数理解析研究所講究録
1009 巻 1997 年 132-140

On a class of nonlinear elliptic systems

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In this survey paper we present some recent results concerning semilinear and quasilinear elliptic systems of the form:

$$(1)\qquad \begin{aligned} Au &= f(u,v) \\ Bv &= g(u,v) \end{aligned}$$

where A and B are (possibly nonlinear) second-order elliptic operators and f, g are given functions satisfying $f(0,0) = g(0,0) = 0$. We also assume $A0 = B0 = 0$. Our main structural assumption on the nonlinearities f and g is the existence of a function $H : \mathbb{R}^2 \to \mathbb{R}$ sufficiently smooth such that

$$\left.\begin{aligned} f(u,v) &= \tfrac{\partial H}{\partial v}(u,v)\ , \\ g(u,v) &= \tfrac{\partial H}{\partial u}(u,v)\ . \end{aligned}\right. \qquad u, v \in \mathbb{R},$$

Moreover we suppose that the operators A and B are invertible (in appropriate function spaces) with monotone (in the sense of order) inverse. Assuming $f(u,v) \geq 0$ and $g(u,v) \geq 0$ for $u, v \geq 0$, it is natural to look for positive solutions to (1). In the first section we shall use a variational approach and in the second section degree arguments together with a priori estimates for positive solutions are used for obtaining existence of nontrivial positive solutions.

1. The variational case.

As a first model problem we consider functions H of the form:

$$(2)\qquad H(u,v) = \frac{1}{p+1}\,|v|^{p+1} + \frac{1}{q+1}\,|u|^{q+1}, \quad u, v \in \mathbb{R}$$

with $p, q > 0$, and as operators A, B, the negative Laplacian operator on a bounded domain of $\mathbb{R}^N$ with zero Dirichlet boundary conditions. We obtain the Lane-Emden type system

$$(3)\qquad \left\{ \begin{array}{llll} -\Delta u = |v|^p \operatorname{sign} v & \text{in } \Omega, & u = 0 & \text{on } \partial\Omega, \\ -\Delta v = |u|^q \operatorname{sign} u & \text{in } \Omega, & v = 0 & \text{on } \partial\Omega. \end{array} \right.$$

This system has been studied by many authors [15], [25], [3], [20].

Existence of positive solutions of (3) can be obtained by using the following abstract result. Let (Σ, μ) be a σ-finite measure space, let X, Y be two real Banach spaces such that X (resp. Y) is continuously imbedded in $L^{q+1}(\Sigma)$ (resp. $L^{p+1}(\Sigma)$) for some $p, q > 0$.
Let A (resp. B) be a linear isomorphism from X onto $L^{1+1/p}(\Sigma)$ (resp. Y onto $L^{1+1/q}(\Sigma)$ satisfying $A^{-1}g \geq 0$ whenever $g \in L^{1+1/p}(\Sigma),\ g \geq 0$.
We consider the system

$$\begin{array}{rcll} Au & = & \phi_p(v), & \text{in } \Sigma, \\ Bv & = & \phi_q(u), & \text{in } \Sigma, \end{array} \tag{4}$$

where $\phi_r(t) = |t|^r \text{ sign } t,\ t \in \mathbb{R}, r > 0$.
Inverting the nonlinearity in the first equation, we obtain

$$B\phi_{1/p}(Au) = \phi_q(u), \text{ in } \Sigma. \tag{5}$$

In analogy with the case $p = 1$, we call problem (5) sublinear if $q < \frac{1}{p}$ and superlinear if $q > \frac{1}{p}$. In case $pq = 1$, it is natural to look instead at the eigenvalue problem

$$\begin{cases} B\phi_q(Au) & = \lambda\phi_q(u), \text{ with } \lambda > 0, \\ \quad \phi_q(u) & = 1. \end{cases} \tag{6}$$

Observe that in this generality problems (4), (5), (6) have no variational structure.
However if the following condition is satisfied

$$\int_\Sigma Au \cdot v \, d\mu = \int_\Sigma u \cdot Bv \, d\mu, \ \forall u \in X, \ \forall v \in Y, \tag{7}$$

then a positive solution of (5) is a critical point of the functional

$$I(u) := \frac{1}{1 + 1/p} \|Au\|^{1+1/p}_{L^{1+1/p}(\Sigma)} - \frac{1}{1+q} \|u^+\|^{1+q}_{L^{1+q}(\Sigma)}.$$

If $pq < 1$, the functional $I : X \to \mathbb{R}$ is bounded from below. In this paper we are mainly interested in the superlinear case $pq > 1$ or equivalently

$$1 > \frac{1}{p+1} + \frac{1}{q+1}. \tag{8}$$

The functional $I : X \to \mathbb{R}$ is C^1 and satisfies "P.S." condition if $pq \neq 1$ and

(9) the imbedding of X into $L^{1+q}(\Sigma)$ is compact.

By using the Mountain pass theorem of Ambrosetti-Rabinowitz in the superlinear case one obtains

Proposition 1. [4] [5]

Under the above conditions, if either $pq < 1$ or $pq > 1$, system (4) possesses at least one nontrivial solution with positive components (u, v) in $X \times Y$.

Returning to the case of Lane-Emden system (3), we notice that if Ω has a C^2 boundary and

$$\begin{aligned} X &= W^{2,1+1/p}(\Omega) \cap W_0^{1,1+1/p}(\Omega), \\ Y &= W^{2,1+1/q}(\Omega) \cap W_0^{1,1+1/q}(\Omega), \\ Au &= -\Delta u, \ u \in X, \\ Bv &= -\Delta v, \ v \in Y, \end{aligned}$$

then the assumptions of proposition 1 are satisfied provided that

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N} \ , \text{ when } N \geq 3 \tag{10}$$

(no conditions on $p, q > 0$, when $N = 1, 2$).

Condition (10) implies the compactness condition (9). Using a bootstrap argument and condition (10) again, one obtains [3] *bounded* positive solutions and if the boundary is $C^{2,\alpha}$, *classical* positive solutions.

Observe that in the special case when $\Omega = \{x \in \mathbb{R}^N; 0 < r < \|x\|_2 < R\}$, then the compactness condition is always satisfied provided we restrict ourselves to radially symmetric functions.

If instead of considering the Laplacian operator for A, we choose the heat operator $Au = u_t - \Delta u$ with appropriate domain, we obtain an unbounded Hamiltonian system

$$\begin{cases} u_t = \Delta u + \phi_p(u) \ , & x \in \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \\ -v_t = \Delta u + \phi_q(u) \ , & x \in \Omega, \quad v = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{11}$$

Looking for positive solutions which are $2T$-periodic in time, in order to apply Proposition 1, we need the stronger condition

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N}{N+2} \ , \tag{12}$$

and T sufficiently large, in order to insure that the solution be not constant in time. As limit of periodic solutions as the period tends to infinity, we obtain a smooth homoclinic connection to the origin. More precisely, we have

Theorem 2. [4] [5]
Let Ω be a bounded domain of $\mathbb{R}^N$ with boundary $C^{2,\alpha}$. Let $p, q > 0$ with $pq > 1$. If condition (12) is satisfied then system (11) possesses a solution $(u, v) \in (C^{1,2}(\mathbb{R} \times \bar{\Omega})^2$ with positive components such that $\lim_{|t|\to\infty} u(t,x) = \lim_{|t|\to\infty} v(t,x) = 0$ uniformly in $x \in \bar{\Omega}$.

We conclude this section by mentioning some results concerning the existence of solutions to (1) by using a variational approach. When the operators A and B are still the negative Laplacian but the Hamiltonian H is more general, an approach based on Benci-Rabinowitz theorem for strongly indefinite functionals in suitable interpolation spaces has been independently considered by [14] and [10]. In [7] more general pairs A, B have been investigated and in [8] a dual approach has been implemented allowing in particular to relax the smoothness condition on $\partial\Omega$ (for bounded solutions).

Variational and Rellich type identities.
A natural question is to know whether conditions (10) and (12) are in some sense necessary for the existence of positive solutions. The first result in this direction has been obtained by Pohozaev [21] for system (3) in case $p = q$. It is easy to see that if $p = q$, then $u = v$ and system (3) reduces to the equation

$$-\Delta u = \phi_q(u) \text{ in } \Omega, \; u = 0 \text{ on } \partial\Omega. \tag{13}$$

By using his famous identity, Pohozaev was able to show among other things that if Ω is star-shaped, then (13) possesses no nontrivial solution if

$$q \geq \frac{N+2}{N-2}, \; N \geq 3, \tag{14}$$

which corresponds to the negation of (10) when $p = q$.
Variational identities have been obtained for more general situations by Pohozaev [22], Pucci-Serrin [23] and more recently by van der Vorst [25]. In a different spirit, Mitidieri [15] developed Rellich type identities also allowing

to prove nonexistence theorems for systems and showing the criticality of the hyperbola defined by (10). In [9] the criticality of the hyperbola defined by (12) is proven.

2. The nonvariational case.
In [6], the following model problem has been investigated:

$$(15)\qquad \begin{cases} -\Delta_\alpha u = \phi_p(v) & \text{in } \Omega, \;\; u = 0 \quad \text{on } \partial\Omega, \\ -\Delta_\beta v = \phi_q(u) & \text{in } \Omega, \;\; v = 0 \quad \text{on } \partial\Omega. \end{cases}$$

where $\Delta_\gamma u = \operatorname{div}\left(|\nabla u|^{\gamma-2}\nabla u\right)$, $\gamma > 1$, and $\Omega = B_R = \{x \in \mathbb{R}^N;\ \|x\|_2 < R\}$, $R > 0$. Apart from the case $\alpha = \beta = 2$, this quasilinear system has no variational structure. The existence of positive ***radially symmetric*** solutions has been obtained by using a priori estimates together with a degree argument in the "superlinear" case, that is

$$(16)\qquad pq > (\alpha - 1)(\beta - 1)\ ,\ \alpha, \beta > 1.$$

L^∞ a priori bounds for positive solutions are derived by using a "blow up" argument (in the spirit of Gidas-Spruck [13]) producing positive radially symmetric solutions on $\mathbb{R}^N$ (if by contradiction such bounds do not exist) together with a Liouville type theorem implying that under certain conditions on α, β, p, q no such positive solutions on $\mathbb{R}^N$ can exist. In the case $\alpha = \beta = 2$, it has been proved in [15] that no positive radially symmetric solutions exist in $\mathbb{R}^N$ if condition (10) holds. It is still an open problem in the non-radial case. Partial results in that direction has been obtained by [11], [16] and [24]. In the non variational case due to the lack of "variational identities" the critical curve for (15) is not yet known (even in the radial case). However using Liouville type theorems for inequalities:

$$(17)\qquad \begin{cases} -\Delta_\alpha u \geq v^p & \text{in } \mathbb{R}^N, \\ -\Delta_\beta v \geq u^q & \text{in } \mathbb{R}^N, \end{cases}$$

with $u, v \geq 0$,
existence results for (15) has been obtained in [6] under the following assumptions:

$$(18)\qquad 1 < \alpha, \beta < N\ ,\ p, q > 0\ ,$$

(19)
$$\max\left\{\frac{\alpha(\beta-1)+p\beta}{pq-(\alpha-1)(\beta-1)}-\frac{N-\alpha}{\alpha-1},\frac{\beta(\alpha-1)+q\alpha}{pq-(\alpha-1)(\beta-1)}-\frac{N-\beta}{\beta-1}\right\}\geq 0$$

together with (16). Observe that these conditions for the Liouville type theorem are optimal in the following sense. If (u,v) are positive solutions to (17) on $\mathbb{R}^N$ with $u,v\in C^2(\mathbb{R}^N\backslash\{0\})\cap C^1(\mathbb{R}^N)$, radially symmetric and (18) holds, then (19) cannot hold. Moreover if $\gamma\geq N$ and

$$-\Delta_\gamma u\geq 0 \quad \text{in } \mathbb{R}^N$$

with $u\in C^2(\mathbb{R}^N\backslash\{0\})\cap C^1(\mathbb{R}^N)$ nonnegative and radially symmetric, then u is constant. Liouville type theorems for inequalities has been introduced for the equation case by Ni and Serrin [18], [19]. We recall that for the inequation

$$-\Delta u\geq u^q \text{ in } \mathbb{R}^N \text{ with } u\geq 0,$$

the critical exponent $q_s=\frac{N}{N-2}$, for $N\geq 3$, which corresponds to the case $\alpha=\beta=2$ and $p=q$.
Recently Liouville type theorems for inequalities on $\mathbb{R}^N$ and on cones for a broader class of operators A,B and nonlinearities f,g have been established by several authors [17], [2], [1],
Finally we mention that existence and non existence results for hyperbolic systems of the form

$$\begin{aligned}\partial_t^2\, u-\Delta\, u&=|v|^p\\ \partial_t^2\, v-\Delta\, v&=|u|^q\end{aligned}\qquad \text{in } [0,\infty)\times\mathbb{R}^N,$$

have been obtained in [12].

Acknowledgment. The first author would like to thank Professor Okazawa for his kind invitation to participate in the Conference on Nonlinear Evolution Equations and Applications, RIMS, Kyoto, October 21-23, 1996 and for his warm hospitality.

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