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NONLINEAR SCHRÖDINGER EQUATIONS IN FRACTIONAL ORDER SOBOLEV SPACES

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In this note I describe some recent work on nonlinear Schrödinger equations, done jointly with M. Nakamura [27, 28]. We consider the nonlinear Schrödinger equations of the form

$$i\partial_t u + \Delta u = f(u), \tag{1}$$

where u is a complex-valued function of $(t, x) \in \mathbf{R} \times \mathbf{R}^n$, $\partial_t = \partial/\partial t, \Delta$ is the Laplacian in $\mathbf{R}^n$, and f is a complex-valued function, a typical form of which is the single power interaction

$$f(u) = \lambda |u|^{p-1} u \tag{2}$$

with $\lambda \in \mathbf{R}$ and $1 < p < \infty$.

There is a large literature on the Cauchy problem for the equation (1) and on the asymptotic behavior in time of the global solutions [2, 4, 5-9, 12-17, 22, 25, and references therein]. The Cauchy problem for (1) has been studied mainly in the Sobolev spaces H^m of integral order m, especially $m = 0, 1, 2$, while there arises a new interest in the treatment of the Cauchy problem in the Sobolev spaces $H^s = (1-\Delta)^{-s/2} L^2(\mathbf{R}^n)$ of fractional order s with $0 \leq s < n/2$. In [5], Cazenave and Weissler proved that the Cauchy problem for (1) with (2) has global solutions in H^s for the data $\phi \in H^s$ with $||(-\Delta)^{s/2}\phi; L^2||$ sufficiently small, provided that $p = 1 + 4/(n-2s)$ and $[s] < p - 1$, where $[s]$ is the greatest integer that is less than or equal to s. In [14], Kato generalized the results in [5] in some directions. In [7], Ginibre, Ozawa, and Velo proved the existence and asymptotic completeness of the wave operators for (1) with a class of interactions including (2) on small asymptotic states in H^s, provided that $1 + 4/n \leq p \leq 1 + 4/(n-2s)$ and $s < \min(2, p)$. In [22], Pecher proved that the Cauchy problem for (1) with (2) has global solutions in H^s for small data in H^s, provided that $1 + 4/n \leq p < 1 + 4/(n-2s)$ and $1 < s < \min(4, p+1)$ or $4 \leq s < p + 2$. In connection with the H^s theory for (1) with (2), a homogeneity argument indicates that the power p in (2) is critical [resp. subcritical] at the level of H^s if and only if $p = 1 + 4/(n-2s)$ [resp.

$p < 1 + 4/(n-2s)$]. To sum up with this definition, the critical case is studied in [5, 7, 14] and the subcritical case is studied [7, 14, 22].

The purpose of this paper is to study the H^s theory for (1) with a class of interactions including (2) in more detail both in the critical and subcritical cases in the framework of low energy scattering. We prove the existence and asymptotic completeness of the wave operators for (1) on small asymptotic states in H^s in the critical case with $s < \min(n/2, p)$ as well as in the subcritical case with $s < p$. Moreover, smallness assumption is shown to be necessary only for the L^2 norm of the fractional derivative $(-\Delta)^{s_0/2}\phi$ of the data $\phi \in H^s$, where $s_0 \equiv n/2 - 2/(p-1)$.

Here, when p is not an odd integer, an additional assumption such as $s < p$ is required to keep the smoothness of f compatible with the behavior at zero. Concerning the number s_0, we notice the following simple facts: (1) $s = s_0$ in the critical case. (2) $s_0 < s$ in the subcritical case. (3) p is critical at the level of H^{s_0}. (4) $0 \leq s_0 < n/2$.

As we see above, as regards the H^s-theory with $0 \leq s < n/2$, the power behavior of the nonlinearity determines the order of the Sobolev space where the smallness of the data is imposed to ensure the existence and uniqueness of global H^s-solutions. This is the right phenomenon, as is usual with other nonlinear evolution equations with dilation structure, such as the heat equation with single power interaction and the Navier-Stokes equations.

In contrast, when $s > n/2$, no specific behavior of the nonlinearity is required of the H^s-theory for (1) at least locally in time. In fact, when $s > n/2$, for the existence and uniqueness of local H^s- solutions one has only to assume that $f \in C^k(\mathbf{C}; \mathbf{C})$ with $f(0) = 0$, where differentiability refers to the real sense and k is the smallest integer greater than or equal to s. The proof depends on the usual Sobolev embedding $H^s \subset L^\infty$ for $s > n/2$ in an essential way.

The case $s = n/2$ may therefore be regarded as the borderline in two aspects: (1) No power behavior of interaction amounts to the critical nonlinearity at the level of $H^{n/2}$. (2) Poinwise control of solutions falls beyond the scope of the $H^{n/2}$-theory, so that any argument similar to that of the H^s-theory with $s > n/2$ breaks down even for local theory without specific behavior of interaction.

In addition to the critical phenomena described above, $H^{n/2}$-solutions deserve attention as finite energy solutions for $n = 2$ and as strong solutions for $n = 4$.

We prove the existence and uniqueness of global $H^{n/2}$-solutions to (1) with small Cauchy data under the nonlinearity of exponential type. This is reminiscent of Trudinger's inequality, which replaces the Sobolev embedding in the limiting case on the basis of the exponential estimates in terms of functions in the critical order Sobolev space $H^{n/2}$.

To state the results precisely, we use the following notation. For any r with $1 \leq r \leq \infty$, $L^r = L^r(\mathbf{R}^n)$ denotes the Lebesgue space on $\mathbf{R}^n$. For any $s \in \mathbf{R}$ and any r with $1 < r < \infty$, $H^s_r = (1-\Delta)^{-s/2}L^r$ denotes the Sobolev space defined in terms of Bessel potentials. For any $s \in \mathbf{R}$ and any r, m with $1 \leq r, m \leq \infty$, $B^s_{r,m}$ denotes the Besov space defined as the space of distributions u such that $\{2^{sj}\|\phi_j * u; L^r\|\}_{j=0}^{\infty} \in \ell^m$, where $\{\phi_j\}$ is a dyadic decomposition on $\mathbf{R}^n$. For any $s \in \mathbf{R}$ and any r with $1 < r < \infty$, $\dot{H}^s_r$ denotes the homogeneous Sobolev space defined as the space of classes of distributions u modulo polynomials such that $(-\Delta)^{s/2}u \in L^r$. For any $s \in \mathbf{R}$ and any r, m with $1 \leq r, m \leq \infty$, $\dot{B}^s_{r,m}$ denotes the homogeneous Besov space defined as the space of classes of distributions u modulo polynomials such that $\{2^{sj}\|\psi_j * u; L^r\|\}_{j=-\infty}^{\infty} \in \ell^m$, where $\{\psi_j\}$ is a dyadic decomposition on $\mathbf{R}^n \backslash \{0\}$. We refer to [1, 10, 24] for general information on Besov and Triebel-Lizorkin spaces and their homogeneous versions. For simplicity, we put $H^s = H^s_2, \dot{H}^s = \dot{H}^s_2, B^s_r = B^s_{r,2}, \dot{B}^s_r = \dot{B}^s_{r,2}$. For any interval $I \subset \mathbf{R}$ and any Banach space X we denote by $C(I; X)$ the space of strongly continuous functions from I to X and by $L^q(I; X)$ the space of strongly measurable functions u from I to X such that $\|u(\cdot); X\| \in L^q(I)$. Let $U(t) = \exp(it\Delta)$ be the free propagator, namely the one parameter group which solves the free Schrödinger equation. For any r with $2 \leq r \leq \infty$, we define $\delta(r) = n/2 - n/r$. Concerning the space-time integrability properties with respect to $U(\cdot)$, it is convenient to call a pair of exponents (q, r) admissible if $0 \leq 2/q = \delta(r) < 1$, which is understood to be $0 \leq 2/q = \delta(r) \leq 1/2$ when $n = 1$. The Cauchy problem for the equation (1) with data $u(t_0) = U(t_0)\phi$ at time t_0 will be treated in the form of the integral equation

$$
\begin{aligned}
u(t) &= U(t-t_0)u(t_0) - i\int_{t_0}^{t} U(t-\tau)f(u(\tau))d\tau \\
&= U(t)\phi - i(G_{t_0}f(u))(t),
\end{aligned} \tag{3}
$$

where the second line is understood to define the integral operator G_{t_0}. The first line of (3) is formally equivalent to (1) with Cauchy data $u(t_0)$ given at finite time t_0, while the second line will be used to describe the Cauchy problem for (1) with data ϕ at time $t_0 = 0$ as well as at $t_0 = \pm\infty$. The integral equation (3) will be studied in the spaces X^s and Y^s with $s \geq 0$ defined as

$$X^s = C(\mathbf{R}; H^s) \cap \bigcap_{0 \leq 2/q = \delta(r) < 1} L^q(\mathbf{R}; B^s_r),$$

$$Y^s = C(\mathbf{R}; H^s) \cap \bigcap_{0 \leq 2/q = \delta(r) < 1} L^q(\mathbf{R}; H^s_r).$$

Note that $X^s \subset Y^s$. For the nonlinear interaction f behaving as a power p at zero, we introduce the following assumptions $(\mathrm{A})_k$ and $(\mathrm{B})_k$ with integer k with $0 \leq k \leq p$.

$(\mathrm{A})_k$ $\quad f \in C^k(\mathbf{C}; \mathbf{C})$ and $f^{(j)}(0) = 0$ for all j with $0 \leq j \leq k$. There exists a constant C such that for all $z_1, z_2 \in \mathbf{C}$

$$|f^{(k)}(z_1) - f^{(k)}(z_2)| \leq \begin{cases} C(|z_1|^{p-k-1} + |z_2|^{p-k-1})|z_1 - z_2| & \text{if } \ p \geq k+1, \\ C|z_1 - z_2|^{p-k} & \text{if } \ p < k+1. \end{cases}$$

$(\mathrm{B})_k$ $\quad f \in C^k(\mathbf{C}; \mathbf{C})$ and $f^{(j)}(0) = 0$ for all j with $0 \leq j \leq \max(k-1, 0)$. There exists a constant C such that for all $z \in \mathbf{C}$

$$|f^{(k)}(z)| \leq C|z|^{p-k}.$$

Here $f^{(j)}$ denotes any of the j-th order derivatives of f with respect to z and $\overline{z}$ and $|f^{(j)}|$ denotes the maximum of the moduli of those derivatives. Note that $(A)_k$ implies $(B)_k$ and that $(A)_k$ [resp. $(B)_k$] implies $(A)_j$ [resp. $(B)_j$] for all j with $0 \leq j \leq k$. Single power interaction (2) satisfies $(A)_k$ with $0 \leq k < p$ (see [11]).

With the notation above we now state the main results in this paper. Theorem 1 is devoted to the critical case and Theorem 2 is devoted to the subcritical case. For any s, p, ϵ with $s \geq s_0 \equiv n/2 - 2/(p-1) \geq 0, \epsilon > 0$, we define

$$\dot{B}_\epsilon = \{\psi \in H^s; ||\psi; \dot{H}^{s_0}|| < \epsilon\}.$$

Theorem 1. *(I) Let s and p satisfy*

$$0 < s < n/2,$$

$$s < p = 1 + 4/(n - 2s).$$

Let f satisfy $(A)_{[s]}$. Then there exists $\epsilon > 0$ with the following property.

(1) For any data $\phi \in \dot{B}_\epsilon$ at time $t_0 = 0$ the equation (3) has a unique solution $u \in X^s$.

(2) For any data $\phi_+ \in \dot{B}_\epsilon$ *at time* $t_0 = +\infty$ *the equation (3) has a unique solution* $u \in X^s$ *such that*

$$||u(t) - U(t)\phi_+; H^s|| \longrightarrow 0 \quad as \quad t \longrightarrow +\infty \tag{4$_+$}$$

(3) For any data $\phi_- \in \dot{B}_\epsilon$ *at time* $t_0 = -\infty$ *the equation (3) has a unique solution* $u \in X^s$ *such that*

$$||u(t) - U(t)\phi_-; H^s|| \longrightarrow 0 \quad as \quad t \longrightarrow -\infty. \tag{4$_-$}$$

(4) For any $\phi \in \dot{B}_\epsilon$ *at time* $t_0 = 0$ *there exists a unique pair of asymptotic states* $\phi_\pm \in H^s$ *satisfying* $(4)_\pm$, *where* u *is the unique solution given by Part (1).*

(II) Let an integer s *and* p *satisfy*

$$0 \le s < n/2,$$
$$s \le p = 1 + 4/(n - 2s).$$

Let f *satisfy* $(B)_s$. *Then all the conclusions of Part (I) hold if* X^s *is replaced by* Y^s *throughout the statement of Part (I).*

Remark 1[2, 6, 15]. The power $p = 1 + 4/(n - 2s)$ comes out as a critical one in H^s in the sense that $||u; \dot{H}^s||$ is invariant under the dilation $u \mapsto u_\lambda$ if and only if $s = n/2 - 2/(p-1)$, where $u_\lambda(t, x) \equiv \lambda^{-2/(p-1)}u(\lambda^{-2}t, \lambda^{-1}x)$ with $\lambda > 0$ and the dilation above leaves (1) with (2) invariant. Another characterization is given as the power which makes the estimates of the form

$$||G_0 f(u); L^q(\mathbf{R}; \dot{H}^s_r)|| \le C||u; L^q(\mathbf{R}; \dot{H}^s_r)||^p$$

with any admissible pair (q, r) invariant under the dilation $u \mapsto u_\lambda$, where $u_\lambda(t, x) \equiv u(\lambda^{-2}t, \lambda^{-1}x)$ with $\lambda > 0$.

Remark 2. In part (I) of Theorem 1, the assumption $s < n/2$ is required to keep the critical power finite, while the assumption $s < p$ is required to keep the smoothness of the nonlinearity f compatible with a power behavior such as (2) at zero when p is not an odd integer. The condition

$$0 < s < \min(n/2, 1 + 4/(n - 2s))$$

is equivalent to:

(a) $s \in (0, n/2)$ if $n \le 7$,

(b) $s \in (0, s_-(n)) \cup (s_+(n), n/2)$ if $n \ge 8$, where

$$s_\pm(n) = (n + 2 \pm (n^2 - 4n - 28)^{1/2})/4.$$

Compare those two conditions (a) and (b) with those given in [14]. The restriction $s < p$ may be partially removed by taking into account the regularity in time direction in more detail(see [22]).

Remark 3. Theorem 1 shows the existence and asymptotic completeness of the wave operators $W_\pm$ defined on $\dot{B}_\epsilon$ as the maps $\phi_\pm \mapsto u(0) = \phi$. The scattering operator operator S is then defined on $\dot{B}_\epsilon$ as $S = W_+^{-1} \circ W_-$. Note that smallness assumption is imposed on the data only through the fractional derivative of critical order $n/2 - 2/(p-1)$, which is equal to s in the critical case.

Remark 4. The existence and asymptotic completeness of the wave operators has been proved in [25] for (1) with (2) with $p = 1 + 4/(n-2), n \geq 3$, on small asymptotic states in H^1. A part of the result in [25] was then reproduced in [17]. Part (1) of Theorem 1 (I) is proved for (2) with $[s] + 1 < p = 1 + 4/(n-2s)$ and $0 < s < n/2$. Related results were proved by Pecher [19, 20] for the nonlinear Klein-Gordon equation in H^1 with $p = 1 + 4/(n-2)$ and $n \geq 3$.

Theorem 2. *(I) Let $s > 0$ and $p > 1 + 4/n$ satisfy*

$$s < p < \begin{cases} \infty & \text{if } \ s \geq n/2, \\ 1 + 4/(n-2s) & \text{if } \ s < n/2. \end{cases}$$

Let f satisfy $(A)_{[s]}$. Then there exists $\epsilon > 0$ with the following property.
(1) For any data $\phi \in \dot{B}_\epsilon$ at time $t_0 = 0$ the equation (3) has a unique solution $u \in X^s$. Moreover, there exists a unique pair of asymptotic states $\phi_\pm \in H^s$ satisfying $(4)_\pm$.
(2) For any data $\phi_+ \in \dot{B}_\epsilon$ at time $t_0 = +\infty$ [resp. $\phi_- \in \dot{B}_\epsilon$ at time $t_0 = -\infty$] the equation (3) has a unique solution $u \in X^s$ satisfying $(4)_+$ [resp. $(4)_-$].
(II) Let $s > 0$ be an integer and let $p > 1 + 4/n$ satisfy

$$s \leq p < \begin{cases} \infty & \text{if } \ s \geq n/2, \\ 1 + 4/(n-2s) & \text{if } \ s < n/2. \end{cases}$$

Let f satisfy $(B)_s$. Then all the conclusions of Part (I) hold if X^s is replaced by Y^s throughout the statement of Part (I).

Remark 5. Theorem 2 shows the existence and asymptotic completeness of the wave operators on $\dot{B}_\epsilon$ for (1) in the subcritical case. Note that smallness assumption is imposed on the data only through the fractional derivative of critical order $n/2 - 2/(p-1)$, which is less than s in the subcritical case. For the Cauchy Problem in both critical and subcritical cases the observation of this kind is made by Kato[14].

Remark 6. The assumptions of Theorem 2 cover for instance the case where $n = 3, p = 1 + 4/(n-2) = 5, s = 2, s_0 = 1$ [25], and therefore the result of Theorem 2 gives a partial answer to Question 4 of Kenig, Ponce, and Vega[15] under the smallness assumption on $||\phi; \dot{H}^1||$. Related results were proved by Rauch[23] for the nonlinear Klein-Gordon equation with $n = 3, p = 5, s = 2, s_0 = 1$.

To describe the nonlinear interaction f with an exponential growth at infinity as well as with a vanishing behavior as a power at zero, for $\lambda > 0$ we introduce the following assumptions $(C)_m$ with $m \geq 1$ and $(D)_m$ with $m \geq 0$.

$(C)_1$: $f \in C^1(\mathbf{C}; \mathbf{C})$ and $f(0) = 0$. There exists a constant C such that for all $z \in \mathbf{C}$

$$|f'(z)| \leq Ce^{\lambda|z|^2}|z|^2.$$

$(C)_m$ for $m \geq 2$: $f \in C^m(\mathbf{C}; \mathbf{C})$ and $f(0) = 0$. There exists a constant C such that for all $z \in \mathbf{C}$ and $2 \leq k \leq m$

$$|f'(z)| \leq Ce^{\lambda|z|^2}|z|,$$
$$|f^{(k)}(z)| \leq Ce^{\lambda|z|^2}.$$

$(D)_0$: $f \in C(\mathbf{C}; \mathbf{C})$ and $f(0) = 0$. There exists a constant C such that for all $z_1, z_2 \in \mathbf{C}$

$$|f(z_1) - f(z_2)| \leq C(e^{\lambda|z_1|^2}|z_1|^4 + e^{\lambda|z_2|^2}|z_2|^4)|z_1 - z_2|.$$

$(D)_m$ for $m \geq 1$: In addition to $(C)_m$, $f^{(m)}$ satisfies the estimate for all $z_1, z_2 \in \mathbf{C}$

$$|f^{(m)}(z_1) - f^{(m)}(z_2)| \leq C(e^{\lambda|z_1|^2} + e^{\lambda|z_2|^2})|z_1 - z_2|.$$

We solve the equation (3) in the Banach space Z defined by

$$Z = C(\mathbf{R}; H^{n/2}) \cap \bigcap_{0 \leq 2/q = \delta(r) < 1} L^q(\mathbf{R}; H_r^{n/2}) \quad \text{if} \quad n \quad \text{is even},$$
$$Z = C(\mathbf{R}; H^{n/2}) \cap \bigcap_{0 \leq 2/q = \delta(r) < 1} L^q(\mathbf{R}; \dot{B}_r^0 \cap B_r^{n/2}) \quad \text{if} \quad n \quad \text{is odd}.$$

Theorem 3. *Let $n \geq 1$. If n is even let f satisfy $(C)_{n/2}$ for some $\lambda > 0$. If n is odd let f satisfy $(D)_{(n-1)/2}$ for some $\lambda > 0$. Then there exists $\epsilon > 0$ with the following property.*

(1) For any data $\phi \in B_\epsilon$ at time $t_0 = 0$ the equation (3) has a unique solution $u \in Z$, where B_ϵ is the ball in $H^{n/2}$ with center 0 and radius ϵ. Moreover there exists a unique pair $\phi_\pm \in H^{n/2}$ satisfying $(4)_\pm$ with s replaced by $n/2$.

(2) For any data $\phi_+ \in B_\epsilon$ at time $t_0 = +\infty$ the equation (3) has a unique solution $u \in Z$ satisfying $(4)_+$ with s replaced by $n/2$.

(3) For any data $\phi_- \in B_\epsilon$ at time $t_0 = -\infty$ the equation (3) has a unique solution $u \in Z$ satisfying $(4)_-$ with s replaced by $n/2$.

Remark 7. The assumptions of the theorem above cover for instance the nonlinearities of the form

$$f(u) = \pm(e^{\lambda|u|^2} - 1 - \lambda|u|^2)u \quad \text{for} \quad n = 1,$$
$$f(u) = \pm(e^{\lambda|u|^2} - 1)u \quad \text{for} \quad n = 2, 3,$$
$$f(u) = \pm(e^{\lambda|u|^2} - 1) \quad \text{for} \quad n \geq 4,$$

with $\lambda > 0$, which need not be the same as that of $(C)_m$ or of $(D)_m$.

Remark 8. In the framework of pure H^s-theory the nonlinearity is required to behave as a power u^p at least $p \geq 1 + 4/n$ at the origin. On the other hand, the nonlinearity is required to have the differentiability of order greater than or equal to $n/2$ at the origin. To take those requirements into account, it is sufficient to suppose that the nonlinearity should behave as a power u^5 for $n = 1, u^3$ for $n = 2, 3$, and u^2 for $n \geq 4$ to keep everything smooth. This is the reason why we have imposed additional power behavior at the origin of the nonlinearity. Although there is a room to reduce the order of power behavior at the origin to the minimal value $1 + 4/n$, that is outside the purpose of this paper since we intend to keep the exposition not too technical.

Remark 9. To our knowledge there is no other work to treat the Schrödinger equation with nonlinearity of exponential growth in the H^s-theory with $s \leq n/2$. In view of Trudinger's inequality the growth rate as $e^{\lambda|z|^2}$ at infinity seems to be optimal at the level of $H^{n/2}$. Note that the L^∞-norm is out of control of the $H^{n/2}$-norm even when the latter is infinitesimally small.

Remark 10. The theorem above proves the existence and asymptotic completeness of the wave operators $W_\pm : \phi_\pm \mapsto u(0) = \phi$ on the small asymptotic states $\phi_\pm$ in $H^{n/2}$.

We now give a brief sketch of the proofs. As usual the method depends on a contraction argument on (3) in $X = X^s, Y^s$ or Z with metric on $L^q(\mathbb{R}; L^r)$ for an admissinle pair of exponents (q,r) in which the Strichartz type estimates for the free propagator fit naturally. For that purpose we prove that all the norms appearing in the definition of X are reproduced by the right hand side of (3) and that the metric on $L^q(\mathbb{R};L^r)$ is contracted. At a technical level we need the following key estimates. We use Lemma 1 for the proof of Theorems 1 and 2, while Lemma 2 is required to estimate the exponential functions from Theorem 3.

Lemma 1. *Let p and s satisfy $1 \le p < \infty$ and $0 \le s \le p$. Let ℓ, r, m satisfy $1 < \ell \le r < \infty, 1 < m \le \infty$, $1/\ell = 1/r + (p-1)/m$. Let $f \in C^{[s]}(\mathbf{C};\mathbf{C})$.*
(1)When s is not an integer, assume in addition that $r, m \ge 2$ and $s < p$ and that f satisfies $(A)_{[s]}$. Then

$$\begin{aligned} \|f(u);\dot{B}^s_\ell\| &\le C\|u;\dot{B}^0_m\|^{p-1}\|u;\dot{B}^s_r\| && \text{if } m<\infty,\\ \|f(u);\dot{B}^s_\ell\| &\le C(\|u;\dot{B}^0_\infty\|+\|u;L^\infty\|)^{p-1}\|u;\dot{B}^s_\ell\| && \text{if } m=\infty. \end{aligned}$$

(2)When s is an integer, assume that f satisfies $(B)_{[s]}$. Then

$$\|f(u);\dot{H}^s_\ell\| \le C\|u;L^m\|^{p-1}\|u;\dot{H}^s_r\|.$$

Lemma 2. *Let $1 < r < \infty$. Then there exists a constant $C_0 > 0$ such that for any q with $r \le q < \infty$ the following estimates hold.*

$$\begin{aligned} \|u;L^q\| &\le C_0 q^{1/2+(r-2)/(2q)}\|u;\dot{H}^{n/2}\|^{1-r/q}\|u;L^r\|^{r/q},\\ \|u;\dot{B}^0_q\| &\le C_0 q^{1/2+(r-2)/(2q)}\|u;\dot{H}^{n/2}\|^{1-r/q}\|u;\dot{B}^0_r\|^{r/q}. \end{aligned}$$

The proof of Lemma 1 follows closely that of [7; Lemma 3.4] in the sense that we make use of an equivalent norm on Besov spaces in terms of modulus of continuity with the second differences, though actual proof is rather involved because of higher derivatives of functions. Lemma 2 follows from [18; Inequality (2.6)] and convexity inequalities between Besov and Sobolev spaces. See [17, 18] for details.

References

[1] J.Bergh, J.Löfström, "Interpolation Spaces," Springer-Verlag, Berlin-Heidelberg-New York, 1976.

[2] B.Birnir, C.E.Kenig, G.Ponce, N.Svanstedt, L.Vega *On the ill-posedness of the IVP for the generalized Korteweg-de Vries and Schrödinger equations*, preprint.

[3] P.Brenner, W.von Wahl, *Global classical solutions of nonlinear wave equations*, Math.Z., **176**(1981), 87-121.

[4] T.Cazenave, "An Introduction to Nonlinear Schrödinger Equations," Textos de Métodos Matemáticos **22**, Instituto de Matemática, Rio de Janeiro, 1989.

[5] T.Cazenave, F.B.Weissler, *The Cauchy problem for the critical nonlinear Schrödinger equation in H^s*, Nonlinear Anal. TMA **14**(1990), 807-836.

[5] J.Ginibre, "Equations d'Evolution Semilinéaires: L'Equation de Schrödinger Nonlinéaire," Cours de DEA, Université de Paris-Sud, 1994.

[6] J. Ginibre, *An Introduction to Nonlinear Schrödinger Equations*, in "Nonlinear Waves," R.Agemi, Y.Giga, T.Ozawa(Eds), GAKUTO International Series, Mathematical Sciences and Applications, Gakkōtosho, Tokyo(in press).

[7] J.Ginibre, T.Ozawa, G.Velo, *On the existence of the wave operators for a class of nonlinear Schrödinger equations*, Ann. Inst. Henri Poincaré, Physique théorique, **60**(1994), 211-239.

[8] J.Ginibre, G.Velo, *On a class of nonlinear Schrödinger equations II. Scattering theory, general case*, J.Funct.Anal., **32**(1979), 33-71.

[9] J.Ginibre, G.Velo, *Scattering theory in the energy space for a class on nonlinear Schrödinger equations*, J.Math.pures et appl., **64**(1984), 363-401.

[10] J.Ginibre, G.Velo, *The global Cauchy problem for the nonlinear Klein-Gordon equation*, Math.Z., **189**(1985), 487-505.

[11] J.Ginibre, G.Velo, *Scattering theory in the energy space for a class of non-linear wave equations*, Commun.Math.Phys., **123**(1989), 535-573.

[12] T.Kato, *On nonlinear Schrödinger equations*, Ann.Inst.Henri Poincaré, Physique théorique, **46**(1987), 113-129.

[13] T.Kato, *Nonlinear Schrödinger Equations*, in "Schrödinger Operators," H.Holden, A.Jensen(Eds), Lecture Notes in Physics, **345**, Springer-Verlag, Berlin-Heidelberg-New York, 1989.

[14] T.Kato, *On nonlinear Schrödinger equations II. H^s-solutions and unconditional well-posedness*, J.d'Anal.Math., **67**(1995), 281-306.

[15] C.E.Kenig, G.Ponce, L.Vega, *On the IVP for the nonlinear Schrödinger equations*, Contemporary Mathematics, **189**(1995), 353-367.

[16] J.E.Lin, W.A.Strauss, *Decay and scattering of solutions of a nonlinear Schrödinger equation*, J.Funct.Anal., **30**(1978), 245-263.

[16] J.E.Lin, W.A.Strauss, *Decay and scattering of solutions of a nonlinear Schrödinger equation*, J.Funct.Anal., **30**(1978), 245-263.

[17] Liu Yue, *Asymptotic behavior of solutions of nonlinear Schrödinger equations*, Chinese Ann.Math.Ser. A **12**(1991), 19-25(in Chinese).

[18] T.Ozawa, *On critical cases of Sobolev's inequalities*, J. Funct. Anal., **127**(1995), 259-269.

[19] H.Pecher, *Nonlinear small data scattering for the wave and Klein-Gordon equation*, Math.Z., **150**(1984), 261-270.

[20] H.Pecher, *Low energy scattering for nonlinear Klein-Gordon equations*, J.Funct.Anal., **63**(1985), 101-122.

[21] H.Pecher, *Local solutions of semilinear wave equations in H^{s+1}*, Math.Methods Appl. Sci., **19**(1996), 145-170.

[22] H.Pecher, *Solutions of semilinear Schrödinger equations in H^s*, Preprint.

[23] J.Rauch, *I. The u^5 Klein-Gordon equation. II. Anomalous singularities for semilinear wave equations*, in "Nonlinear Partial Differential Equations and Their Applications," Collège de France Seminar, Vol.I, Pitman, Boston, 1976.

[24] H.Triebel, "Theory of Function Spaces," Birkhäuser, 1983.

[25] Y.Tsutsumi, "Global Existence and Asymptotic Behavior of Solutions for Nonlinear Schrödinger Equations," Doctoral Thesis, University of Tokyo, 1985.

[26] K.Yajima, *Existence of solutions for Schrödinger evolution equations*, Commun.Math. Phys., **110**(1987), 415-426.

[27] M.Nakamura, T.Ozawa, *Low energy scattering for nonlinear Schrödinger equations in fractional order Sobolev spaces*, Rev. Math. Phys. (in press)

[28] M.Nakamura, T.Ozawa, *Nonlinear Schrödinger equations in the Sobolev space of critical order*, preprint.