数理解析研究所講究録
1007 巻 1997 年 11-34

Weights of $\bar{\chi}^2$ distribution for smooth or piecewise smooth cone alternatives†

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Abstract

We study the problem of testing a simple null hypothesis on multivariate normal mean vector against smooth or piecewise smooth cone alternatives. We show that the mixture weights of the $\bar{\chi}^2$ distribution of the likelihood ratio test can be characterized as mixed volumes of the cone and its dual. The weights can be calculated by integration involving the second fundamental form on the boundary of the cone. We illustrate our technique by spherical cone, cone of non-negative definite matrices, and two other cones which were not treated before.

Key words: multivariate one-sided alternative, one-sided simultaneous confidence region, mixed volume, second fundamental form, volume element, internal angle, external angle, Gauss-Bonnet theorem, Shapiro's conjecture.

1 Introduction

We first state our problem and then give outline of the paper. In Section 1.2 we prepare basic material from convex analysis.

1.1 The problem

We consider the problem of testing a simple null hypothesis on multivariate normal mean vector against a convex cone alternative in the following canonical form. Let $Z \in R^p$ be distributed according to the p-dimensional multivariate normal distribution with mean vector μ and the identity covariance matrix $N_p(\mu, I_p)$. Let K be a closed convex cone with non-empty interior in R^p. Our testing problem in the canonical form is

$$H_0 : \mu = 0 \quad \text{vs.} \quad H_1 : \mu \in K. \tag{1}$$

The main objective of this paper is to study the null distribution of the likelihood ratio statistic for K with smooth or piecewise smooth boundary using techniques of convex analysis and differential geometry.

In addition to (1) consider a complementary testing problem

$$H_1 : \mu \in K \quad \text{vs.} \quad H_2 : \mu \in R^p. \tag{2}$$

† Currently submitted for publication.

In describing the complementary testing problem we need to use the dual cone K^* of K :

$$K^* = \{y \mid \langle y, x\rangle \leq 0, \ \forall x \in K\},$$

where $\langle \ , \ \rangle$ denotes the inner product.

For $x \in R^p$ let x_K denote the orthogonal projection of x onto K and x_{K^*} denote the orthogonal projection of x onto K^* . Then the likelihood ratio test of (1) is equivalent to rejecting H_0 when

$$\bar{\chi}^2_{01} = \|Z_K\|^2 \tag{3}$$

is large and the likelihood ratio test of (2) is equivalent to rejecting H_1 when

$$\bar{\chi}^2_{12} = \|Z_{K^*}\|^2 \tag{4}$$

is large. We consider the joint distribution of $\bar{\chi}^2_{01}$ and $\bar{\chi}^2_{12}$ under H_0 .

The statistics $\bar{\chi}^2_{01}$ and $\bar{\chi}^2_{12}$ in (3) and (4) are called *chi-bar-square statistics*, and known to have a finite mixture of the chi-square distributions when H_0 is true. In this paper we call the mixing probabilities the *weights*. Generally, it is not easy to derive the explicit expression of the weights. Here we list some examples of cones whose weights are known explicitly or can be easily calculated numerically. The following are such examples of practical importance:

$$\begin{aligned} K_1 &= \{\mu \mid \mu_1 \leq \cdots \leq \mu_p\} \\ K_2 &= \{\mu \mid \mu_1 \leq \mu_j, \ j = 2, \ldots, p\} \\ K_3 &= \left\{\mu \mid \frac{\mu_1 + \cdots + \mu_j}{j} \leq \frac{\mu_{j+1} + \cdots + \mu_p}{p - j}, \quad j = 1, \ldots, p-1\right\}. \end{aligned}$$

K_1 and K_2 are defined by the partial orders referred to as *simple order* and *simple tree order*, respectively. For these three cones the null hypothesis is usually $\mu_1 = \cdots = \mu_p$, the hypothesis of homogeneity. However, the testing problems can be easily reduced to the canonical form in (1). The corresponding weights are given by recurrence formulas. In particular, the weights for K_1 are known to be expressed in terms of the Stirling number of the first kind. The weights for K_3 are obtained as the reverse sequence of those of K_1 . See Section 3 of Barlow et al. (1972), Section 2 of Robertson et al. (1988), and their references for the weights of these cones as well as the related statistical inference. See also Bohrer and Francis (1972a, b) and Wynn (1975), in which $\bar{\chi}^2$ distributions are treated in the context of constructing the one-sided Scheffé-type simultaneous confidence regions.

The cones K_1 , K_2 and K_3 above are polyhedral, i.e., the cones defined by a finite number of linear constraints. The following are examples of non-polyhedral cones whose weights are known:

$$\begin{aligned} K_4 &= \{\, \mu \mid \mu_1 \geq \|\mu\| \cos\psi \,\} \\ K_5 &= \{\, M \ : \ p \times p \text{ symmetric} \mid M \text{ is non-negative definite} \,\}. \end{aligned}$$

K_4 is the spherical cone which is smooth in the sense of Section 2.2 with no singularities except for the origin. K_5 is a piecewise smooth cone in the sense of Section 2.3. In Section 2.4 we show that the singularities of K_5 exhibit a beautiful recurrence structure. The weights for K_4 and K_5 were given by Pincus (1975) and Kuriki (1993), respectively.

For the polyhedral cone, the geometrical meaning of the weights is clear, since the weights can be expressed in terms of the internal and external angles. Compared with the polyhedral cone, the meaning of the weights for non-polyhedral cones is not clear. In this paper we clarify the geometrical meaning of the weights in the case that the boundary of the cone is smooth or piecewise smooth.

In Section 2 we prove our basic theorem relating the weights to the mixed volumes of K and its dual K^*. For smooth or piecewise smooth cones we express the mixed volumes as integrals involving the second fundamental form on the boundary of the cone. We apply our technique to the cones K_4 and K_5 and clarify the geometrical meanings. Also, we obtain the weights for two other cones which were not known.

Throughout this paper by "smooth" we mean class C^2.

1.2 Preparation from convex analysis

Here we summarize basic results from convex analysis. These results are taken from Webster (1994). Let $U = U_p$ be the closed unit ball and K be a convex set in R^p. For $\lambda \geq 0$, λ*-neighborhood of* K or *outer parallel set of* K *at distance* λ is defined as

$$(K)_\lambda = K + \lambda U,$$

where the addition is the vector sum. The *Hausdorff distance* between two non-empty compact convex sets K_1, K_2 is defined by

$$\rho(K_1, K_2) = \inf\{\lambda \geq 0 \mid K_1 \subset (K_2)_\lambda \text{ and } K_2 \subset (K_1)_\lambda\}.$$

Endowed with the Hausdorff distance, the set of compact convex sets becomes a complete metric space (Section 1.8 of Schneider (1993a)).

A *polytope* is the convex hull of a finite number of points. Any compact convex set can be approximated by polytopes.

Lemma 1.1 *(Theorem 3.1.6 of Webster (1994)) Let* K *be a non-empty compact convex set in* R^p *and let* $\epsilon > 0$. *Then there exist polytopes* P, Q *in* R^p *such that* $P \subset K \subset Q$ *and* $\rho(K, P) \leq \epsilon$, $\rho(K, Q) \leq \epsilon$.

We deal with convex cones which are not bounded. However uniform convergence on any bounded region is sufficient for us because we are concerned with the standard normal probabilities of the cones. Let K be a convex cone and denote $K_{(\lambda)} = K \cap \lambda U$. In view of the fact that polytopes are bounded polyhedral sets (Theorem 3.2.5 of Webster (1994)) the next lemma follows easily from Lemma 1.1.

Lemma 1.2 *Let* K *be a closed convex cone in* R^p *and let* $\lambda \geq 0$, $\epsilon > 0$. *Then there exist polyhedral cones* P, Q *in* R^p *such that* $P \subset K \subset Q$ *and* $\rho(K_{(\lambda)}, P_{(\lambda)}) \leq \epsilon$, $\rho(K_{(\lambda)}, Q_{(\lambda)}) \leq \epsilon$.

Now we introduce the notion of *mixed volumes* of two compact convex sets K_1, K_2 in R^p. Let $v_p(\cdot)$ denote the volume in R^p and consider $v_p(\nu K_1 + \lambda K_2)$ for $\nu, \lambda \geq 0$. Mixed volumes $v_{p-i,i}(K_1, K_2)$, $i = 0, \ldots, p$, are defined implicitly by the following lemma.

Lemma 1.3 *(Theorem 6.4.3 of Webster (1994))* $v_p(\nu K_1+\lambda K_2)$ *is a homogeneous polynomial of degree* p *in* ν *and* λ *with non-negative coefficients, i.e.,*

$$\begin{aligned} v_p(\nu K_1 + \lambda K_2) &= \nu^p v_{p,0}(K_1, K_2) + p\nu^{p-1}\lambda v_{p-1,1}(K_1, K_2) + \cdots + \lambda^p v_{0,p}(K_1, K_2) \\ &= \sum_{i=0}^{p} \binom{p}{i} \nu^{p-i}\lambda^i v_{p-i,i}(K_1, K_2), \end{aligned}$$

where $v_{p,0}(K_1, K_2) = v_p(K_1)$ *and* $v_{0,p}(K_1, K_2) = v_p(K_2)$.

In the particular case $\nu = 1$ and $K_2 = U$, i.e., when we are considering the outer parallel set of K_1 , $v_{p-i,i}(K_1, U)$ is called *quermassintegral* of K_1 and $\binom{p}{i} v_{i,p-i}(K_1, U)/\omega_{p-i}$ is called *intrinsic volume* of K_1 , where

$$\omega_q = \frac{\pi^{q/2}}{\Gamma(\frac{q}{2} + 1)} \tag{5}$$

is the volume of the unit ball U_q in R^q . It is also known that mixed volumes are continuous in K_1, K_2 with respect to Hausdorff metric (Theorem 6.4.7 of Webster (1994)).

2 Weights of $\bar{\chi}^2$ distribution as mixed volumes

In this section we first prove our basic theorem which states that the weights of the $\bar{\chi}^2$ distribution are the mixed volumes of the convex cone K and its dual cone K^* . Then we apply the basic theorem to the case of smooth convex cone using the fact that mixed volumes can be evaluated as integrals involving the second fundamental form on the boundary of K . Our result for the case of R^3 is very easily stated and connection to the classical Gauss-Bonnet theorem will be discussed. We illustrate our result for the smooth cone with the cases of elliptical cone in R^3 and spherical cone in R^p . Finally we discuss the case of "piecewise smooth" cone. Full treatment of piecewise smooth cone is needed to discuss the cone of non-negative definite matrices in Section 2.4.

2.1 Basic theorem

Here we prove our basic theorem stating that the weights of $\bar{\chi}^2$ distributions are mixed volumes. Since the concept of mixed volumes applies equally to polyhedral as well as smooth cones, our Theorem 2.1 covers both cases.

Theorem 2.1 *Consider the testing problems (1) and (2). Let* $K_{(1)} = K \cap U$ *and* $K^*_{(1)} = K^* \cap U$ *and let* $v_{p-i,i}(K_{(1)}, K^*_{(1)}), i = 0, \ldots, p,$ *be the mixed volumes of* $K_{(1)}$ *and* $K^*_{(1)}$. *Then under* H_0

$$P(\bar{\chi}^2_{01} \le a, \bar{\chi}^2_{12} \le b) = \sum_{i=0}^{p} \binom{p}{i} \frac{v_{p-i,i}(K_{(1)}, K^*_{(1)})}{\omega_i \omega_{p-i}} G_{p-i}(a) G_i(b), \tag{6}$$

where ω_q *is the volume of the unit ball in* R^q *given in (5) and* $G_q(t)$ *is the cumulative distribution function of chi-square distribution with* q *degrees of freedom.*

Proof. Let $P_n, n = 1, 2, \ldots,$ be a sequence of polyhedral cones converging to K in the sense of Lemma 1.2. For a given point $x \in R^p$ let x_{P_n} denote the orthogonal projection onto P_n. Then it is easy to show that x_{P_n} converges to x_K. At the same time the dual cone P_n^* converges to K^* and the projection $x_{P_n^*}$ converges to x_{K^*}. Since pointwise convergence implies convergence in law we have

$$\begin{aligned} P(\bar{\chi}^2_{01} \le a, \bar{\chi}^2_{12} \le b) &= P(\|Z_K\|^2 \le a, \|Z_{K^*}\|^2 \le b) \\ &= \lim_{n \to \infty} P(\|Z_{P_n}\|^2 \le a, \|Z_{P_n^*}\|^2 \le b). \end{aligned} \tag{7}$$

In view of the continuity of the mixed volumes, (7) shows that it is enough to prove our theorem for polyhedral cones.

From now on let K be a polyhedral cone. In this case the weights of $\bar{\chi}^2$ distribution is well understood in terms of the internal and external angles. Therefore we only need to verify that these angles can be expressed in terms of mixed volumes. Let F be a (closed) face of K and let $\beta(0, F)$ and $\gamma(F, K)$ be the internal angle and the external angle. See the Appendix for precise definition. Then it is well known that the joint distribution of $\bar{\chi}^2_{01}$ and $\bar{\chi}^2_{12}$ is a mixture of independent chi-square distributions

$$P(\bar{\chi}^2_{01} \le a, \bar{\chi}^2_{12} \le b) = \sum_{i=0}^{p} w_{p-i} G_{p-i}(a) G_i(b).$$

The mixture weight is expressed as

$$w_d = \sum_{\substack{F \in \mathcal{F}(K) \\ \dim F = d}} \beta(0, F)\gamma(F, K),$$

where $\mathcal{F}(K)$ is the set of faces of K and $\dim F$ is the dimension of the affine hull of F (Bohrer and Francis (1972b), Wynn (1975)).

Let F^* be the face of K^* dual to the face F of K. Then $\dim F^* = p - \dim F$, and F is orthogonal to F^*. Consider the orthogonal sum $F \oplus F^*$. For different faces F_1, F_2, interiors of the sets $F_1 \oplus F_1^*$, $F_2 \oplus F_2^*$ are disjoint and R^p is covered by these sets

$$R^p = \bigcup_{F \in \mathcal{F}(K)} F \oplus F^*$$

(Lemma 3 of McMullen (1975), Wynn (1975)). Then

$$\begin{aligned} \nu K_{(1)} + \lambda K^*_{(1)} &= (\nu K_{(1)} + \lambda K^*_{(1)}) \cap \Big(\bigcup_{F \in \mathcal{F}(K)} F \oplus F^* \Big) \\ &= \bigcup_{F \in \mathcal{F}(K)} (F \oplus F^*) \cap (\nu K_{(1)} + \lambda K^*_{(1)}) \\ &= \bigcup_{F \in \mathcal{F}(K)} (F \cap \nu U) \oplus (F^* \cap \lambda U). \end{aligned}$$

Therefore

$$v_p(\nu K_{(1)} + \lambda K^*_{(1)}) = \sum_{F \in \mathcal{F}(K)} v_p((F \cap \nu U) \oplus (F^* \cap \lambda U)).$$

Because of the orthogonality

$$\begin{aligned} v_p((F \cap \nu U) \oplus (F^* \cap \lambda U)) &= v_d(F \cap \nu U) \times v_{p-d}(F^* \cap \lambda U) \\ &= \nu^d \omega_d \beta(0, F) \times \lambda^{p-d} \omega_{p-d} \gamma(F, K), \end{aligned}$$

where $d = \dim F$. Therefore

$$v_p(\nu K_{(1)} + \lambda K^*_{(1)}) = \sum_{d=0}^{p} \sum_{\dim F = d} \nu^d \lambda^{p-d} \omega_d \omega_{p-d} \beta(0, F) \gamma(F, K)$$

and

$$\binom{p}{i} v_{p-i,i}(K_{(1)}, K^*_{(1)}) = \omega_i \omega_{p-i} \sum_{\dim F = p-i} \beta(0, F) \gamma(F, K) = \omega_i \omega_{p-i} \times w_{p-i},$$

or

$$w_{p-i} = \binom{p}{i} \frac{v_{p-i,i}(K_{(1)}, K^*_{(1)})}{\omega_i \omega_{p-i}}.$$

Therefore (6) holds for polyhedral cones. This proves the theorem for general cones as well by the argument given at the beginning of the proof. ∎

Remark 2.1 *The argument of approximating a non-polyhedral cone with a sequence of polyhedral cones is also found in Theorem 3.1 of Shapiro (1985).*

To characterize the set $\nu K_{(1)} + \lambda K^*_{(1)}$ the following lemma is useful.

Lemma 2.1 *Let K be a closed convex cone in R^p and K^* be its dual. Then for $\nu, \lambda \geq 0$,*

$$\nu K_{(1)} + \lambda K^*_{(1)} = (K + \lambda U) \cap (K^* + \nu U).$$

Proof. Note that $\nu K_{(1)} = \nu(K \cap U) = K \cap (\nu U)$ and $\lambda K^*_{(1)} = K^* \cap (\lambda U)$. Now suppose that $x \in K \cap \nu U$ and $y \in K^* \cap \lambda U$. Then $x \in K$, $y \in \lambda U$ and $x + y \in K + \lambda U$. At the same time $x \in \nu U$, $y \in K^*$ and $x + y \in K^* + \nu U$. Therefore $x + y \in (K + \lambda U) \cap (K^* + \nu U)$. This implies

$$(K \cap \nu U) + (K^* \cap \lambda U) \subset (K + \lambda U) \cap (K^* + \nu U).$$

To prove the converse let $z \in (K + \lambda U) \cap (K^* + \nu U)$. Since $z \in K^* + \nu U$ there exist x and y such that $z = x + y$ and $x \in K^*, \|y\| \leq \nu$. Write $z = z_K + z_{K^*}$ and $y = y_K + y_{K^*}$. Then

$$\begin{aligned} \|z_K\|^2 &= \|z - z_{K^*}\|^2 \leq \|z - x - y_{K^*}\|^2 = \|y_K\|^2 \\ &= \|y\|^2 - \|y_{K^*}\|^2 \leq \|y\|^2 \leq \nu^2. \end{aligned}$$

Therefore $z_K \in K \cap \nu U$. Similarly $z_{K^*} \in K^* \cap \lambda U$. Hence $z = z_K + z_{K^*} \in (K \cap \nu U) + (K^* \cap \lambda U)$ and this implies

$$(K + \lambda U) \cap (K^* + \nu U) \subset (K \cap \nu U) + (K^* \cap \lambda U).$$

$\blacksquare$

In evaluating mixed volumes, the p-dimensional volumes $v_{p,0}(K_{(1)}, K^*_{(1)}) = v_p(K_{(1)})$ and $v_{0,p}(K_{(1)}, K^*_{(1)}) = v_p(K^*_{(1)})$ have to be evaluated individually. Other mixed volumes turn out to be easier to evaluate. Consider

$$(\nu K_{(1)} + \lambda K^*_{(1)}) \cap (\nu K_{(1)})^C \cap (\lambda K^*_{(1)})^C \tag{8}$$

where A^C is the complement of A. By Lemma 2.1, $x \notin K, \notin K^*$ belongs to the set (8) if and only if $\|x - x_K\| \le \lambda$ and $\|x - x_{K^*}\| \le \nu$, i.e., x is not more than λ distant from the boundary surface ∂K of K and $\|x_K\| \le \nu$. Therefore the evaluation of mixed volumes is reduced to the evaluation of quermassintegrals, or more precisely, the volume of "local parallel sets" defined below in (9). In the case of polyhedral cones, the evaluation reduces to the evaluation of lower dimensional internal and external angles. On the other hand in the case of the smooth cone the evaluation reduces to integral of principal curvatures on ∂K.

2.2 The case of smooth cone

One of the main objectives of this research is to characterize the weights of $\bar{\chi}^2$ distributions for cones with smooth boundaries. Although the characterization by the mixed volumes in Theorem 2.1 applies to smooth cones, the definition of mixed volumes is not necessarily easy to work with for computational purposes. Here we can use the result that the volume of local parallel set of a smooth cone K can be expressed as an integral of principal curvatures on ∂K. See Section III.13.5 of Santaló (1976), Section 2.5 of Schneider (1993a), or Schneider (1993b). We summarize the result below.

Let K be a closed convex set with boundary ∂K. For a relatively open subset S of ∂K the *local parallel set of S at distance λ* is defined as

$$A_\lambda(K, S) = \{x \mid x_K \in S \text{ and } 0 < \|x - x_K\| \le \lambda\}. \tag{9}$$

Assume that ∂K is of class C^2. Let $H = H(s)$ be the positive semidefinite matrix of the second fundamental form at $s \in \partial K$ with respect to an orthonormal basis. The *principal curvatures* $\kappa_1, \ldots, \kappa_{p-1}$ are the eigenvalues of H. Denote the j-th trace of H, i.e., the j-th elementary symmetric function of the eigenvalues of H, by

$$\begin{aligned} \mathrm{tr}_j H = \mathrm{tr}_j H(s) &= \sum_{1 \le i_1 < \cdots < i_j \le p-1} \kappa_{i_1} \cdots \kappa_{i_j}, \quad j = 1, \ldots, p-1, \\ \mathrm{tr}_0 H &\equiv 1, \end{aligned} \tag{10}$$

and let ds denote the ($p-1$ dimensional) volume element of ∂K. Then we have the following lemma.

Lemma 2.2 *(Steiner's formula, (2.5.31) of Schneider (1993a))*

$$v_p(A_\lambda(K, S)) = \sum_{j=1}^{p} \lambda^j \frac{1}{j} \int_S \mathrm{tr}_{j-1} H(s) ds. \tag{11}$$

We now apply Lemma 2.2 to our problem. Let K be a closed convex cone with smooth boundary ∂K and $\mathrm{tr}_j H(s)$ be defined by (10) on ∂K. Consider the base set

$$S = \{s \mid s \in \partial K \text{ and } 0 < \|s\| \le \nu\},$$

then $A_\lambda(K,S)$ is equal to the set (8) except for the boundary, i.e.,

$$\mathrm{int} A_\lambda(K,S) = \mathrm{int}\Big((\nu K_{(1)} + \lambda K^*_{(1)}) \cap (\nu K_{(1)})^C \cap (\lambda K^*_{(1)})^C\Big).$$

Note that for each $s \in \partial K$, ∂K contains a half line starting at the origin in the direction of s. Therefore the principal curvature for the direction s is 0 and $\mathrm{tr}_{p-1} H(s) = 0$. Other principal directions lie in the tangent space $T_s(\partial K \cap \partial(lU))$, where $l = \|s\|$. Furthermore because of the cone structure the integration on ∂K can be reduced to the product of integration on $\partial K \cap \partial U$ and the 1-dimensional integration with respect to l.

Theorem 2.2 *Let K be a closed convex cone whose boundary ∂K is of class C^2 except for the origin. Then the mixed volumes $v_{p-i,i}(K_{(1)}, K^*_{(1)})$, $1 \le i \le p-1$, in (6) of Theorem 2.1 are expressed as*

$$\binom{p}{i} v_{p-i,i}(K_{(1)}, K^*_{(1)}) = \frac{1}{i(p-i)} \int_{\partial K \cap \partial U} \mathrm{tr}_{i-1} H(u) du,$$

where du denotes the ($p-2$ dimensional) volume element of $\partial K \cap \partial U$.

Proof. Let $l = \|s\|$ for $s \in \partial K$. The half line in the direction of s and $T_s(\partial K \cap \partial(lU))$ are orthogonal and the volume element of $\partial K \cap \partial(lU)$ is $l^{p-2} du$. Therefore

$$ds = dl \times (l^{p-2} du).$$

The principal curvatures are inversely proportional to l, i.e., $\kappa_i(s) = \kappa_i(u)/l$, where $u = s/l$. Therefore

$$\mathrm{tr}_j H(s) = \mathrm{tr}_j H(u)/l^j, \qquad l = \|s\|,\ u = s/l.$$

Then

$$\int_S \mathrm{tr}_{j-1} H(s) ds = \int_0^\nu \frac{l^{p-2}}{l^{j-1}} dl \int_{\partial K \cap \partial U} \mathrm{tr}_{j-1} H(u) du = \frac{\nu^{p-j}}{p-j} \int_{\partial K \cap \partial U} \mathrm{tr}_{j-1} H(u) du.$$

By (11)

$$v_p(A_\lambda(K,S)) = \sum_{j=1}^{p-1} \frac{\lambda^j \nu^{p-j}}{j(p-j)} \int_{\partial K \cap \partial U} \mathrm{tr}_{j-1} H(u) du.$$

Therefore

$$\binom{p}{j} v_{p-j,j}(K_{(1)}, K^*_{(1)}) = \frac{1}{j(p-j)} \int_{\partial K \cap \partial U} \mathrm{tr}_{j-1} H(u) du$$

and this proves the theorem. ∎

Remark 2.2 *Theorem 2.2 is stated in terms of* K . *However because of the duality of* K *and* K^* , *equivalent statement can be made in terms of* K^* .

Remark 2.3 *(The case of* R^3 *and the classical Gauss-Bonnet theorem)*
For $p = 3$ *the mixed volumes take particularly simple forms. Let*

$$P(\bar{\chi}_{01}^2 \le a, \bar{\chi}_{12}^2 \le b) = w_3 G_3(a) + w_2 G_2(a) G_1(b) + w_1 G_1(a) G_2(b) + w_0 G_3(b).$$

Then clearly

$$w_3 = \frac{\text{total area of } K \cap \partial U}{4\pi}, \quad w_0 = \frac{\text{total area of } K^* \cap \partial U}{4\pi},$$

where 4π *is the total surface area of the unit sphere* ∂U *in* R^3 . *By Theorem 2.2*

$$\begin{aligned} w_2 &= \frac{1}{2\omega_1\omega_2} \int_{\partial K \cap \partial U} \mathrm{tr}_0 H(u) du = \frac{1}{4\pi} \int_{\partial K \cap \partial U} 1 du \\ &= \frac{\text{total length of the curve } \partial K \cap \partial U}{4\pi} \end{aligned}$$

and considering K^*

$$w_1 = \frac{\text{total length of the curve } \partial K^* \cap \partial U}{4\pi}.$$

On the other hand by Theorem 2.2

$$w_1 = \frac{1}{4\pi} \int_{\partial K \cap \partial U} \kappa(u) du,$$

where $\kappa(u) = \mathrm{tr}_1 H(u)$ *is the geodesic curvature of the curve* $\partial K \cap \partial U$ *on* ∂U . *Since the Gaussian curvature is 1 on* ∂U , *the classical Gauss-Bonnet theorem states*

$$2\pi = \int_{\partial K \cap \partial U} \kappa(u) du + (\text{total area of } K \cap \partial U \text{ }).$$

Therefore

$$\frac{1}{2} = w_1 + w_3,$$

which is a particular case of Shapiro's conjecture that $\sum_{i=0}^{p} (-1)^i w_i = 0$ *(Shapiro (1987)).*

Remark 2.4 *Shapiro's conjecture is known to hold for polyhedral cones. Because of the continuity of mixed volumes, Shapiro's conjecture holds for smooth or piecewise smooth cones as well.*

Example 2.1 *Elliptical cone in* R^3

$$K = \{(\mu_1, \mu_2, \mu_3) \mid {\mu_1}^2 \ge (\frac{\mu_2}{a})^2 + (\frac{\mu_3}{b})^2, \ \mu_1 \ge 0\}, \qquad a, b > 0.$$

This is a special case of Remark 2.3. Using a local coordinate system, $\partial K \cap \partial U$ is expressed as

$$\{s(\theta) \in R^3 \mid 0 \le \theta < 2\pi\},$$

where

$$s(\theta) = \frac{1}{\sqrt{1 + a^2\cos^2\theta + b^2\sin^2\theta}} \begin{pmatrix} 1 \\ a\cos\theta \\ b\sin\theta \end{pmatrix}.$$

The total length of the curve $\partial K \cap \partial U$ is

$$\int_0^{2\pi} \left\| \frac{ds}{d\theta} \right\| d\theta = f(a,b),$$

where

$$f(a,b) = \int_0^{2\pi} \frac{\sqrt{a^2b^2 + b^2\cos^2\theta + a^2\sin^2\theta}}{1 + a^2\cos^2\theta + b^2\sin^2\theta} d\theta,$$

and therefore we have $w_2 = f(a,b)/4\pi$, $w_0 = 1/2 - f(a,b)/4\pi$. The dual of K is

$$K^* = \{(\mu_1, \mu_2, \mu_3) \mid {\mu_1}^2 \ge (a\mu_2)^2 + (b\mu_3)^2, \ \mu_1 \le 0\},$$

and hence we have $w_1 = f(a^{-1}, b^{-1})/4\pi$, $w_3 = 1/2 - f(a^{-1}, b^{-1})/4\pi$.

Example 2.2 *Spherical cone in* R^p *(Pincus (1975), Akkerboom (1990))*

$$K = \{ \mu = (\mu_1, \dots, \mu_p) \mid \mu_1 \ge \|\mu\| \cos\psi \}, \qquad 0 < \psi < \frac{\pi}{2}.$$

This is the spherical cone K_4 mentioned in Section 1.1. Let

$$F(x) = F(x_1, \dots, x_p) = x_1^2 \sin^2\psi - (x_2^2 + \dots + x_p^2)\cos^2\psi. \tag{12}$$

Then the boundary ∂K of K can be written as

$$\partial K = \{x \mid F(x) = 0, \ x_1 \ge 0\}.$$

By our Theorem 2.2 we consider a point $s \in \partial K, \ \|s\| = 1$. Because of spherical symmetry with respect to $x_2, \dots, x_p$ we take $s^0 = (\cos\psi, \sin\psi, 0, \dots, 0)$ as a representative point. The values of $\mathrm{tr}_j H(u)$ are the same for all $u \in \partial K \cap \partial U$. The outward unit normal vector at s^0 is easily seen to be

$$N_p = (-\sin\psi, \cos\psi, 0, \dots, 0).$$

Consider the rotation of coordinates $(x_1, \dots, x_p) \mapsto (u_1, \dots, u_p)$

$$\begin{aligned} u_1 &= -\sin\psi \ x_1 + \cos\psi \ x_2, \\ u_2 &= \cos\psi \ x_1 + \sin\psi \ x_2, \\ u_i &= x_i, \quad i = 3, \dots, p. \end{aligned}$$

Note that u_2 is the coordinate for the direction of s^0 . Substituting the inverse rotation $x_1 = -\sin\psi\ u_1 + \cos\psi\ u_2$, $x_2 = \cos\psi\ u_1 + \sin\psi\ u_2$ into (12), ∂K can be written as

$$\begin{aligned} F &= x_1^2 \sin^2\psi - x_2^2\cos^2\psi - (x_3^2 + \cdots + x_p^2)\cos^2\psi \\ &= -u_1^2\cos 2\psi - u_1u_2\sin 2\psi - (u_3^2+\cdots+u_p^2)\cos^2\psi \\ &= 0. \end{aligned} \tag{13}$$

The particular point s^0 expressed in the new coordinates is $u^0 = (0,1,0,\ldots,0)$. Now we want to calculate the elements of the second fundamental form

$$-\frac{\partial^2 u_1}{\partial u_i \partial u_j}, \quad i,j \geq 2. \tag{14}$$

Recall that s^0 itself is the principal direction with zero principal curvature and u_2 is the coordinate for this direction. Therefore actually we only need to calculate (14) for $i,j = 3,\ldots,p$. (Or one can easily verify that derivatives with respect to u_2 are indeed 0.) Now regard (13) as an equation determining u_1 in terms of $u_2,\ldots,u_p$. Taking partial derivative of (13) with respect to $u_i,\ i \geq 3$, we have

$$0 = \frac{\partial F}{\partial u_i} = -2\frac{\partial u_1}{\partial u_i}u_1\cos 2\psi - \frac{\partial u_1}{\partial u_i}u_2\sin 2\psi - 2u_i\cos^2\psi.$$

Differentiating this once more we obtain

$$0 = -2\frac{\partial^2 u_1}{\partial u_i\partial u_j}u_1\cos 2\psi - 2\frac{\partial u_1}{\partial u_i}\frac{\partial u_1}{\partial u_j}\cos 2\psi - \frac{\partial^2 u_1}{\partial u_i \partial u_j}u_2 \sin 2\psi - 2\delta_{ij}\cos^2\psi,$$

where δ_{ij} is the Kronecker's delta. Evaluating this at u^0 we obtain

$$H(u^0) = \mathrm{diag}\Big(0, \underbrace{\frac{1}{\tan\psi},\ldots,\frac{1}{\tan\psi}}_{p-2}\Big).$$

Therefore

$$\mathrm{tr}_j H(u^0) = \binom{p-2}{j}\frac{1}{\tan^j\psi}.$$

As mentioned earlier this value is the same for all u , i.e., $\mathrm{tr}_j H(u^0) = \mathrm{tr}_j H(u),\ \forall u \in \partial K \cap \partial U$. Furthermore

$$\partial K \cap \partial U = \{x \mid x_1 = \cos\psi,\ x_2^2 + \cdots + x_p^2 = 1 - \cos^2\psi = \sin^2\psi\}.$$

Therefore the ($p-2$ dimensional) total volume of $\partial K \cap \partial U$ equals the total surface volume of sphere of radius $\sin\psi$ in R^{p-1} , i.e.,

$$v_{p-2}(\partial K \cap \partial U) = v_{p-2}(\partial(\sin\psi U_{p-1})) = (p-1)\sin^{p-2}\psi\ \omega_{p-1}.$$

Combining the above results the weights of $\bar{\chi}^2$ distribution are

$$\begin{aligned} \binom{p}{i} v_{p-i,i}(K_{(1)}, K^*_{(1)}) &= \frac{1}{i(p-i)}\binom{p-2}{i-1}\frac{1}{\tan^{i-1}\psi} \times (p-1)\sin^{p-2}\psi\ \omega_{p-1} \\ &= \frac{(p-1)!}{i!(p-i)!}\omega_{p-1}\sin^{p-i-1}\psi\ \cos^{i-1}\psi. \end{aligned} \tag{15}$$

Further manipulation of (15) shows that

$$w_{p-i} = \binom{p}{i} \frac{v_{p-i,i}(K_{(1)}, K^*_{(1)})}{\omega_i \omega_{p-i}} = \frac{1}{2}\binom{p-2}{i-1} \frac{B(\frac{p-i}{2}, \frac{i}{2})}{B(\frac{1}{2}, \frac{p-1}{2})} \sin^{p-i-1}\psi \; \cos^{i-1}\psi,$$

which coincides with the result given by Pincus (1975).

Remark 2.5 *After completing this paper in a form of discussion paper, we found that Lin and Lindsay (1995) derived essentially the same result as Theorem 2.2 using the formula in Weyl (1939). They also calculated the weights for the spherical cone as an example.*

2.3 The case of piecewise smooth cone

Here we consider an intermediate case between the polyhedral cone and everywhere smooth cone, namely a cone K whose boundary ∂K consists of both smooth surfaces and edges. To fix ideas let us consider a generalization of Example 2.2.

Example 2.3 *Let K be defined as*

$$K = \{\, \mu \in R^p \mid \mu_1 \geq \|\mu\| \cos\psi_1 \;\text{ and }\; \mu_2 \geq \|\mu\| \cos\psi_2 \,\},$$

where

$$\cos^2\psi_1 + \cos^2\psi_2 < 1, \quad 0 < \psi_i < \frac{\pi}{2}, \; i = 1, 2, \quad p \geq 3.$$

In this example $K = K_1 \cap K_2$ where

$$K_i = \{\, \mu \mid \mu_i \geq \|\mu\| \cos\psi_i \,\}, \quad i = 1, 2,$$

are cones of Example 2.2. Note that ∂K is no longer smooth at the common boundary $\partial K_1 \cap \partial K_2$. At a point s of $\partial K_1 \cap \partial K_2$, the outward unit normal vector is no longer unique and contribution to the mixed volume from $s \in \partial K_1 \cap \partial K_2$ can not be expressed as an integral with respect to the volume element of the $p-1$ dimensional surface of ∂K.

Let K be a convex set. For each point s on the boundary ∂K of K, the *normal cone* $N(K, s)$ is defined as

$$N(K, s) = \{y \mid \langle y, z - s\rangle \leq 0, \; \forall z \in K\} \tag{16}$$

(see Section 2.2 of Schneider (1993a)). Define

$$D_m(\partial K) = \{s \in \partial K \mid \dim N(K, s) = m\}, \quad m = 1, \ldots, p.$$

Then

$$\partial K = D_1(\partial K) \cup \cdots \cup D_p(\partial K).$$

In Example 2.3, $D_2(\partial K) = \mathrm{relint}(\partial K_1 \cap \partial K_2)$, and $D_1(\partial K)$ consists of 2 relatively open connected components $\mathrm{relint}(\partial K_1 \cap \partial K)$, $\mathrm{relint}(\partial K_2 \cap \partial K)$, where $\mathrm{relint}(\cdot)$ denotes the relative interior. $D_p(\partial K) = \{0\}$, and other D_i 's are empty. With Example 2.3 in mind, we make the following assumption on convex set K and we call such K *piecewise smooth*.

Assumption 2.1 *$D_m(\partial K)$ is a $p-m$ dimensional C^2-manifold consisting of a finite number of relatively open connected components. Furthermore $N(K,s)$ is continuous in s on $D_m(\partial K)$ in the sense of Lemma 1.2.*

Let $s \in D_m(\partial K)$. In a neighborhood of s we take an orthonormal system of vectors $e_1,\dots,e_{p-m},N_{p-m+1},\dots,N_p$ where $e_1,\dots,e_{p-m}$ constitute an orthonormal basis for the tangent space $T_s(D_m(\partial K))$ and $N_{p-m+1},\dots,N_p$ constitute an orthonormal basis for the orthogonal complement $T_s(D_m(\partial K))^\perp$ of $T_s(D_m(\partial K))$. Clearly $N(K,s) \subset T_s(D_m(\partial K))^\perp$.

Let

$$H_{ij\alpha}, \quad i,j=1,\dots,p-m, \ \alpha=p-m+1,\dots,p,$$

be the element of the second fundamental tensor with respect to the chosen coordinate system. For a unit vector v in $T_s(D_m(\partial K))^\perp$

$$v = \sum_{\alpha=p-m+1}^{p} v^\alpha N_\alpha, \qquad \|v\|=1,$$

define

$$H_{ij}(s,v) = \sum_{\alpha=p-m+1}^{p} v^\alpha H_{ij\alpha}.$$

Furthermore let

$$\mathrm{tr}_j H(s,v) = \sum_{1\le i_1<\cdots<i_j\le p-m} \kappa_{i_1}(s,v)\cdots\kappa_{i_j}(s,v), \qquad j=1,\dots,p-m,$$

where $\kappa_1(s,v),\dots,\kappa_{p-m}(s,v)$ are eigenvalues of the $(p-m)\times(p-m)$ matrix $H(s,v)=(H_{ij}(s,v))$, i.e., the principal curvatures against a particular normal direction v at s.

We now generalize Lemma 2.2 to the case of piecewise smooth convex set. We use the same notation as in Lemma 2.2

Theorem 2.3 *Let K be a piecewise smooth closed convex set satisfying Assumption 2.1. Let ds_{p-m} denote the ($p-m$ dimensional) volume element of $D_m(\partial K)$ and let dv_{m-1} denote the $m-1$ dimensional volume element of the surface ∂U_m. Then*

$$v_p(A_\lambda(K,S)) = \sum_{m=1}^{p}\sum_{j=m}^{p} \lambda^j \frac{1}{j} \int_{S\cap D_m(\partial K)} \Big[\int_{N(K,s_{p-m})\cap\partial U} \mathrm{tr}_{j-m} H(s_{p-m},v_{m-1})dv_{m-1}\Big] ds_{p-m}. \tag{17}$$

For a sketch of the proof see the Appendix. From Theorem 2.3 we obtain the corresponding result for our problem.

Theorem 2.4 *Let K be a closed convex cone satisfying Assumption 2.1. Let du_{p-m-1} denote the ($p-m-1$ dimensional) volume element of $D_m(\partial K)\cap\partial U$, $m=1,\dots,p-1$.*

Then the mixed volumes $v_{p-i,i}(K_{(1)}, K^*_{(1)})$, $1 \le i \le p-1$, *in (6) of Theorem 2.1 is expressed as*

$$\binom{p}{i} v_{p-i,i}(K_{(1)}, K^*_{(1)}) = \frac{1}{i(p-i)} \times \sum_{m=1}^{i} \int_{D_m(\partial K)\cap\partial U} \Big[\int_{N(K,u_{p-m-1})\cap\partial U} \mathrm{tr}_{i-m} H(u_{p-m-1}, v_{m-1}) dv_{m-1} \Big] du_{p-m-1}. \tag{18}$$

Proof. It is easy to show that

$$N(K,s) = N(K,u), \qquad l = \|s\|,\ u = s/l.$$

As in the proof of Theorem 2.2

$$\mathrm{tr}_{j-m} H(s,v) = \mathrm{tr}_{j-m} H(u,v)/l^{j-m}.$$

Therefore in (17)

$$\int_{N(K,s_{p-m})\cap\partial U} \mathrm{tr}_{j-m} H(s_{p-m}, v_{m-1}) dv_{m-1} = \frac{1}{l^{j-m}} \int_{N(K,u_{p-m-1})\cap\partial U} \mathrm{tr}_{j-m} H(u_{p-m-1}, v_{m-1}) dv_{m-1}.$$

Moreover

$$ds_{p-m} = dl \times (l^{p-m-1} du_{p-m-1}).$$

Therefore for $S = \{s \mid s \in \partial K \text{ and } 0 < \|s\| \le \nu\}$

$$\begin{aligned}
&\int_{S\cap D_m(\partial K)} \Big[\int_{N(K,s_{p-m})\cap\partial U} \mathrm{tr}_{j-m} H(s_{p-m}, v_{m-1}) dv_{m-1} \Big] ds_{p-m} \\
&= \int_0^{\nu} l^{p-j-1} dl \int_{D_m(\partial K)\cap\partial U} \Big[\int_{N(K,u_{p-m-1})\cap\partial U} \mathrm{tr}_{j-m} H(u_{p-m-1}, v_{m-1}) dv_{m-1} \Big] du_{p-m-1} \\
&= \frac{\nu^{p-j}}{p-j} \int_{D_m(\partial K)\cap\partial U} \Big[\int_{N(K,u_{p-m-1})\cap\partial U} \mathrm{tr}_{j-m} H(u_{p-m-1}, v_{m-1}) dv_{m-1} \Big] du_{p-m-1}.
\end{aligned}$$

It follows that

$$v_p(A_\lambda(K,S)) = \sum_{m=1}^{p} \sum_{j=m}^{p} \frac{\lambda^j \nu^{p-j}}{j(p-j)} \times \int_{D_m(\partial K)\cap\partial U} \Big[\int_{N(K,u_{p-m-1})\cap\partial U} \mathrm{tr}_{j-m} H(u_{p-m-1}, v_{m-1}) dv_{m-1} \Big] du_{p-m-1}$$

and this proves the theorem. ∎

Example 2.3 *(continued)*

Using Theorem 2.4 we evaluate the weights of $\bar{\chi}^2$ distribution. First we consider $D_1(\partial K) = \mathrm{relint}(\partial K_1 \cap \partial K) \cup \mathrm{relint}(\partial K_2 \cap \partial K)$. Note that $\mathrm{relint}(\partial K_1 \cap \partial K) = \partial K_1 \cap \mathrm{int} K_2$. Therefore

$$\mathrm{relint}(\partial K_1 \cap \partial K) \cap \partial U = \{x \mid x_1 = \cos\psi_1,\ x_2 > \cos\psi_2,\ \|x\| = 1\}.$$

Now consider the following ratio of volumes

$$\frac{v_{p-2}(\{(x_2,\ldots,x_p) \mid x_2 > \cos\psi_2, x_2^2+\cdots+x_p^2 = \sin^2\psi_1\})}{v_{p-2}(\{(x_2,\ldots,x_p) \mid x_2^2+\cdots+x_p^2 = \sin^2\psi_1\})}.$$

This is obviously equal to the following incomplete beta function

$$\beta_1 = \frac{1}{2}\int_{\cos^2\psi_2/\sin^2\psi_1}^{1} u^{-\frac{1}{2}}(1-u)^{\frac{p-4}{2}}du. \tag{19}$$

The contribution to the weights from $\partial K_1 \cap \partial K \cap \partial U$ is just (15) multiplied by β_1 with $\psi = \psi_1$. Similarly the contribution from $\partial K_2 \cap \partial K \cap \partial U$ is (15) multiplied by β_2 with $\psi = \psi_2$, where

$$\beta_2 = \frac{1}{2}\int_{\cos^2\psi_1/\sin^2\psi_2}^{1} u^{-\frac{1}{2}}(1-u)^{\frac{p-4}{2}}du. \tag{20}$$

It remains to evaluate the contribution from $\partial K_1 \cap \partial K_2$. Consider a representative point

$$s^0 = (\cos\psi_1, \cos\psi_2, \tau, 0, \ldots, 0),$$

where

$$\tau = \sqrt{1-\cos^2\psi_1-\cos^2\psi_2}. \tag{21}$$

The outward unit normal vector to K_1 at s^0 is

$$n_1 = \Big(-\sin\psi_1, \frac{\cos\psi_2}{\tan\psi_1}, \frac{\tau}{\tan\psi_1}, 0, \ldots, 0\Big).$$

Similarly the outward unit normal vector to K_2 at s^0 is

$$n_2 = \Big(\frac{\cos\psi_1}{\tan\psi_2}, -\sin\psi_2, \frac{\tau}{\tan\psi_2}, 0, \ldots, 0\Big).$$

The normal cone $N(K, s^0)$ is the non-negative combination of these two vectors

$$N(K, s^0) = an_1 + bn_2, \quad a, b \geq 0.$$

The inner product of these two vectors is

$$\langle n_1, n_2\rangle = -\frac{1}{\tan\psi_1\tan\psi_2}.$$

Let

$$N_{p-1} = n_1, \quad N_p = \Big(0, -\frac{\tau}{\sin\psi_1}, \frac{\cos\psi_2}{\sin\psi_1}, 0, \ldots, 0\Big).$$

Then N_{p-1}, N_p form an orthonormal basis of $T_{s^0}(D_2(\partial K))^{\perp}$. Now consider the rotation of coordinates based on N_{p-1}, N_p and s^0 :

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -\sin\psi_1 & \frac{\cos\psi_2}{\tan\psi_1} & \frac{\tau}{\tan\psi_1} \\ 0 & -\frac{\tau}{\sin\psi_1} & \frac{\cos\psi_2}{\sin\psi_1} \\ \cos\psi_1 & \cos\psi_2 & \tau \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and $u_i = x_i, i = 4, \ldots, p$. s^0 in the new coordinates is $u^0 = (0, 0, 1, 0, \ldots, 0)$.
Now consider (12) for K_1 and K_2:

$$\begin{aligned} 0 &= F_1 = x_1^2 \sin^2 \psi_1 - (x_2^2 + x_3^2) \cos^2 \psi_1 - (u_4^2 + \cdots + u_p^2) \cos^2 \psi_1, && (22) \\ 0 &= F_2 = x_2^2 \sin^2 \psi_2 - (x_1^2 + x_3^2) \cos^2 \psi_2 - (u_4^2 + \cdots + u_p^2) \cos^2 \psi_2. && (23) \end{aligned}$$

In (22) and (23) x_1, x_2, x_3 are functions of u_1, u_2, u_3. We regard (22) and (23) as a system of equations for determining u_1, u_2 in terms of $u_3, \ldots, u_p$. Furthermore as in Example 2.2 we can ignore differentiation with respect to u_3 and we differentiate (22) and (23) with respect to $u_4, \ldots, u_p$. At u^0

$$0 = \frac{\partial u_1}{\partial u_i}\Big|_{u^0} = \frac{\partial u_2}{\partial u_i}\Big|_{u^0}, \quad i \geq 4.$$

Therefore

$$\frac{\partial x_j}{\partial u_i}\Big|_{u^0} = 0, \quad i \geq 4,\ j = 1, 2, 3.$$

Using this it can be easily shown that $0 = \partial^2 F_1 / (\partial u_i \partial u_j)$, $i, j \geq 4$, evaluated at u^0 reduces to

$$0 = -2 \frac{\partial^2 u_1}{\partial u_i \partial u_j} \cos \psi_1 \sin \psi_1 - 2\delta_{ij} \cos^2 \psi_1 \tag{24}$$

and that $0 = \partial^2 F_2 / (\partial u_i \partial u_j)$ evaluated at u^0 reduces to

$$0 = 2 \frac{\partial^2 u_1}{\partial u_i \partial u_j} \frac{\cos^2 \psi_2}{\tan \psi_1} - 2 \frac{\partial^2 u_2}{\partial u_i \partial u_j} \frac{\tau \cos \psi_2}{\sin \psi_1} - 2\delta_{ij} \cos^2 \psi_2. \tag{25}$$

Solving (24) and (25) we obtain

$$-\frac{\partial^2 u_1}{\partial u_i^2} = \frac{1}{\tan \psi_1}, \qquad -\frac{\partial^2 u_2}{\partial u_i^2} = \frac{\cos \psi_2}{\tau \sin \psi_1}.$$

All the other second order derivatives evaluated at u^0 are 0.
Let

$$\theta_0 = \arccos\Big(-\frac{1}{\tan \psi_1 \tan \psi_2} \Big), \qquad \frac{\pi}{2} < \theta_0 < \pi.$$

Then $v \in N(K, s^0)$, $\|v\| = 1$, can be written as

$$v = \cos \theta N_{p-1} + \sin \theta N_p, \qquad 0 \leq \theta \leq \theta_0.$$

Therefore

$$H(s^0, v) = \mathrm{diag}(0, \underbrace{h(\theta, \psi_1, \psi_2), \ldots, h(\theta, \psi_1, \psi_2)}_{p-3}),$$

where

$$h(\theta, \psi_1, \psi_2) = \cos \theta \frac{1}{\tan \psi_1} + \sin \theta \frac{\cos \psi_2}{\tau \sin \psi_1}$$

and we obtain

$$\mathrm{tr}_j H(s^0, v) = \binom{p-3}{j} h(\theta, \psi_1, \psi_2)^j.$$

Therefore

$$\int_{N(K,s^0)\cap\partial U} \mathrm{tr}_j H(s^0, v_1) dv_1 = \binom{p-3}{j} \int_0^{\theta_0} h(\theta, \psi_1, \psi_2)^j d\theta. \tag{26}$$

The value of (26) is the same for all $s \in \partial K_1 \cap \partial K_2 \cap \partial U$, and

$$v_{p-3}(\partial K_1 \cap \partial K_2 \cap \partial U) = (p-2)\tau^{p-3}\omega_{p-2}.$$

Therefore the contribution from $\partial K_1 \cap \partial K_2$ to the mixed volume $\binom{p}{i} v_{p-i,i}(K_{(1)}, K^*_{(1)})$ is obtained as

$$\binom{p-3}{i-2} \frac{1}{i(p-i)} \int_0^{\theta_0} h(\theta, \psi_1, \psi_2)^{i-2} d\theta \times (p-2)\tau^{p-3}\omega_{p-2}.$$

Summarizing the above calculations the mixed volume is

$$\begin{aligned}\binom{p}{i} v_{p-i,i}(K_{(1)}, K^*_{(1)}) &= \frac{(p-1)!}{i!(p-i)!}\omega_{p-1}(\beta_1 \sin^{p-i-1}\psi_1 \cos^{i-1}\psi_1 \\ &\qquad + \beta_2 \sin^{p-i-1}\psi_2 \cos^{i-1}\psi_2) \\ &\quad + \frac{(i-1)(p-2)!}{i!(p-i)!}\tau^{p-3}\omega_{p-2} \int_0^{\theta_0} h(\theta, \psi_1, \psi_2)^{i-2} d\theta,\end{aligned}$$

where τ is defined in (21) and β_1, β_2 are defined in (19),(20). Note that the last term vanishes for $i = 1$, and that it can be expressed using the incomplete beta functions.

2.4 The cone of non-negative definite matrices

In this subsection, we treat the cone of non-negative definite matrices, which is a typical example of the piecewise smooth cone. This cone is needed to discuss multivariate one-sided alternatives for covariance matrices (Kuriki (1993)). By deriving the normal cone and the second fundamental form at the boundary of the cone, we reveal "recurrence structure" of the singularities.

Let $\mathcal{S}_p$ be the set of $p \times p$ symmetric matrices. We identify $\mathcal{S}_p$ with $R^{p(p+1)/2}$ by the map

$$W = (w_{ij}) \in \mathcal{S}_p \leftrightarrow (w_{11}, \ldots, w_{pp}, \sqrt{2}w_{12}, \ldots, \sqrt{2}w_{p-1,p}) \in R^{p(p+1)/2}$$

and the corresponding inner product

$$\langle W_1, W_2 \rangle = \mathrm{tr}\, W_1 W_2 = \sum_i w_{1ii} w_{2ii} + \sum_{i<j} (\sqrt{2}w_{1ij})(\sqrt{2}w_{2ij}) \tag{27}$$

for $W_1 = (w_{1ij})$, $W_2 = (w_{2ij}) \in \mathcal{S}_p$.

Let K be the cone formed by the $p \times p$ non-negative definite matrices, i.e.,

$$K = \{W \in \mathcal{S}_p \mid W \geq O\},$$

where $\geq$ denotes the *Löwner order.*

Define

$$\mathcal{S}_{r,p} = \{W \in \mathcal{S}_p \mid \mathrm{rank}\, W = r\},$$

and

$$\mathcal{S}_{r,p}^+ = \mathcal{S}_{r,p} \cap K = \{W \in \mathcal{S}_p \mid W \geq O,\ \operatorname{rank} W = r\}.$$

Denote the spectral decomposition of $W_0 \in \mathcal{S}_{r,p}^+$ as $W_0 = H_{10}\Lambda_0 H_{10}'$, where $\Lambda_0 = \operatorname{diag}(l_{10},\dots,l_{r0})$ with $l_{10} \geq \cdots \geq l_{r0} > 0$ and H_{10} is a $p \times r$ matrix such that $H_{10}'H_{10} = I_r$. Let H_{20} be a $p \times (p-r)$ matrix such that $H_0 = (H_{10}, H_{20})$ is $p \times p$ orthogonal. It is straightforward to show that the normal cone (16) of K at $W_0 \in \mathcal{S}_{r,p}^+$ is given by

$$N(K, W_0) = \{-H_{20} Y H_{20}' \mid Y \in \mathcal{S}_{p-r},\ Y \geq O\}. \tag{28}$$

We see that this normal cone is a lower dimensional replica of the original cone K. The dimension of the cone (28) is $(p-r)(p-r+1)/2$, which is 1 iff $r = p-1$. In other words, $\mathcal{S}_{p-1,p}^+$ is the smooth surface, and $\mathcal{S}_{r,p}^+$, $r = 0,\dots,p-2$, form singularities of ∂K.

Now we will derive the second fundamental form at W_0. In order to do this we introduce a local coordinate system $X = (x_{ij}) = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}' & X_{22} \end{pmatrix}$ of $\mathcal{S}_p$ in the neighborhood of W_0 as

$$\mathcal{S}_p \ni W = W_0 + H_0 X H_0' = (\,H_{10} \quad H_{20}\,) \begin{pmatrix} \Lambda_0 + X_{11} & X_{12} \\ X_{12}' & X_{22} \end{pmatrix} \begin{pmatrix} H_{10}' \\ H_{20}' \end{pmatrix}.$$

We note here that for a $p \times p$ orthogonal matrix H, the transform $W \mapsto HWH'$ is orthogonal and preserves the inner product (27), because $\operatorname{tr}(HW_1H')(HW_2H') = \operatorname{tr} W_1 W_2$. So, the new coordinate system X, i.e., $(x_{11},\dots,x_{pp},\sqrt{2}x_{12},\dots,\sqrt{2}x_{p-1,p})$, is also orthonormal.

Here we can take $\partial/\partial x_{ii}$ $(r+1 \leq i \leq p)$, $\partial/\partial(\sqrt{2}x_{ij})$ $(r+1 \leq i < j \leq p)$ as an orthonormal basis of $N(K, W_0)$, and therefore, $\partial/\partial x_{ii}$ $(1 \leq i \leq r)$, $\partial/\partial(\sqrt{2}x_{ij})$ $(1 \leq i \leq r,\ i < j \leq p)$ as an orthonormal basis of $N(K, W_0)^\perp = T_{W_0}(\mathcal{S}_{r,p}^+)$.

In the neighborhood of W_0, $W \in \mathcal{S}_{r,p}^+$ is equivalent to

$$X_{22} = X_{12}'(\Lambda_0 + X_{11})^{-1} X_{12},$$

because $\Lambda_0 + X_{11}$ is positive definite in the neighborhood of W_0. Fix $\tilde{W} = -H_{20} Y H_{20}' \in N(K, W_0)$. Then, the second fundamental form with respect to the normal direction $\tilde{W}$ becomes

$$H(W_0, \tilde{W}) = \left.\frac{\partial^2 \operatorname{tr}(Y X_{22})}{\partial((x_{ii})_{1\leq i\leq r}, (\sqrt{2}x_{ij})_{1\leq i\leq r,\ i<j\leq p})^2}\right|_{W_0}. \tag{29}$$

The $(k-r, l-r)$-th element of X_{22} is

$$\begin{aligned} x_{kl} &= (X_{12}'(\Lambda_0 + X_{11})^{-1} X_{12})_{k-r,l-r} \\ &= (\,x_{1k} \ \cdots \ x_{rk}\,)(\Lambda_0 + X_{11})^{-1} \begin{pmatrix} x_{1l} \\ \vdots \\ x_{rl} \end{pmatrix}, \quad r+1 \leq k \leq l \leq p. \end{aligned} \tag{30}$$

Differentiating (30) twice with respect to $(x_{ii})_{1\leq i\leq r}$, $(\sqrt{2}x_{ij})_{1\leq i\leq r,\ i<j\leq p}$, and putting $X_{11} = O$ and $X_{12} = O$, we see that the non-vanishing terms of (29) are only

$$\left.\frac{\partial^2 x_{kl}}{\partial(\sqrt{2}x_{ik})\partial(\sqrt{2}x_{jl})}\right|_{W_0} = \frac{\delta_{ij}}{l_{i0}} \cdot \frac{1+\delta_{kl}}{2},$$

$1 \le i \le j \le r$, $r+1 \le k \le l \le p$. So

$$\frac{\partial^2 \operatorname{tr}(YX_{22})}{\partial(\sqrt{2}x_{ik})\partial(\sqrt{2}x_{jl})}\bigg|_{W_0} = \frac{\delta_{ij}}{l_{i0}} \cdot y_{kl}$$

with $Y = (y_{kl})$, and other contributions are 0. Now we have established the following.

Lemma 2.3 *The non-vanishing part of the second fundamental form at* $W_0 = H_{10}\Lambda_0 H_{10}' \in \mathcal{S}^+_{r,p}$ *with respect to the direction* $\tilde{W} = -H_{20}YH_{20}' \in N(K, W_0)$ *is*

$$H(W_0, \tilde{W}) = \Big(\frac{\delta_{ij}}{l_{i0}} \cdot y_{kl}\Big) = {\Lambda_0}^{-1} \otimes Y.$$

Here $H_0 = (H_{10}, H_{20})$ *is* $p \times p$ *orthogonal, and* $\otimes$ *denotes the Kronecker product.*

Let $\tilde{\Lambda} = \operatorname{diag}(\tilde{l}_1, \dots, \tilde{l}_{p-r})$ be the eigenvalues of Y. Concerning the m-th trace

$$\operatorname{tr}_m H = \operatorname{tr}_m({\Lambda_0}^{-1} \otimes Y) = \operatorname{tr}_m({\Lambda_0}^{-1} \otimes \tilde{\Lambda}),$$

the following lemma holds.

Lemma 2.4 *For* $\Lambda = \operatorname{diag}(l_i)_{1\le i \le r}$ *and* $\tilde{\Lambda} = \operatorname{diag}(\tilde{l}_i)_{1 \le i \le p-r}$

$$\det(\Lambda)^{p-r}\operatorname{tr}_m(\Lambda^{-1} \otimes \tilde{\Lambda}) = \sum_{(q,\bar{q})} \frac{\det({l_i}^{q_j})_{1\le i,j\le r}}{\prod_{1\le i<j\le r}(l_i - l_j)} \cdot \frac{\det(\tilde{l}_i^{\bar{q}_j})_{1\le i,j\le p-r}}{\prod_{1\le i<j\le p-r}(\tilde{l}_i - \tilde{l}_j)},$$

where the summation $\sum_{(q,\bar{q})}$ *is over the set of integers*

$$(q_1, \dots, q_r, \bar{q}_1, \dots, \bar{q}_{p-r}) \in Q_{r,p}(-m + r(p-r) + r(r-1)/2)$$

with

$$Q_{r,p}(n) = \{(q_1, \dots, q_r, \bar{q}_1, \dots, \bar{q}_{p-r}) \in \pi_p \mid q_1 > \cdots > q_r,\ \bar{q}_1 > \cdots > \bar{q}_{p-r},\ \sum_{j=1}^r q_j = n\}$$

and π_p *denotes the set of all permutations of* $\{p-1, p-2, \dots, 0\}$.

Proof. Define the generating function by

$$\Phi(x) = \sum_{m=0}^{r(p-r)} (-1)^m x^{r(p-r)-m} \det(\Lambda)^{p-r}\operatorname{tr}_m(\Lambda^{-1} \otimes \tilde{\Lambda}).$$

Then

$$\begin{aligned}
\Phi(x) &= \det(\Lambda)^{p-r}\det(xI_r \otimes I_{p-r} - \Lambda^{-1} \otimes \tilde{\Lambda}) \\
&= \Big(\prod_{i=1}^r l_i^{p-r}\Big) \cdot \prod_{i=1}^r \prod_{j=1}^{p-r}\Big(x - \frac{\tilde{l}_j}{l_i}\Big) = \prod_{i=1}^r \prod_{j=1}^{p-r}(l_i x - \tilde{l}_j) \\
&= \det\begin{pmatrix} (xl_1)^{p-1} & \cdots & xl_1 & 1 \\ \vdots & & \vdots & \vdots \\ (xl_r)^{p-1} & \cdots & xl_r & 1 \\ \tilde{l}_1^{p-1} & \cdots & \tilde{l}_1 & 1 \\ \vdots & & \vdots & \vdots \\ \tilde{l}_{p-r}^{p-1} & \cdots & \tilde{l}_{p-r} & 1 \end{pmatrix} \Big/ \prod_{1\le i<j\le r}(xl_i - xl_j) \prod_{1\le i<j\le p-r}(\tilde{l}_i - \tilde{l}_j). \quad (31)
\end{aligned}$$

By the Laplace expansion of the determinant in (31), and by comparing the coefficient of the term $(-1)^m x^{r(p-r)-m}$, we prove the lemma. ∎

To evaluate the mixed volumes by virtue of Theorem 2.3 or 2.4, we have to know the concrete forms of the volume elements of $\mathcal{S}_{r,p}$ or $\mathcal{S}_{r,p} \cap \partial U$.

Before proceeding we prepare several facts on Stiefel manifolds. Let $\mathcal{V}_{r,p} = \{H_1 : p \times r \,|\, H_1{}'H_1 = I_r\}$ be the Stiefel manifold. Let H_2 be $p \times (p-r)$ such that $H = (H_1, H_2) = (h_1, \ldots, h_r, h_{r+1}, \ldots, h_p)$ is $p \times p$ orthogonal. Then the differential form for the invariant measure on $\mathcal{V}_{r,p}$ is

$$dH_1 = \bigwedge_{i=1}^{r} \bigwedge_{j=i+1}^{p} h_j{}' dh_i.$$

The integral over $\mathcal{V}_{r,p}$ is

$$\int_{\mathcal{V}_{r,p}} dH_1 = \frac{2^r \pi^{pr/2}}{\Gamma_r(p/2)}, \qquad \Gamma_r\Big(\frac{p}{2}\Big) = \pi^{r(r-1)/4} \prod_{i=1}^{r} \Gamma\Big(\frac{p-i+1}{2}\Big).$$

Lemma 2.5 *(Theorem 2 of Uhlig (1994)) Let*

$$W = H_1 \Lambda H_1{}' \in \mathcal{S}_{r,p},$$

where $\Lambda = \mathrm{diag}(l_i)_{1 \le i \le r}$, $l_1 \ge \cdots \ge l_r$, *and* $H_1 \in \mathcal{V}_{r,p}$. *Then, the volume element of* $\mathcal{S}_{r,p}$ *at* W *is*

$$dW_{r,p} = 2^{r(r-1)/4 + r(p-r)/2} \prod_{1 \le i < j \le r} (l_i - l_j) \prod_{i=1}^{r} l_i^{p-r} \prod_{i=1}^{r} dl_i \, dH_1.$$

Corollary 2.1 *The volume element of* $\mathcal{S}_{r,p} \cap \partial U$ *is*

$$dU_{r,p} = 2^{r(r-1)/4 + r(p-r)/2} \prod_{1 \le i < j \le r} (l_i - l_j) \prod_{i=1}^{r} l_i^{p-r} d\mu_r(l) \, dH_1,$$

where $d\mu_r(l)$ *is the volume element of the surface of the unit ball* $\{l \mid l_1{}^2 + \cdots + l_r{}^2 = 1\}$.

Remark 2.6 *In Uhlig (1994), the inner product of* $\mathcal{S}_p$ *is not defined explicitly. If we adopt (27) as the inner product of* $\mathcal{S}_p$ *and regard* $\mathcal{S}_{r,p}$ *as a subspace of* $\mathcal{S}_p$, *the constant* $2^{r(r-1)/4 + r(p-r)/2}$ *is necessary in the expression of the volume element which does not appear in Theorem 2 of Uhlig (1994).*

Remark 2.7 *As mentioned in Muirhead (1982) and Uhlig (1994), we have to be careful because the sign of each* h_i *is not uniquely determined. If we integrate with respect to* dH_1 *over the whole* $\mathcal{V}_{r,p}$, *we have to divide by* 2^r.

Now we can evaluate the weights. In this case, the double integral in (18) reduces to

$$I_{r,p}(i) = \int_{\mathcal{S}^+_{r,p} \cap \partial U} \Big[\int_{\mathcal{S}^+_{p-r,p-r} \cap \partial U} \mathrm{tr}_{i-(p-r)(p-r+1)/2} \, H(\Lambda, \tilde{\Lambda}) \, dU_{p-r,p-r} \Big] dU_{r,p}, \tag{32}$$

where $H(\Lambda, \tilde{\Lambda}) = \Lambda^{-1} \otimes \tilde{\Lambda}$. Note that $\mathcal{S}_{r,p}^{+} \cap \partial U = \partial\mathcal{L}_r^{+} \times \mathcal{V}_{r,p}$ with

$$\partial\mathcal{L}_r^{+} = \{(l_1, \dots, l_r) \mid l_1 \geq \cdots \geq l_r > 0,\ {l_1}^2 + \cdots + {l_r}^2 = 1\}.$$

From Lemma 2.4 and Remark 2.7, the integral (32) is separated into two parts as

$$I_{r,p}(i) = c_p \sum_{(q,\bar{q})} \int_{\partial\mathcal{L}_r^{+}} \det({l_k}^{q_j})_{1\leq k,j\leq r} d\mu_r(l) \cdot \int_{\partial\mathcal{L}_{p-r}^{+}} \det(\tilde{l}_k^{\bar{q}_j})_{1\leq k,j\leq p-r} d\mu_{p-r}(\tilde{l}),$$

where the summation $\sum_{(q,\bar{q})}$ is over

$$(q_1, \dots, q_r, \bar{q}_1, \dots, \bar{q}_{p-r}) \in Q_{r,p}(-i - r + p(p+1)/2), \tag{33}$$

and $c_p = 2^{p(p-1)/4}\pi^{p(p+1)/4} / \prod_{k=1}^{p} \Gamma(k/2)$. Then, the mixed volume in (18) is

$$v_{p(p+1)/2-i,i} = \frac{(i-1)!\{p(p+1)/2 - i - 1\}!}{\{p(p+1)/2\}!} \sum_r I_{r,p}(i),$$

where the summation $\sum_r$ is over

$$r \in R_p(i) = \{r \mid 0 \leq i - (p-r)(p-r+1)/2 \leq r(p-r)\}, \tag{34}$$

since $\mathrm{tr}_{m'} H(\Lambda, \tilde{\Lambda}) = 0$ for $m' > r(p-r)$. From Theorem 2.1, we obtain the weights as

$$\begin{aligned} w_{p(p+1)/2-i} &= \binom{p(p+1)/2}{i} \frac{v_{p(p+1)/2-i,i}}{\omega_i\, \omega_{p(p+1)/2-i}} \\ &= \frac{1}{i\{p(p+1)/2 - i\}} \Gamma\Big(\frac{i}{2} + 1\Big) \Gamma\Big(\frac{p(p+1)/2 - i}{2} + 1\Big) \frac{2^{p(p-1)/4}}{\prod_{k=1}^{p}(k/2)} \\ &\quad \times \sum_r \sum_{(q,\bar{q})} \int_{\partial\mathcal{L}_r^{+}} \det({l_k}^{q_j})_{1\leq k,j\leq r} d\mu_r(l) \cdot \int_{\partial\mathcal{L}_{p-r}^{+}} \det(\tilde{l}_k^{\bar{q}_j})_{1\leq k,j\leq p-r} d\mu_{p-r}(\tilde{l}), \end{aligned} \tag{35}$$

where the summations $\sum_r$ and $\sum_{(q,\bar{q})}$ are over (34) and (33), respectively. We can easily see that the weights (35) coincide with Theorem 2.1 of Kuriki (1993).

Remark 2.8 *We conclude this paper by making a brief comment on the Weyl's tube formula (Weyl (1939)) and Naiman's inequality (Johnston and Siegmund (1989), Naiman (1990)). We have obtained the expressions for weights by evaluating the volume of the local parallel set, whose definition is similar to the Weyl's tube. In fact, our proof of Theorem 2.3, the extension of the Steiner's formula, is essentially equivalent to the method in Weyl (1939) (see the Appendix). Unlike the Naiman's inequality, we can restrict our attention to the local parallel sets which are defined by the projection onto the convex surface, and therefore the problem of overlapping does not matter.*

A Appendix

Internal angle and external angle

Let F be a face of a closed polyhedral convex cone K in R^p. The internal angle $\beta(0,F)$ of F at 0 (the origin) is defined as

$$\beta(0,F) = \frac{v_d(U \cap F)}{\omega_d},$$

where v_d is restricted to the affine hull $L(F)$ of F. Let $C(F,K)$ be the smallest cone containing K and $L(F)$, and let $F^* = C(F,K)^*$. F^* can also be written as

$$F^* = \{y \mid y \in K^* \text{ and } \langle x, y\rangle = 0, \ \forall x \in F\}.$$

Therefore F^* is the face of K^* dual to F of K. The external angle $\gamma(F,K)$ of K at F is defined as

$$\gamma(F,K) = \frac{v_{p-d}(U \cap F^*)}{\omega_{p-d}} = \beta(0,F^*),$$

where v_{p-d} is restricted to the affine hull $L(F^*)$. See McMullen (1975) and Section 2.4 of Schneider (1993a) for more detail.

Sketch of the Proof of Theorem 2.3

Let $s \in D_m(\partial K)$ and consider an infinitesimal spherical neighborhood $B(s) \subset D_m(\partial K)$ of s of radius Δ. The essential step of the proof is evaluating the infinitesimal contribution $v_p(A_\lambda(K, B(s)))$ of $B(s)$ to $v_p(A_\lambda(K,S))$. The rest of the proof is just integration similar to the proof of Theorem 2.2 or Theorem 2.4. Note that we only need to evaluate terms of order $O(\Delta^{p-m})$.

Now fix $y \in N(K,s)$, $l = \|y\| \le \lambda$. Define

$$B(s,y) = (y + D_m(\partial K)) \cap A_\lambda(K, B(s))$$

where $y + D_m(\partial K)$ is $D_m(\partial K)$ translated to go through the point $P = s + y$. $B(s,y)$ is orthogonal to $N(K,s)$ and hence $v_p(A_\lambda(K,B(s)))$ can be evaluated as

$$v_p(A_\lambda(K,B(s))) = \int_{N(K,s)\cap\lambda U} v_{p-m}(B(s,y))dy$$

where dy is the standard volume element of R^m.

For $v = y/l$ and let $G = G_v$ be the associated Weingarten map. By definition of G_v

$$B(s,y) = P + \cup_{s' \in B(s)}(s' - s + lG_v(s' - s)) + o(\Delta).$$

With respect to an appropriate orthonormal basis around s, the elements of G_v are the elements of the second fundamental form $H(s,v)$. Hence

$$\begin{aligned} v_{p-m}(B(s,y)) &= \det(I_{p-m} + lH(s,v))\, v_{p-m}(B(s)) + o(\Delta^{p-m}) \\ &= (1 + l\mathrm{tr}_1 H(s,v) + \cdots + l^{p-m}\mathrm{tr}_{p-m}H(s,v))v_{p-m}(B(s)) + o(\Delta^{p-m}). \end{aligned}$$

The rest of the proof is integration similar to the proof of Theorem 2.2 or Theorem 2.4 and omitted.

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