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The structure of the center of the universal enveloping algebra for the Lie superalgebra $\mathfrak{sl}(m,1)$

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1 Introduction

One of the fundamental tools in the representation theory of finite-dimensional Lie algebras is the Harish-Chandra isomorphism. It gives an identification between the center of the universal enveloping algebra of a simple finite-dimensional Lie algebra and a certain algebra of symmetric polynomials. It is natural to ask if the similar result holds for simple finite-dimensional Lie superalgebras. Unfortunately the Harish-Chandra homomorphism is not necessarily an isomorphism for Lie superalgebras. The lack of reflections attached to roots of length zero causes the situation where the Harish-Chandra homomorphism is not surjective. Thus for Lie superalgebras, the determination of the image of the Harish-Chandra homomorphism is a real problem. There is a general result in this direction obtained by F.A.Berezin [1] and V.G.Kac [5]. In this talk we shall give more explicit and elementary description of the image of Harish-Chandra homomorphism for $\mathfrak{sl}(m,1)$.

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2 Preliminaries

As for the elementary facts about Lie superalgebras we refer to [2].

Let $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$ be a finite-dimensional Lie superalgebra $\mathfrak{sl}(m,1)$ $(m\geq 2)$ over $\mathbf{C}$. We write $\mathfrak{h}$ for a Cartan subalgebra of $\mathfrak{g}_0$ and $\Pi=\{\alpha_1,\ldots,\alpha_m\}\subset\mathfrak{h}^*$ for the set of simple roots. $\Pi^\vee=\{h_1,\ldots,h_m\}\subset\mathfrak{h}$ denotes the set of corresponding simple coroots. We denote by Δ_+^{even} and Δ_+^{odd} the sets of even and odd positive roots, respectively.

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The generators $\{e_i, f_i, h_i \mid (1 \leq i \leq m)\}$ is so chosen that e_m and f_m are the only odd generators. The defining relations are:

$$[e_i, f_j] = \delta_{i,j} h_i, \quad [h_i, h_j] = 0, \quad [h_i, e_j] = a_{i,j} e_j, \quad [h_i, f_j] = -a_{i,j} f_j,$$

where

$$a_{ij} = \begin{cases} 2 & i = j, \\ -1 & j = i+1 \text{ or } i-1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $(x|y) = \phi(x,y)/2h^\vee$ be the non-degenerate even invariant bilinear form on $\mathfrak{g}$, where ϕ is the Killing form and $h^\vee = m-1$ is the dual Coxeter number . We have a triangular decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+,$$

where $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) is the subalgebra of $\mathfrak{g}$ generated by $e_1, \ldots, e_m$ (resp. $f_1, \ldots, f_m$).

For a Lie superalgebra $\mathfrak{s}$, we write $U(\mathfrak{s})$ for its universal enveloping algebra. Let δ be the projection:

$$\delta : U(\mathfrak{g}) = (U(\mathfrak{g})\mathfrak{n}_+ + \mathfrak{n}_- U(\mathfrak{g})) \oplus U(\mathfrak{h}) \longrightarrow U(\mathfrak{h}).$$

We define $\gamma : \mathfrak{h} \to U(\mathfrak{h})$ by

$$\gamma(h) := h - (\rho|h) \cdot 1,$$

where $\rho := (\sum_{\alpha \in \Delta_+^{\text{even}}} \alpha - \sum_{\alpha \in \Delta_+^{\text{odd}}} \alpha)/2$. Extend this to an algebra automorphism of $U(\mathfrak{h})$. Then the composite $\gamma \circ \delta$ induces a homomorphism

$$\iota : U(\mathfrak{g})^{\mathfrak{g}} \longrightarrow U(\mathfrak{h})^W.$$

Here the center of $U(\mathfrak{g})$ denotes

$$U(\mathfrak{g})^{\mathfrak{g}} := \{f \in U(\mathfrak{g}) \mid [f, x] = 0 \text{ for any } x \in \mathfrak{g}\},$$

and $U(\mathfrak{h})^W$ stands for the set of elements of $U(\mathfrak{h})$ fixed by the Weyl group W. This ι is called the *Harish-Chandra homomorphism* for $\mathfrak{g}$.

3 An "odd roots condition" for the image

Here we shall prove a key lemma. This is inspired by the proof of Lemma 3 in [3].

Lemma 1 *Let $\mathfrak{g}$ be a finite-dimensional simple Lie superalgebra and ι the Harish-Chandra homomorphism. We denote by $(\cdot|\cdot)$ the non-degenerate even invariant bilinear form defined in [2]. We write $\mathfrak{C}$ for the algebra consisting of $f \in U(\mathfrak{h})^W$ with the property*

$$f(\Lambda + \rho) = f(\Lambda - k\beta + \rho), \quad \forall k \in \mathbf{Z}$$

for any $\beta \in \Delta_+^{\text{odd}} \cap \Pi$ and $\Lambda \in \mathfrak{h}^*$ satisfying $(\beta|\beta) = (\beta|\Lambda + \rho) = 0$. Then the image of ι is contained in $\mathfrak{C}$.

Proof. Let $\beta \in \Delta_+^{\mathrm{odd}} \cap \Pi$ and $\Lambda \in \mathfrak{h}^*$ be such that $(\beta|\beta) = (\beta|\Lambda+\rho) = 0$. Let $M(\Lambda)$ (resp. $M(\Lambda-\beta)$) be the Verma module with the highest weight Λ (resp. $\Lambda-\beta$) and $v_\Lambda \in M(\Lambda)_\Lambda$ (resp. $u_{\Lambda-\beta} \in M(\Lambda-\beta)_{\Lambda-\beta}$) its highest weight vector.

For each $z \in U(\mathfrak{g})^{\mathfrak{g}}$ we write $f_z \in P(\mathfrak{h}^*) = S(\mathfrak{h}) = U(\mathfrak{h})$ for the image $\iota(z)$. Here $S(\mathfrak{h})$ denotes the symmetric algebra over $\mathfrak{h}$ which is canonically isomorphic to the algebra of polynomial functions $P(\mathfrak{h}^*)$ over $\mathfrak{h}^*$. As is well known, each $z \in U(\mathfrak{g})^{\mathfrak{g}}$ acts on v_Λ and $u_{\Lambda-\beta}$ by $f_z(\Lambda+\rho)$ and $f_z(\Lambda-\beta+\rho)$, respectively. Thus z acts on $v_{\Lambda-\beta} \in M(\Lambda)_{\Lambda-\beta}$ as $f_z(\Lambda+\rho)$-multiplication also. Since $v_{\Lambda-\beta}$ is a singular vector in $M(\Lambda)$ we must have $M(\Lambda) \supset M(\Lambda-\beta)$. It follows that

$$f_z(\Lambda+\rho) = f_z(\Lambda+\rho-\beta), \quad \forall z \in U(\mathfrak{g})^{\mathfrak{g}}.$$

This formula is valid for any $M(\Lambda-k\beta)$ $(k \in \mathbf{Z})$. Hence we must have

$$f_z(\Lambda+\rho) = f_z(\Lambda+\rho-k\beta), \quad \forall k \in \mathbf{Z}.$$

□

Next we give an explicit description of this $\mathfrak{C}$ in the case of $\mathfrak{sl}(m,1)$. The only simple odd root β of length zero is α_m. Any $\Lambda+\rho \in \mathfrak{h}^*$ orthogonal to β is of the form $\sum_{i=1}^{m-1} a_i\varepsilon_i$, where $\{\varepsilon_i\}_{i=1}^m$ is the standard basis of the weight lattice of $\mathfrak{h}$:

$$\varepsilon_i : \mathfrak{h} \ni \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_{m+1} \end{pmatrix} \longmapsto x_i \in \mathbf{C}.$$

Then the condition on $f \in \mathfrak{C}$ reads

$$f\Big(\sum_{i=1}^{m-1} a_i\varepsilon_i\Big) = f\Big(\sum_{i=1}^{m-1} (a_i+k)\varepsilon_i\Big), \quad \forall k \in \mathbf{Z}.$$

Write $\lambda+\rho \in \mathfrak{h}^*$ as $\sum_{i=1}^m z_i\varepsilon_i$. This identifies $U(\mathfrak{h})^W$ with the space of symmetric polynomials in $z_1,\ldots,z_m$. Then $f(z_1,\ldots,z_m) \in U(\mathfrak{h})^W$ belongs to $\mathfrak{C}$ if and only if

$$(1) \qquad f(z_1,\ldots,z_{m-1},0) = f(z_1+k,\ldots,z_{m-1}+k,0), \quad \forall k \in \mathbf{Z}.$$

This condition is automatically satisfied if $f(z_1,\ldots,z_m)$ is divisible by z_m. Since $f(z_1,\ldots,z_m)$ is symmetric, this implies that $f(z_1,\ldots,z_m)$ is divisible by $z_1\cdots z_m$.

Noting that $U(\mathfrak{h})^W = \mathbf{C}[\mu_1,\ldots,\mu_m]$ with

$$\mu_j(\lambda) := \sum_{1\le i_1<\cdots<i_j\le m} z_{i_1}\cdots z_{i_j}, \quad (1 \le j \le m),$$

we have

$$(2) \qquad \mathfrak{C} = \mu_m \cdot \mathbf{C}[\mu_1,\ldots,\mu_m] \oplus \Big(\mathbf{C}[\mu_1,\ldots,\mu_{m-1}] \cap \mathfrak{C}\Big).$$

4 Image of the Harish-Chandra homomorphism

Theorem 2 *Let* $\mathfrak{g} := \mathfrak{sl}(m,1)$ *and* ι *its Harish-Chandra homomorphism. Then the image of* ι *coincides with the algebra* $\mathfrak{C}$ *of Lemma 1 .*

The rest of this note will be devoted to the proof of this theorem. We use the following well-known construction of elements of $U(\mathfrak{g})^{\mathfrak{g}}$ via the supertrace of representations of $\mathfrak{g}$.

4.1 Supertraces as central elements

Let $V = V_0 \oplus V_1$ be a superspace, i.e. a $\mathbf{Z}_2$-graded $\mathbf{C}$-vector space. $T(V) = \bigoplus_{k=0}^{\infty} T^k(V)$ denotes its tensor algebra. We write $S(V) = \bigoplus_{k=0}^{\infty} S^k(V)$ for the super symmetric algebra of V, which is the quotient algebra of $T(V)$ by the ideal $\mathcal{I}(V)$ generated by elements of the form

$$x \otimes y - (-1)^{p(x)p(y)} y \otimes x, \quad (x,\, y \in \mathfrak{g}_0 \text{ or } \mathfrak{g}_1),$$

where $p(a) := i$ for $a \in \mathfrak{g}_i$. We write X_S for the image of $X \in T(V)$ in $S(V)$ by the projection

$$T(V) \twoheadrightarrow T(V)/\mathcal{I}(V) = S(V). \tag{3}$$

$S(V)$ can also be realized as the subspace of $T(V)$ spanned by elements of the form

$$(X_1 \otimes \cdots \otimes X_k)^S := \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} (\pm 1) X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(k)}, \quad X_i \in \mathfrak{g}_0 \text{ or } \mathfrak{g}_1 \ (1 \le i \le k).$$

Here the sign (± 1) is determined by the super rule: transposition of elements X_i and X_j causes $(-1)^{p(X_i)p(X_j)}$-multiplication on the sign.

We now return to the general Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. We write $\mathrm{gr}U(\mathfrak{g})$ for the graded algebra of $U(\mathfrak{g})$ with respect to the standard filtration. Just as in the Lie algebra case, $\mathrm{gr}U(\mathfrak{g})$ is isomorphic to the super symmetric algebra $S(\mathfrak{g})$. Furthermore the choice of $\mathfrak{g}$-invariant pairing on $\mathfrak{g}$ enables us to identify $S(\mathfrak{g})$ with $S(\mathfrak{g}^*)$. Since all of these isomorphisms are $\mathfrak{g}$-equivariant, the composite of them gives rise to an isomorphism:

$$U(\mathfrak{g})^{\mathfrak{g}} \underset{\mathfrak{g}\text{-module}}{\xrightarrow{\sim}} S(\mathfrak{g}^*)^{\mathfrak{g}}.$$

Thus we are reduced to construct elements in $S(\mathfrak{g}^*)^{\mathfrak{g}}$.

Let (π, V) be a finite-dimensional representation of $\mathfrak{g}$. This gives a linear form on $T^k(\mathfrak{g})$:

$$\Phi_k(\pi) : T^k(\mathfrak{g}) \ni (X_1 \otimes \cdots \otimes X_k) \longmapsto \mathrm{str}(\pi(X_1) \circ \cdots \circ \pi(X_k)) \in \mathbf{C}, \tag{4}$$

which is obviously $\mathfrak{g}$-invariant. (Recall that $\mathfrak{g}$-invariance means

$$\Phi_k(\pi)(\mathrm{ad}^{\otimes k}(Y)(X_1 \otimes \cdots \otimes X_k)) = 0, \quad \forall Y \in \mathfrak{g}.)$$

Restriction of this to the subspace

$$S^k(\mathfrak{g}) = \mathrm{Span}\{(X_1 \otimes \cdots \otimes X_k)^S \mid X_i \in \mathfrak{g}_0 \text{ or } \mathfrak{g}_1 \ (1 \le i \le k)\}$$

gives a desired element $\Phi_k(\pi) \in [S^k(\mathfrak{g})^*]^{\mathfrak{g}} = S^k(\mathfrak{g}^*)^{\mathfrak{g}} \subset S(\mathfrak{g}^*)^{\mathfrak{g}}$.

4.2 The image of supertraces under ι

To describe the image of $\Phi_k(\pi) \in S(\mathfrak{g}^*)^{\mathfrak{g}} \simeq U(\mathfrak{g})^{\mathfrak{g}}$ under ι we need to transport $\iota : U(\mathfrak{g})^{\mathfrak{g}} \to U(\mathfrak{h})^W$ to $\iota : S(\mathfrak{g}^*)^{\mathfrak{g}} \to S(\mathfrak{h}^*)^W$. The well-known decomposition:

$$\mathrm{gr}U(\mathfrak{g})^{\mathfrak{g}} \subset U(\mathfrak{h}) \oplus \mathrm{gr}U(\mathfrak{g})\mathfrak{n}_+$$

restricted to the degree k component projects to

$$S^k(\mathfrak{g})^{\mathfrak{g}} \subset S^k(\mathfrak{h}) \oplus (S(\mathfrak{g})\mathfrak{n}_+)_S.$$

Here $(S(\mathfrak{g})\mathfrak{n}_+)_S$ is the image of $S(\mathfrak{g})\mathfrak{n}_+$ by the map (3). The identification $S^k(\mathfrak{g}) \simeq S^k(\mathfrak{g}^*)$ composed with the canonical isomorphism $S^k(\mathfrak{g}^*) = S^k(\mathfrak{g})^*$ sends this to

$$[S^k(\mathfrak{g})^*]^{\mathfrak{g}} \subset S^k(\mathfrak{h})^* \oplus \left((S(\mathfrak{g})\mathfrak{n}_+)^S\right)^*.$$

This consideration combined with the definition of ι yields that $\iota : S(\mathfrak{g}^*)^{\mathfrak{g}} \to S(\mathfrak{h}^*)^W$ equals the composite

$$S(\mathfrak{g}^*)^{\mathfrak{g}} \xrightarrow{=} [S(\mathfrak{g})^*]^{\mathfrak{g}} \ni \Phi \longmapsto \Phi|_{S(\mathfrak{h})} \in [S(\mathfrak{h})^*]^W \xrightarrow{=} S(\mathfrak{h}^*)^W.$$

We apply this construction to the case when $\mathfrak{g} = \mathfrak{sl}(m,1)$ and π is the standard representation. Then $\Phi_k(\pi)$ $(k \geq 2)$ in (4) restricted to $S^k(\mathfrak{h})$ is simply

$$S^k(\mathfrak{h}) \ni (X_1 \otimes \cdots \otimes X_k)^S \longmapsto \mathrm{str}(X_1 \cdots X_k) \in \mathbf{C}.$$

As an element of $S(\mathfrak{h}^*)$, this can be expressed in terms of the basis $\{\varepsilon_i\}_{1\leq i\leq m}$ as

$$c_k := \varepsilon_1^{\otimes k} + \cdots + \varepsilon_m^{\otimes k} - \Big(\sum_{i=1}^{m} \varepsilon_i\Big)^{\otimes k}, \quad k \geq 2.$$

4.3 Proof of Theorem 2

Lemma 1 implies

$$\langle c_k \mid k \geq 2\rangle_{\mathbf{C}} \subset \mathrm{Im}\iota \subset \mathfrak{C},$$

where $\langle c_k \mid k \geq 2\rangle_{\mathbf{C}}$ denotes the algebra generated by $\{c_k\}_{k\geq 2}$ over $\mathbf{C}$. Our goal is to show $\langle c_k \mid k \geq 2\rangle_{\mathbf{C}} = \mathfrak{C}$.

Lemma 3 *We have the following decomposition*

$$\mathfrak{C} = \mu_m \cdot \mathbf{C}[\mu_1, c_2, \ldots, c_m] \oplus \mathbf{C}[c_2, \ldots, c_{m-1}].$$

Proof. We can rewrite (2) as

$$\mathfrak{C} = \mu_m \cdot \mathbf{C}[\mu_1, c_2, \ldots, c_m] \oplus \left(\mathbf{C}[\mu_1, c_2, \ldots, c_{m-1}] \cap \mathfrak{C}\right).$$

Thus we have only to check that $\mathbf{C}[\mu_1, c_2, \ldots, c_{m-1}] \cap \mathfrak{C}$ coincides with $\mathbf{C}[c_2, \ldots, c_{m-1}]$. Note that our form of $f(z_1, \ldots, z_m)$ allows us to replace $k \in \mathbf{Z}$ with $k \in \mathbf{R}$ in (1). Thus for $f \in \mathbf{C}[\mu_1, c_2 \ldots, c_m]$ to belong to $\mathfrak{C}$ it is necessary and sufficient that

$$f(z_1, \ldots, z_{m-1}, 0) = f(z_1 + k, \ldots, z_{m-1} + k, 0), \quad \forall k \in \mathbf{R}.$$

By differentiating this in k we have

$$\mathbf{C}[\mu_1, \ldots, \mu_{m-1}] \cap \mathfrak{C} \subset \{f \in \mathbf{C}[\mu_1, \ldots, \mu_{m-1}] \mid Df(z_1, \ldots, z_{m-1}, 0) = 0\},$$

where $D := \sum_{j=1}^{m-1} \frac{\partial}{\partial z_j}$. If we write $f \in \mathbf{C}[\mu_1, \ldots, \mu_{m-1}]$ as $\sum_{j=0}^{n} b_j \mu_1^j$ ($b_j \in \mathbf{C}[c_2, \ldots, c_{m-1}] \subset \mathfrak{C}$), then

$$\begin{aligned} Df(z_1, \ldots, z_{m-1}, 0) = &\sum_{j=0}^{n} \big(Db_j(z_1, \ldots, z_{m-1}, 0)\big)\mu_1^j(z_1, \ldots, z_{m-1}, 0) \\ &+ \sum_{j=0}^{n} b_j(z_1, \ldots, z_{m-1}, 0)(m-1)j \cdot \mu_1^{j-1}(z_1, \ldots, z_{m-1}, 0). \end{aligned}$$

This is identically zero if and only if

$$b_j(z_1, \ldots, z_{m-1}, 0) = 0, \quad (1 \leq j).$$

Hence the assertion follows. □

Lemma 4 *We have*

$$\mu_1^k \mu_m \in \langle c_j \mid j \geq 2 \rangle_{\mathbf{C}}, \quad (0 \leq k).$$

Proof.

It is sufficient to show the following formula of symmetric polynomials:

(5)

$$\left(\frac{\mu_1}{m-1}\right)^k \mu_m = \sum_{\substack{i_2, \ldots, i_{m+k} \in \mathbf{Z}_{\geq 0} \\ 2i_2 + 3i_3 + \cdots + (m+k)i_{m+k} = m+k}} \frac{1}{i_2! i_3! \cdots i_{m+k}!} \left(\frac{c_2}{2}\right)^{i_2} \cdots \left(\frac{c_{m+k}}{m+k}\right)^{i_{m+k}} \quad 0 \leq k.$$

We consider ε_i as indeterminates.

Set

$$\varphi_0 := 1, \qquad \varphi_k := \sum_{1 \leq i_1 < \cdots < i_k \leq m} \varepsilon_{i_1} \otimes \cdots \otimes \varepsilon_{i_k} \quad k \geq 1.$$

Let $\{\varepsilon_i^\vee\}_{1 \leq i \leq m}$ be the basis of $\mathfrak{h}$ which is dual to $\{\varepsilon_i\}_{1 \leq i \leq m}$. Then our identification yields $\varepsilon_j^\vee = \varepsilon_j - \sum_{i=1}^{m} \varepsilon_i$ and we have

$$\begin{aligned} \mu_m &= \prod_{j=1}^{m} \left(\varepsilon_j - \sum_{i=1}^{m} \varepsilon_i\right) = \sum_{j=0}^{m} (-1)^j \Big(\sum_{i=1}^{m} \varepsilon_i\Big)^j \varphi_{m-j} \\ &= \sum_{j=0}^{m} (-1)^j \varphi_1^j \varphi_{m-j}. \end{aligned} \tag{6}$$

Next we note

$$\sum_{n=1}^{\infty}(\varepsilon_1^n+\cdots+\varepsilon_m^n)\frac{t^n}{n}=-\log\Bigl(\prod_{j=1}^{m}(1-\varepsilon_j t)\Bigr).$$

The left hand side reads:

$$\sum_{n=1}^{\infty}\frac{t^n}{n}\bigl(c_n+\varphi_1^n\bigr)=\sum_{n=1}^{\infty}\frac{t^n}{n}c_n-\log(1-\varphi_1 t),\quad (c_1:=0).$$

Thus

$$\log\left(\frac{\prod_{j=1}^{m}(1-\varepsilon_j t)}{1-\varphi_1 t}\right)=-\sum_{n=2}^{\infty}\frac{c_n}{n}t^n.$$

Exponentiating this and expanding it in t, we have

$$\left(\sum_{j=0}^{m}(-1)^j\varphi_j t^j\right)\left(\sum_{j=0}^{\infty}\varphi_1^j t^j\right)$$
$$=\sum_{n=2}^{\infty}\sum_{\substack{i_2,\ldots,i_n\in\mathbf{Z}_{\geq 0}\\ 2i_2+3i_3+\cdots+(n)i_n=n}}\frac{t^n}{i_2!i_3!\cdots i_n!}\left(-\frac{c_2}{2}\right)^{i_2}\cdots\left(-\frac{c_n}{n}\right)^{i_n},\quad (0\leq k).$$

Using (6) the coefficient of $t^{m+k}(0\leq k)$ in the left hand side becomes:

$$\sum_{j=0}^{m}(-1)^{m-j}\varphi_{m-j}\varphi_1^{k+j}=(-1)^m\varphi_1^k\mu_m=(-1)^{m+k}\left(\frac{\mu_1}{m-1}\right)^k\mu_m,$$

and (5) follows. □

Lemmas 3 and 4 show that $\mathfrak{C}=\langle c_j\,|\,j\geq 2\rangle_{\mathbf{C}}$. Hence Theorem 2 is proved. This also gives an explicit description of Imι. Moreover we can deduce the Euler-Poincaré series of Imι from Theorem 2 and Lemma 3.

Corollary 5 *The Euler-Poincaré series $P(t)$ of* Imι *for $\mathfrak{g}=\mathfrak{sl}(m,1)$ is given by:*

$$P(t)=\prod_{j=2}^{m-1}\sum_{n=0}^{\infty}t^{nj}+t^m\prod_{j=1}^{m}\sum_{n=0}^{\infty}t^{nj}.$$

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