M_2 = 2 < \dim M_1$.

Proof. Let π and τ be C^ω diffeomorphisms of M_1 and M_2, respectively, such that $f \circ \pi = \tau \circ g$. Since a C^ω map between compact Nash manifolds can be approximated by a Nash map, we can assume π, τ and g are sufficiently close to id, id and f, respectively. As in the proof of 1.7 we can suppose

$$\Sigma_f = \Sigma_g, \qquad \Sigma_{f^\mathbf{C}} = \Sigma_{g^\mathbf{C}}, \qquad f(\Sigma_f) = g(\Sigma_g), \qquad f^\mathbf{C}(\Sigma_{f^\mathbf{C}}) = g^\mathbf{C}(\Sigma_{g^\mathbf{C}}).$$

Set $m_i = \dim M_i$, $i = 1, 2$, and let $M_i \subset \mathbf{R}^{n_i}$.

Case of $2 < m_2 < m_1$. We reduce the problem to the case where $f^\mathbf{C} = g^\mathbf{C}$ on $\Sigma_{f^\mathbf{C}}$, which is similar to the proof of 1.7. We can assume there exists a complex Nash set germ $S \subset \Sigma_{f^\mathbf{C}}$ at M_1 of dimension $< m_2 - 1$, defined by real polynomial functions, such that $\mathcal{O}^\mathbf{C}/J_{f^\mathbf{C}}$ and $\mathcal{O}^\mathbf{C}/J_{g^\mathbf{C}}$ are regular on $\Sigma_{f^\mathbf{C}} - S$,

$$f^\mathbf{C}(S) = g^\mathbf{C}(S), \qquad \Sigma_{f^\mathbf{C}} \cap f^{\mathbf{C}-1}(f^\mathbf{C}(S)) = \Sigma_{f^\mathbf{C}} \cap g^{\mathbf{C}-1}(f^\mathbf{C}(S)) = S,$$

and $f^\mathbf{C}|_{\Sigma_{f^\mathbf{C}}-S}$ and $g^\mathbf{C}|_{\Sigma_{f^\mathbf{C}}-S}$ are complex Nash coverings to $f^\mathbf{C}(\Sigma_{f^\mathbf{C}} - S)$. Then there exists a unique complex Nash diffeomorphism germ ρ of $\Sigma_{f^\mathbf{C}} - S$ close to id in the C^0 topology such that $f^\mathbf{C} \circ \rho = g^\mathbf{C}$ on $\Sigma_{f^\mathbf{C}} - S$.

Since $\mathcal{O}/J_f$ and hence $\mathcal{O}^\mathbf{C}/J_{f^\mathbf{C}}$ are normal, we can extend ρ to a complex C^ω map germ $P\colon M_1^\mathbf{C} \to \mathbf{C}^{n_1}$. We can choose P so that
(1) $P|_{M_1}$ is close to id,
(2) $P(M_1) = M_1$, and
(3) P is semialgebraic for the following reason.

Assume such a P. If we replace g with $g \circ (P|_{M_1})^{-1}$ then the required property $f^\mathbf{C} = g^\mathbf{C}$ on $\Sigma_{f^\mathbf{C}}$ is satisfied.

***Proof of* (1).** We have

$$\pi^\mathbf{C}(\Sigma_{f^\mathbf{C}}) = \Sigma_{f^\mathbf{C}}, \qquad \tau^\mathbf{C}(f^\mathbf{C}(\Sigma_{f^\mathbf{C}})) = f^\mathbf{C}(\Sigma_{f^\mathbf{C}}),$$

and we can assume

$$\pi^\mathbf{C}(S) = S, \qquad \tau^\mathbf{C}(f^\mathbf{C}(S)) = f^\mathbf{C}(S).$$

Define a complex C^ω diffeomorphism germ χ of $\Sigma_{f^\mathbf{C}} - S$ by $f^\mathbf{C} \circ \chi = \tau^\mathbf{C} \circ f^\mathbf{C}$. We will construct its extension $X\colon M_1^\mathbf{C} \to M_1^\mathbf{C}$ so that $X|_{M_1}$ is close to id. Then $\rho = \chi^{-1} \circ \pi^\mathbf{C}$, and $P = X^{-1} \circ \pi^\mathbf{C}$ fulfills the requirement.

We claim that there exists a C^ω isotopy τ_t, $t \in [0,1]$, of M_2 such that $\tau_0 = \text{id}$, $\tau_1 = \tau$, and

$$\tau_t^\mathbf{C}(f^\mathbf{C}(\Sigma_{f^\mathbf{C}})) = f^\mathbf{C}(\Sigma_{f^\mathbf{C}}),\ \ \tau_t^\mathbf{C}(f^\mathbf{C}(S)) = f^\mathbf{C}(S) \quad \text{for } t \in [0,1]. \tag{4}$$

As shown in the proof of 1.7, (4) is equivalent to that $z = \tau_t(y)$ is a solution of an equation:

$$F_i(y, z) = 0, \quad i = 1, \dots, l, \tag{5}$$

where F_i are Nash functions on M_2^2. Recall the proof of Theorem 0.0 in [C-R-S]. There are a compact Nash manifold M_3 in some $\mathbf{R}^{n_3}$, a Nash map $h\colon M_3 \to M_2$ and a Nash submersion $h'\colon M_3 \to M_2$ such that a C^ω map $z = \tau_t(y)$ satisfies (5) if and only if there exists a C^ω map $\zeta\colon M_2 \to M_3$ such that $\tau_t = h \circ \zeta$ and $h' \circ \zeta = \mathrm{id}$. Let $\zeta_t\colon M_2 \to \mathbf{R}^{n_3}$, $t \in [0,1]$, be a C^ω homotopy such that $h \circ \zeta_0 = \tau$, ζ_1 is of class Nash, and each ζ_t is close to ζ_0, and let q_3 be the orthogonal projection of a tubular neighborhood of M_3 in $\mathbf{R}^{n_3}$. Set $\tau_t' = h \circ q_3 \circ \zeta_t$. It is easy to modify ζ_t so that $h' \circ q_3 \circ \zeta_t = \mathrm{id}$. Thus we obtain a C^ω isotopy τ_t' of M_2 such that $\tau_0' = \tau$, τ_1' is of class Nash and τ_t' satisfies (4). Replace g and τ with $\tau_1' \circ g$ and $\tau \circ \tau_1'^{-1}$, respectively. Then $\tau_t = \tau \circ \tau_t'^{-1}$ is what we wanted.

Set

$$\tilde{M}_i = M_i \times [0,1], \quad i = 1,2, \qquad \tilde{f} = f \times \mathrm{id}\colon \tilde{M}_1 \to \tilde{M}_2.$$

Let $p_i\colon T\tilde{M}_i \to \tilde{M}_i$, $i = 1,2$, denote the tangent bundles. We call a tangent vector and a vector field on $\tilde{M}_i$ *canonical* if their $T[0,1]$-factors are $\partial/\partial t$, where t is the variable of $[0,1]$. Define θ_f to be C^∞ maps $\psi\colon \tilde{M}_1 \to T\tilde{M}_2$ such that $p_2 \circ \psi = \tilde{f}$ and the $T[0,1]$-factor of $\psi(x,t)$ is 0 for each $(x,t) \in \tilde{M}_1$, and let θ_i, $i = 1,2$, denote the space of canonical C^∞ vector fields on $\tilde{M}_i$. These vector spaces are Fréchet spaces. Define continuous homomorphisms

$$tf\colon \theta_1 \longrightarrow \theta_f + \frac{\partial}{\partial t} \quad \text{by } tf(\xi) = T\tilde{f} \circ \xi,$$
$$wf\colon \theta_2 \longrightarrow \theta_f + \frac{\partial}{\partial t} \quad \text{by } wf(\eta) = \eta \circ \tilde{f}.$$

Let $\phi_1, \dots, \phi_l$ be generators of $H^0(M_1, J_f)$. (By the fundamental theorem A, $\phi_{1,x}, \dots, \phi_{l,x}$ are generators of $J_{f,x}$ for each $x \in M_1$.) Let $\tilde{\phi}_i$ denote the C^ω function on $\tilde{M}_1$ naturally induced by ϕ_i. Set

$$\Theta_f = \{(\xi, \eta, \alpha_1, \dots, \alpha_l) \in \theta_1 \times \theta_2 \times \theta_f^l : tf(\xi) + \sum_{i=1}^{l} \tilde{\phi}_i \alpha_i = wf(\eta)\}.$$

Then for construction of X it suffices to show that the image of Θ_f under the projection $\theta_1 \times \theta_2 \times \theta_f^l \to \theta_2$ is closed in θ_2 for the following reason.

Assume the image is closed. Then by the open mapping theorem for Fréchet spaces, the map : $\Theta_f \to$ the image is open. On the other hand, we have $\eta \in \theta_2$ whose integral curves equal $\{(\tau_t(y), t) : t \in [0,1]\}$, $y \in M_2$. Here η can be arbitrarily close to $\partial/\partial t$ by the above construction of τ_t, and is an element of the image because there exists a C^ω isotopy $\lambda_t^{\mathbf{C}}$ of $\Sigma_{f^{\mathbf{C}}}$ real valued on Σ_f such that $f^{\mathbf{C}} \circ \lambda_t^{\mathbf{C}} = \tau_t^{\mathbf{C}} \circ f^{\mathbf{C}}$ on $\Sigma_{f^{\mathbf{C}}}$. Hence there exists an element $(\xi, \alpha_1, \dots, \alpha_l)$ of $\theta_1 \times \theta_f^l$ close to $(\partial/\partial t, 0, \dots, 0)$ such that $(\xi, \eta, \alpha_1, \dots, \alpha_l) \in \Theta_f$. If we can C^ω-smooth $\xi, \alpha, \dots, \alpha_l$, then integrating ξ, we obtain a C^ω diffeomorphism π' of M_1 close to id such that $f^{\mathbf{C}} \circ \pi'^{\mathbf{C}} = \tau^{\mathbf{C}} \circ f^{\mathbf{C}}$ on $\Sigma_{f^{\mathbf{C}}}$. Hence $\pi'^{\mathbf{C}}$ is the required extension X.

Fix η as above, and write

$$\tilde{\Phi} = (\tilde{\phi}_1, \dots, \tilde{\phi}_l), \qquad \xi = (\xi_1, \dots, \xi_{n_1}) \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_{n_1}} \end{pmatrix} + \frac{\partial}{\partial t} = \Xi \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_{n_1}} \end{pmatrix} + \frac{\partial}{\partial t},$$

$$\eta = (\eta_1, \dots, \eta_{n_2}) \begin{pmatrix} \frac{\partial}{\partial y_1} \\ \vdots \\ \frac{\partial}{\partial y_{n_2}} \end{pmatrix} + \frac{\partial}{\partial t} = H \begin{pmatrix} \frac{\partial}{\partial y_1} \\ \vdots \\ \frac{\partial}{\partial y_{n_2}} \end{pmatrix} + \frac{\partial}{\partial t}, \qquad \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_l \end{pmatrix} = A \begin{pmatrix} \frac{\partial}{\partial y_1} \\ \vdots \\ \frac{\partial}{\partial y_{n_2}} \end{pmatrix},$$

where the elements of Ξ and A are C^∞ functions on $\tilde{M}_1$, and the ones of H are C^ω functions on $\tilde{M}_2$. Then

$$\Xi \widetilde{\frac{D(f)}{D(x)}} + \tilde{\Phi} A = H \circ \tilde{f}, \qquad \Xi \widetilde{\frac{D(\mu)}{D(x)}} = 0, \tag{6}$$

and reversely a solution (Ξ', A') of (6) together with η induces an element of Θ_f, where $\widetilde{\ }$ indicates extension to $\tilde{M}_i$ such as $\tilde{f}$, $\frac{D(f)}{D(x)}$ means the restriction to M_1 of the Jacobian matrix of a C^ω extension of f to : $\mathbf{R}^{n_1} \to \mathbf{R}^{n_2}$, $\mu = (\mu_1, \dots, \mu_l)$, $\mu_1, \dots, \mu_l$ are generators of $H^0(\mathbf{R}^{n_1}, \mathcal{I}(M_1))$, and $\frac{D(\mu)}{D(x)}$ means the restriction to M_1 of the Jacobian matrix of μ. We regard (Ξ, A) as a solution of (6). We want to find a C^ω solution. Let $\mathcal{B}$ and $\psi\colon \mathcal{B} \to (\mathcal{O}^{\tilde{M}_1})^{n_2+l}$ denote the sheaf of free $\mathcal{O}^{\tilde{M}_1}$-modules and a $\mathcal{O}$-homomorphism defined by

$$\mathcal{B}_{x,t} = \{(B, C)\colon B = (b_1, \dots, b_{n_1}),\ C = \begin{pmatrix} c_{1,1} & \dots & c_{1,n_2} \\ \vdots & & \vdots \\ c_{l,1} & \dots & c_{l,n_2} \end{pmatrix},$$
$$b_i \text{ and } c_{i,j} \text{ are elements of } \mathcal{O}_{x,t}\},$$

$$\psi(B, C) = (B \widetilde{\frac{D(f)}{D(x)}}_{x,t} + \tilde{\Phi}_{x,t} C, B \widetilde{\frac{D(\mu)}{D(x)}}_{x,t}).$$

Then by the fundamental theorem B, the following sequence is exact :

$$0 \longrightarrow H^0(\tilde{M}_1, \operatorname{Ker}\psi) \longrightarrow H^0(\tilde{M}_1, \mathcal{B}) \longrightarrow H^0(\tilde{M}_1, \operatorname{Im}\psi) \longrightarrow 0,$$

and by Artin Approximation Theorem, for each $(x,t) \in \tilde{M}_1$, $(H \circ f)_{x,t}$ is an element of $H^0(\tilde{M}_1, \operatorname{Im}\psi)_{x,t}$. Hence there exists a C^ω solution (Ξ', A') of (6). It suffices to approximate $(\Xi - \Xi', A - A')$ by an element of $H^0(\tilde{M}_1, \operatorname{Ker}\psi)$. By Artin Approximation Theorem, for each $(x,t) \in \tilde{M}_1$, the Taylor expansion of $(\Xi - \Xi', A - A')$ at (x,t) is an element of the completion of $H^0(\tilde{M}_1, \operatorname{Ker}\psi)_{x,t}$. Hence by Theorem VI.1.1′ in [Mal],

$$(\Xi - \Xi', A - A') \in C^\infty(\tilde{M}_1) H^0(\tilde{M}_1, \operatorname{Ker}\psi).$$

Let $h_1, \dots, h_k$ be generators of $H^0(\tilde{M}_1, \operatorname{Ker}\psi)$, and let $\gamma_1, \dots, \gamma_k$ be C^∞ functions on $\tilde{M}_1$ such that

$$(\Xi - \Xi', A - A') = \sum_{i=1}^{k} \gamma_i h_i.$$

Let γ_i' denote C^ω approximations of γ_i. Then $\sum_{i=1}^k \gamma_i' h_i$ is an approximation of $(\Xi - \Xi', A - A')$ and an element of $H^0(\tilde{M}_1, \operatorname{Ker}\psi)$. Thus we C^ω-smooth $\xi, \alpha, \dots, \alpha_l$.

It remains to show that the image of Θ_f in θ_2 is closed. Do not fix now η, i.e. H. Let $\mu_1', \dots, \mu_{l'}'$ be generators of $H^0(\mathbf{R}^{n_2}, \mathcal{I}(M_2))$, and let $\frac{D(\mu')}{D(y)}$ mean the restriction to M_2 of the Jacobian matrix of $\mu' = (\mu_1', \dots, \mu_{l'}')$. Set

(6)′ $$H \widetilde{\frac{D(\mu')}{D(y)}} = 0.$$

Regard (Ξ, H, A) as a solution of (6) and (6)′, and let $D \subset C^\infty(\tilde{M}_1)^{n_1} \times C^\infty(\tilde{M}_2)^{n_2} \times C^\infty(\tilde{M}_1)^{ln_2}$ denote the solutions. Then what we prove is that the image of D under the projection $\nu_2 \colon C^\infty(\tilde{M}_1)^{n_1} \times C^\infty(\tilde{M}_2)^{n_2} \times C^\infty(\tilde{M}_1)^{ln_2} \to C^\infty(\tilde{M}_2)^{n_2}$ is closed. If the image of $D' = \{$solutions of (6)$\}$ is closed, then $\nu_2(D)$ is closed because $\nu_2(D) = \nu_2(D') \cap \{$solutions of (6)′$\}$. Hence we can forget (6)′. Moreover, it suffices to prove that the image of $C^\infty(\tilde{M}_1)^{n_1} \times C^\infty(\tilde{M}_1)^{ln_2}$ under the map:

$$C^\infty(\tilde{M}_1)^{n_1} \times C^\infty(\tilde{M}_1)^{ln_2} \ni (\Xi, A) \longrightarrow \Xi \widetilde{\frac{D(f)}{D(x)}} + \tilde{\Phi} A \in C^\infty(\tilde{M}_1)^{n_2}$$

is closed because D' is the inverse image of this image under the induced map

$$(f \times \mathrm{id})^{*n_2} \colon C^\infty(\tilde{M}_2)^{n_2} \longrightarrow C^\infty(\tilde{M}_1)^{n_2}.$$

The image is closed if the following statement is true.

An ideal of $C^\infty(M_1)$ generated by a finite number of C^ω functions is closed in $C^\infty(M_1)$.

This follows from Theorem VI.1.1′ in [Mal] and a theorem of Krull which states that any ideal of a power series ring is closed, which completes the proof of (1).

(2) is clear by the above proof of (1). Using the global approximation theorem as in the proof of 1.7, we can modify P to be of class Nash (3). But then (2) may fails. If it fails, it suffices to replace P with $q_1^{\mathbf{C}} \circ P$, where q_1 denotes the orthogonal projection of a tubular neighborhood of M_1 in $\mathbf{R}^{n_1}$.

To complete the proof in the case of $2 < m_2 < m_1$, we need only show that f and g are C^ω $\mathcal{R}$ equivalent because of 3.1. Define a C^ω map $F \colon \tilde{M}_1 \to M_2$ by

$$F(x,t) = q_2(g(x)t + f(x)(1-t)) \quad \text{for } (x,t) \in \tilde{M}_1,$$

where q_2 is the orthogonal projection of a tubular neighborhood of M_2 in $\mathbf{R}^{n_2}$. Then it suffices to find a canonical C^ω vector field $\xi = \sum_{i=1}^{n_1} \xi_i \partial/\partial x_i + \partial/\partial t$ on $\tilde{M}_1$ such that $\xi F = 0$. Moreover, by the same reason as above, a vector field of class

C^∞ is sufficient. Hence the problem becomes local, and we assume $M_i = \mathbf{R}^{n_i}$. Set $\Xi = (\xi_1, \dots, \xi_{n_1})$. Then what we do is to find a C^ω solution Ξ of

$$\tilde{f} - \tilde{g} = \Xi\left(\widetilde{\frac{D(g)}{D(x)}}t + \widetilde{\frac{D(f)}{D(x)}}(1-t)\right). \tag{7}$$

Set $f = (f_1, \dots, f_{n_2})$ and $g = (g_1, \dots, g_{n_2})$. Since $f^{\mathbf{C}} = g^{\mathbf{C}}$ on $\Sigma_{f^{\mathbf{C}}}$,

$$f_j - g_j \in H^0(\mathbf{R}^{n_1}, J_f), \quad j = 1, \dots, n_2.$$

As above, let $\phi_1 \dots, \phi_l$ be generators of $H^0(\mathbf{R}^{n_1}, J_f)$. Let $\alpha_{i,j}$ be C^∞ functions on $\mathbf{R}^{n_1}$ such that

$$f_j - g_j = \sum_{i=1}^{l} \alpha_{i,j}\phi_i, \quad j = 1, \dots, n_2. \tag{8}$$

Here we can choose sufficiently small $\alpha_{i,j}$ as above. Set

$$\Phi = (\phi_1, \dots, \phi_l), \qquad A = \begin{pmatrix} \alpha_{1,1} & \dots & \alpha_{1,n_2} \\ \vdots & & \vdots \\ \alpha_{l,1} & \dots & \alpha_{l,n_2} \end{pmatrix}.$$

Then (8) becomes

$$f - g = \Phi A. \tag{8$'$}$$

Hence the problem is

$$\tilde{\Phi}\tilde{A} = \Xi\left(\widetilde{\frac{D(g)}{D(x)}}t + \widetilde{\frac{D(f)}{D(x)}}(1-t)\right). \tag{7$'$}$$

In the same way as in the proof of 1.7, we see that each $\partial(f_j - g_j)/\partial x_i$ is a linear combination of $\phi_1, \dots, \phi_l$ with small coefficients in $C^\infty(\mathbf{R}^{n_1})$, and then $J_{\tilde{f}}$ is generated by the minors of $\widetilde{\frac{D(g)}{D(x)}}t + \widetilde{\frac{D(f)}{D(x)}}(1-t)$ of degree n_2. Hence we can assume $\tilde{\phi}_1, \dots, \tilde{\phi}_l$ are the minors. Then we can solve (7)$'$ easily as in the proof of 1.7. Thus we prove the first case.

Case of $m_2 = 2 < m_1$. We reduce the problem to the case $f^{\mathbf{C}} = g^{\mathbf{C}}$ on $\Sigma_{f^{\mathbf{C}}}$ as in the proof of 1.7 and the above first case. Define a complex C^ω diffeomorphism germ χ of $\Sigma_{f^{\mathbf{C}}} - S$ so that

$$f^{\mathbf{C}} \circ \chi = \tau^{\mathbf{C}} \circ f^{\mathbf{C}} \quad \text{on } \Sigma_{f^{\mathbf{C}}} - S,$$

where S denotes a finite point set, as in the first case. Then by the above proof it suffices to show that χ is extensible to $M_1^{\mathbf{C}}$. Moreover, the proof of 1.7 of extendability works if for each $y \in f(S)$, τ_y is sufficiently close to id (in the Krull topology). Hence we need only construct a Nash diffeomorphism τ' of M_2 such that

$$(9) \quad \tau' \circ f(S) = f(S), \qquad \tau'(f(\Sigma_f)) = f(\Sigma_f), \qquad \tau'^{\mathbf{C}}(f^{\mathbf{C}}(\Sigma_{f^{\mathbf{C}}})) = f^{\mathbf{C}}(\Sigma_{f^{\mathbf{C}}}),$$

and for each $y \in f(S)$, τ'_y is sufficiently close to τ_y because we can replace g and τ with $\tau' \circ g$ and $\tau \circ \tau'^{-1}$, respectively. As shown already, (9) holds true if and only if there exists a C^ω map $\zeta'\colon M_2 \to M_3$ such that $\tau' = h \circ \zeta'$ and $h' \circ \zeta' = \mathrm{id}$, where $M_3 \subset \mathbf{R}^{n_3}$ is a certain compact Nash manifold and $h, h'\colon M_3 \to M_2$ are certain Nash map and submersion. Hence what to prove is the following assertion.

For a C^ω map $\zeta\colon M_2 \to M_3$ there exists a Nash map $\zeta'\colon M_2 \to M_3$ close to ζ such that for each $y \in f(S)$, ζ'_y is sufficiently close to ζ_y.

Clearly we can suppose $M_2 = \mathbf{R}^{n_2}$ and ζ is of class C^∞. (Then we replace the germs with the Taylor expansions in the assertion, and we apply the compact-open C^∞ topology.) Using a tubular neighborhood of M_3 in the ambient Euclidean space, we can assume also $M_3 = \mathbf{R}$, i.e., ζ and ζ' are functions. Moreover, we do not need to require ζ' to be close to ζ because if ζ is not so then we can modify ζ so that this requirement is satisfied by the arguments in the first case. Let $f(S) = \{a_1, \dots, a_k\}$. The last reduction is to the case where $\zeta_{a_i} = 0$, $i = 2, \dots, k$, which is clearly possible by a C^∞ partition of unity. Now we construct ζ'. Let ζ'' be a polynomial function on $\mathbf{R}^{n_2}$ such that $\zeta''(a_1) \neq 0$ and $\zeta''(a_i) = 0$, $i = 2, \dots, k$. Let n be a large integer, and let $\zeta^{(3)}$ be a polynomial function on $\mathbf{R}^{n_2}$ such that the Taylor expansion $T_{a_1}\zeta^{(3)}$ is close to $T_{a_1}\zeta/(\zeta'')^n$. Then $\zeta' = (\zeta'')^n\zeta^{(3)}$ fulfills the requirements.

The rest of the proof is the same as in the first case.

Case of $m_2 \geq m_1$. We can prove this case as in the proof of 1.7. □

A C^∞ map is called C^∞ *$\mathcal{R}$-$\mathcal{L}$ stable* if it is C^∞ $\mathcal{R}$-$\mathcal{L}$ equivalent to its small C^∞ perturbation. In the same way we define C^ω and Nash *$\mathcal{R}$-$\mathcal{L}$ stable* maps.

A Nash $\mathcal{R}$-$\mathcal{L}$ stable proper Nash map is C^ω $\mathcal{R}$-$\mathcal{L}$ stable and a C^ω $\mathcal{R}$-$\mathcal{L}$ stable proper C^ω map is C^∞ $\mathcal{R}$-$\mathcal{L}$ stable because they are infinitesimally stable (see [Ma$_1$] for the definition). Conversely, a C^ω $\mathcal{R}$-$\mathcal{L}$ stable proper Nash map is Nash $\mathcal{R}$-$\mathcal{L}$ stable by 3.7.

Fact 3.8. *The map $f\colon \mathbf{R}^2 \ni (x_1, x_2) \to (x_1, x_1x_2, x_2^2) \in \mathbf{R}^3$ is proper C^∞ $\mathcal{R}$-$\mathcal{L}$ stable but not C^ω $\mathcal{R}$-$\mathcal{L}$ stable nor Nash $\mathcal{R}$-$\mathcal{L}$ stable. We can modify the map to be a Nash map between compact Nash manifolds.*

Proof. It is easy to see that f is infinitesimally stable. Hence f is C^∞ $\mathcal{R}$-$\mathcal{L}$ stable by [Ma$_1$].

We will construct a C^ω perturbation g of f which is not C^ω $\mathcal{R}$-$\mathcal{L}$ equivalent to f. Note

$$\operatorname{Im} f = \{(y_1, y_2, y_3) \in \mathbf{R}^3\colon y_1^2 y_3 = y_2^2,\ y_3 \geq 0\},$$
$$\operatorname{Im} f \cup \{y_1 = y_2 = 0,\ y_3 < 0\} = \text{the Whitney umbrella},$$
$$\operatorname{Im} f \cap \{y_1 = y_2 = 0,\ y_3 < 0\} = \emptyset.$$

Let $h\colon \mathbf{R}^2 \to \mathbf{R}^2$ be a C^∞ map such that $h = \mathrm{id}$ on $\{z_2 \geq 0\}$, $h|_{\{z_1=0\}}$ is not an imbedding but a C^∞ immersion and the curve $h(\{z_1 = 0,\ z_2 < 0\})$ intersects transversally. Let $\tilde{h}$ denote a C^ω approximation of h, and set $g = (f_1, \tilde{h}(f_2, f_3))$, where $f = (f_1, f_2, f_3)$. (Note that $\tilde{h}(\{z_1 = 0,\ z_2 < 0\})$ is not a simple curve.) Then g is close to f because f is proper, and not C^ω $\mathcal{R}$-$\mathcal{L}$ equivalent to f for the following reason.

Assume f and g are C^ω $\mathcal{R}$-$\mathcal{L}$ equivalent. Then there exists a C^ω diffeomorphism τ of $\mathbf{R}^3$ such that $\tau(\operatorname{Im} f) = \operatorname{Im} g$. It follows that τ carries the analytic closure of $\operatorname{Im} f$ to it of $\operatorname{Im} g$. But the former analytic closure is the Whitney umbrella, and the latter is not homeomorphic to the Whitney umbrella because the latter$-\operatorname{Im} g = 0 \times \tilde{h}(\{z_1 = 0,\ z_2 < 0\})$, which is not a simple curve.

We can construct a Nash perturbation of f which is not Nash $\mathcal{R}$-$\mathcal{L}$ equivalent to f in the same way.

By the same reason, the following map between compact Nash manifolds is C^∞ $\mathcal{R}$-$\mathcal{L}$ stable, but not C^ω $\mathcal{R}$-$\mathcal{L}$ stable nor Nash $\mathcal{R}$-$\mathcal{L}$ stable:

$$\{x \in \mathbf{R}^3\colon |x| = 1\} \ni (x_1, x_2, x_3) \longrightarrow (x_1, x_1x_2, x_3, \sqrt{4 - x_1^2 - x_1^2x_2^2 - x_3^2}) \in \{y \in \mathbf{R}^4\colon |y| = 2\}.$$

□

Fact 3.9. By the above proof and 4.3 it may be natural to conjecture the following assertion

Let $f, g\colon M_1 \to M_2$ be C^∞ $\mathcal{L}$ equivalent Nash maps between compact Nash manifolds. Then there exist semialgebraic open neighborhoods U of $\operatorname{Im} f$ and V of $\operatorname{Im} g$ such that $f\colon M_1 \to U$ and $g\colon M_1 \to V$ are Nash $\mathcal{L}$ equivalent.

But we can construct a counter-example by modifying the latter example in the above proof. We omit the construction.

Fact 3.10. *There exist two Nash maps between compact Nash manifolds which are C^0 $\mathcal{L}$ equivalent but not semialgebraically $\mathcal{R}$-$\mathcal{L}$ equivalent.*

Proof. Let N be a C^∞ manifold homeomorphic to $S^3 \times S^3$ such that N and $S^3 \times S^3$ have distinct PL structures, whose existence follows from [K-S]. Set

$$M_1 = S^5 \times S^1, \qquad M_2 = S^3 \times S^3 \sharp N,$$

where $\sharp$ indicates the connected sum. Give to N and M_2 Nash manifold structures. Here the part of connection of M_2 is C^∞ diffeomorphic to $S^5 \times [-1,1]$. By uniqueness of a Nash structure of a compact C^∞ manifold possibly with boundary [S$_2$], the part of connection is Nash diffeomorphic to $S^5 \times [-1,1]$. Hence we identify the part with $S^5 \times [-1,1]$. Let $f_2\colon S^1 \to [-1,1]$ be a Nash map such that f_2 and $-f_2$ can not be C^0 $\mathcal{R}$-$\mathcal{L}$ equivalent by any orientation preserving homeomorphism of $[-1,1]$. Set

$$f = \operatorname{id} \times f_2,\ g = \operatorname{id} \times (-f_2)\colon S^5 \times S^1 \longrightarrow S^5 \times [-1,1],$$

and regard them as Nash maps from M_1 to M_2.

Since M_2 is homeomorphic to $S^3 \times S^3 \sharp S^3 \times S^3$, f and g are C^0 $\mathcal{L}$ equivalent. On the other hand, they are not semialgebraically $\mathcal{R}$-$\mathcal{L}$ equivalent for the following reason. Assume they are so. Then there exists a semialgebraic homeomorphism from $M_2 \cap N$ to $M_2 \cap S^3 \times S^3$. The homeomorphism can be extended to : $N \to S^3 \times S^3$, i.e., N and $S^3 \times S^3$ are semialgebraically homeomorphic. It follows from Hauptvermutung Theorem III.1.4 in [S₃] that N and $S^3 \times S^3$ have the same PL structure, which is a contradiction. □

§4. Global functions

Fact 4.1 (Example II.7.13 in [S₃]). *There exist two polynomial functions on* $\mathbf{R}^8$ *which are* C^ω $\mathcal{R}$ *equivalent but not semialgebraically* $\mathcal{R}$-$\mathcal{L}$ *equivalent.*

Fact 4.2 (Corollary II.7.6 and Theorem II.7.7 in [S₃]). *Two* C^1 $\mathcal{R}$ *equivalent* C^ω *functions on a* C^ω *manifold are subanalytically* $\mathcal{R}$ *equivalent. Two* C^1 $\mathcal{R}$ *equivalent Nash functions on a compact Nash manifold are semialgebraically* $\mathcal{R}$ *equivalent. Two subanalytic* $\mathcal{R}$ *equivalent semialgebraic functions on a compact semialgebraic set are semialgebraically* $\mathcal{R}$ *equivalent.*

Fact 4.3. *If* C^ω *(Nash) functions* $f, g\colon M \to \mathbf{R}$ *are* C^∞ $\mathcal{L}$ *equivalent then there exist open interval neighborhoods* U *of* $\operatorname{Im} f$ *and* V *of* $\operatorname{Im} g$ *such that* $f\colon M \to U$ *and* $g\colon M \to V$ *are* C^ω *(Nash, respectively)* $\mathcal{L}$ *equivalent.*

Proof. A homeomorphism $\tau\colon \operatorname{Im} g \to \operatorname{Im} f$ such that $f = \tau \circ g$ is unique. On the other hand, by 1.3 and 1.6 we can choose τ of class C^ω (Nash) locally at each point of $\operatorname{Im} g$. Hence τ is of class C^ω (Nash). □

Fact 4.4. *Two* C^ω $\mathcal{R}$-$\mathcal{L}$ *equivalent Nash functions are Nash* $\mathcal{R}$-$\mathcal{L}$ *equivalent if the domain is compact.*

Proof. Let f and g be C^ω $\mathcal{R}$-$\mathcal{L}$ equivalent Nash functions on a compact Nash manifold M. Let π and τ be C^ω diffeomorphisms of M and $\mathbf{R}$, respectively, such that $f \circ \pi = \tau \circ g$. Assume τ is orientation preserving. (The other case is proved similarly.) By 3.3 it suffices to find a Nash diffeomorphism τ_1 of $\mathbf{R}$ such that $\tau \circ g$ and $\tau_1 \circ g$ are C^ω $\mathcal{R}$ equivalent (i.e., g and $\tau_1^{-1} \circ \tau \circ g$ are so). Let S denote the critical value set of g, and let $\phi_1, \dots, \phi_k$ be generators of $H^0(M, J_g)$. If we have a Nash diffeomorphism τ_1 of $\mathbf{R}$ such that the Taylor expansion of $\tau - \tau_1$ at each point of S is close to 0, by the proofs of 2.4 and 3.5 $g - \tau_2 \circ g$ is a linear combination of $\phi_i \phi_j$ with small Nash function coefficients, where $\tau_2 = \tau_1^{-1} \circ \tau$, and by the proof of 3.7 g and $\tau_2 \circ g$ are Nash $\mathcal{R}$ equivalent.

We construct τ_1 as follows. Let s_1 and s_2 be the minimum and the maximum of S, respectively. Let r be a sufficiently large integer. We have a C^r Nash diffeomorphism τ_3 of $\mathbf{R}$ such that the derivatives of $\tau_3 - \tau$ of order $\le r$ vanish on S and

$$\tau_3(x) = \begin{cases} x + \tau(s_1) - s_1 & \text{on } \;]-\infty, s_1 - 1] \\ x + \tau(s_2) - s_2 & \text{on } \;[s_2 + 1, \infty[. \end{cases}$$

Fixing the derivatives of τ_3 of order $\le r$ at S we can approximate τ_3 by a Nash function τ_1 (Theorems II.4.1 and II.5.2 in [S₂]). □

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