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Characterization of the Domain of Fractional Powers of a Class of Elliptic Differential Operators with Feedback Boundary Conditions

神戸大学工学部 南部隆夫 (Takao Nambu)

1. Introduction

We consider in this paper a system of linear differential operators $(\mathcal{L}, \tau)$ in a bounded domain Ω of $\mathbb{R}^m$ with the boundary Γ which consists of a finite number of smooth components of $(m-1)$-dimension. Actually, let $\mathcal{L}$ denote a uniformly elliptic differential operator of order 2 in Ω defined by

$$\mathcal{L}u = -\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{m} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u,$$

where $a_{ij}(x) = a_{ji}(x)$ for $1 \le i,\, j \le m$, $x \in \overline{\Omega}$, and for some positive δ

$$\sum_{i,j=1}^{m} a_{ij}(x)\xi_i\xi_j \ge \delta|\xi|^2, \quad {}^\forall\xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m; \quad {}^\forall x \in \overline{\Omega}.$$

Associated with $\mathcal{L}$ is a boundary operator τ_1 of the Dirichlet type (case I) or τ_2 of the generalized Neumann type (case II) defined by

$$\tau_1 u = u\big|_\Gamma, \quad \text{and}$$
$$\tau_2 u = \frac{\partial u}{\partial \nu} + \sigma(\xi)u = \sum_{i,j=1}^{m} a_{ij}(\xi)\nu_i(\xi)\frac{\partial u}{\partial x_j}\bigg|_\Gamma + \sigma(\xi)u\bigg|_\Gamma,$$

respectively, where $(\nu_1(\xi), \dots, \nu_m(\xi))$ denotes the unit outer normal at $\xi \in \Gamma$. Necessary regularity on $\overline{\Omega}$ and on Γ of coefficients of $\mathcal{L}$ and τ_2 is assumed tacitly (see, e.g., [1, 4, 8, 9, 14]). Let us define the linear operators L_1 and L_2 in $L^2(\Omega)$ by

$$L_1 u = \mathcal{L}u, \quad u \in \mathcal{D}(L_1) = \left\{u \in H^2(\Omega);\ \tau_1 u = 0\right\}$$

and

$$L_2 u = \mathcal{L}u, \quad u \in \mathcal{D}(L_2) = \left\{u \in H^2(\Omega);\ \tau_2 u = 0\right\},$$

respectively. The operators L_1 and L_2 are classical and very standard. Among the well known properties, their fractional powers are of our special interest. In [3, 5], a concrete characterization of the domain of fractional powers of L_1 and L_2 is obtained. A part of these results played an important role in some problems of boundary control systems [10, 11, 12]: The boundary control problem is reduced to a distributed control problem, i.e., a problem with a *homogeneous* bounday condition, by a simple transformation of the state. However, they do not provide us a satisfactory means, for example, in stability analysis of boundary feedback control systems [13]. The study of $\mathcal{L}$ with feedback boundary condition and its fractional powers then becomes necessary. The objective of this paper is to develop the study of fractional powers of linear operators M_1 and M_2 introduced just below. As far as the author's knowledge, basic properties of M_1 and M_2 are not well known, in contrast to the case of L_1 and L_2.

Let us define the linear operators M_1 and M_2 in $L^2(\Omega)$ by

$$M_1 u = \mathcal{L}u, \quad u \in \mathcal{D}(M_1) = \Big\{ u \in H^2(\Omega);\ \tau_1 u = \sum_{k=1}^{p} \langle u, w_k \rangle_\Omega h_k \text{ on } \Gamma \Big\}, \tag{1.1}$$

and

$$M_2 u = \mathcal{L}u, \quad u \in \mathcal{D}(M_2) = \Big\{ u \in H^2(\Omega);\ \tau_2 u = \sum_{k=1}^{p} \langle u, w_k \rangle_\Gamma h_k \text{ on } \Gamma \Big\}, \tag{1.2}$$

respectively. Here, $\langle \cdot, \cdot \rangle_\Omega$ and $\langle \cdot, \cdot \rangle_\Gamma$ denote the inner products in $L^2(\Omega)$ and $L^2(\Gamma)$, respectively, p a positive integer depending on the control problems under consideration, and necessary regularities for the functions w_k and h_k are assumed in the following sections. Thus, the boundary conditions for M_1 and M_2 are described as a feedback type. The boundary control system corresponding to, for example, M_1 is described by

$$\frac{du}{dt} + M_1 u = 0, \quad t > 0, \quad u(0) = u_0 \tag{1.3}$$

in $L^2(\Omega)$, or

$$\begin{aligned} &\frac{\partial u}{\partial t} + \mathcal{L}u = 0 \quad \text{in } \Omega, \\ &u\Big|_\Gamma = \sum_{k=1}^{p} \langle u, w_k \rangle_\Omega h_k \quad \text{on } \Gamma, \\ &u(0, \cdot) = u_0(\cdot) \quad \text{in } \Omega. \end{aligned} \tag{1.4}$$

The operators M_1 and M_2 are not a standard type in the sense that the boundary conditions are composed of terms of local nature (τ_1 and τ_2) and those of global nature ($\langle \cdot, w_k \rangle_\Omega$ and $\langle \cdot, w_k \rangle_\Gamma$). A particular difference between M_1 and M_2 lies in *accretiveness*. In fact, it is easily shown that M_2 (or its right shift $M_2 + c$, $c > 0$, if necessary) is m-accretive, while M_1 is not! Thus, different approaches are necessary for M_1 and M_2.

Throughout the paper, all norms will denote $L^2(\Omega)$ - or $\mathcal{L}(L^2(\Omega))$ - norms. In Section 2, some well known facts are reviewed and preliminary results for M_1 and M_2 are developed, where basic assumptions and notations are introduced. In Section 3, the main results and their proofs are stated, where the domains of fractional powers for M_1 and M_2 are characterized in terms of Sobolev spaces. Since m-accretiveness for $M_1 + c$ is *not* expected, the reader will find a considerable difference between M_1 and M_2 in studying their structures. The results turns out to be a striking extension of Fujiwara's and Grisvard's characterization [3, 5] stated in Section 2. Based on the main results, an application to robustness analysis of a boundary feedback control system is briefly stated in Section 4. Finally the concluding remarks are stated in Section 5, where we discuss versions of the main results occuring due to the replacement of some parameters in M_1 and M_2.

2. Preliminary results

Let us begin with reviewing the well known spectral property for L_1 and L_2. There is a sector $\overline{\sum}_{-\alpha} = \overline{\sum} - \alpha$, $\alpha > 0$, such that $\overline{\sum}_{-\alpha}$ is contained in the resolvent sets $\rho(L_i)$, $i = 1, 2$ and that the following estimates hold:

$$\|(\lambda - L_i)^{-1}\| \leq \frac{\text{const}}{1 + |\lambda|}, \quad \lambda \in \overline{\sum}_{-\alpha}, \quad i = 1, 2 \tag{2.1}$$

where $\overline{\sum} = \{\lambda \in \mathbb{C};\ \theta_0 \leq |\arg \lambda| \leq \pi\}$, $0 < \theta_0 < \pi/2$, and the upper bar means the closure of a set. Choose a positive constant $c\ (> \alpha)$, and set $L_{ic} = L_i + c$, $i = 1, 2$. Then, fractional powers of the operators L_{1c} and L_{2c} are well defined. In order to characterize the domains of L_{1c}^{θ} and L_{2c}^{θ}, $0 \leq \theta \leq 1$, it is assumed in the rest of the paper that $\sigma(\xi)$ appearing in the boundary operator τ_2 has a suitable smooth extension to $\overline{\Omega}$. The distance from $x \in \mathbb{R}^m$ to Γ is denoted by $\zeta(x)$. Then we have the following two fundamental theorems of [3, 5]:

Theorem 2.1. *Case* I (*the Dirichlet boundary condition*). *The domain of the fractional powers* L_{1c}^{θ} *is characterized as follows*:

(i) $\mathcal{D}(L_{1c}^{\theta}) = H^{2\theta}(\Omega), \quad 0 \leq \theta < \frac{1}{4}$;

(ii) $\mathcal{D}(L_{1c}^{1/4}) = \left\{u \in H^{1/2}(\Omega);\ \int_{\Omega} \frac{1}{\zeta(x)} |u|^2\, dx < \infty\right\}$; *and*

(iii) $\mathcal{D}(L_{1c}^{\theta}) = H_{\gamma_0}^{2\theta}(\Omega), \quad \frac{1}{4} < \theta \leq 1$,

where the space $H_{\gamma_0}^{\alpha}(\Omega)$ *is defined by*

$$H_{\gamma_0}^{\alpha}(\Omega) = \{u \in H^{\alpha}(\Omega);\ u\big|_{\Gamma} = 0 \ \ on\ \Gamma\}, \quad \alpha > \frac{1}{2}. \tag{2.2}$$

The generalized Neumann case is somewhat simpler than the Dirichlet case:

Theorem 2.2. *Case* II (*the generalized Neumann boundary condition*). *The domain of the fractional powers* L_{1c}^{θ} *is characterized as follows*:

(i) $\mathcal{D}(L_{2c}^{\theta}) = H^{2\theta}(\Omega), \quad 0 \leq \theta < \frac{3}{4}$;

(ii) $\mathcal{D}(L_{2c}^{3/4}) = \left\{u \in H^{3/2}(\Omega);\ \int_{\Omega} \frac{1}{\zeta(x)} |\tau_{\Omega} u|^2\, dx < \infty\right\}$; *and*

(iii) $\mathcal{D}(L_{2c}^{\theta}) = \{u \in H^{\theta}(\Omega);\ \tau_2 u = 0 \ \ on\ \Gamma\}, \quad \frac{3}{4} < \theta \leq 1$,

where τ_{Ω} *is a first order differential operator given by*

$$\tau_{\Omega} u = \frac{\partial u}{\partial \zeta} + \sigma(x) u. \tag{2.3}$$

The proof of Theorems 2.1 and 2.2 is carried out by transforming first a class of functions in a neighborhood of Γ into functions on the half space $\mathbb{R}^m_{y+}$ and then introducing operators of extension to the whole space $\mathbb{R}^m_y$, e.g., a reflection operator with respect to the hypersurface $\{y_m = 0\}$ and operators of restriction to $\mathbb{R}^m_{y+}$.

When L_1 and L_2 are replaced by M_1 and M_2, respectively, it is natural to expect that the feedback boundary condition would appear in the above theorems. In fact, this expectation is true, and the corresponding results are stated in Section 3. We develop here some basic properties of M_1 and M_2. Most fundamental is the existence of the resolvents and their decay estimates. Henceforth c denotes a various positive constant independent of arguments under consideration unless otherwise indicated. Our first result is stated as follows:

Theorem 2.3. (i) *Case* I (*the Dirichlet boundary condition*). *Let us suppose that* w_k*'s and* h_k*'s in* M_1 *satisfy the assumption*

$$w_k \in L^2(\Omega), \quad \textit{and} \quad h_k \in H^{3/2}(\Gamma), \quad 1 \le k \le p. \tag{2.4}$$

Then the domain $\mathcal{D}(M_1)$ *is dense. There is a sector* $\overline{\sum}_{-\beta} = \overline{\sum} - \beta$, $\beta > \alpha$, *such that* $\overline{\sum}_{-\beta}$ *is contained in the resolvent set* $\rho(M_1)$ *and that the following estimate holds*:

$$\|(\lambda - M_1)^{-1}\| \le \frac{\text{const}}{1 + |\lambda|}, \quad \lambda \in \overline{\sum}_{-\beta}. \tag{2.5}$$

(ii) *Case* II (*the generalized Neumann boundary condition*). *Let us suppose that* w_k*'s and* h_k*'s in* M_2 *satisfy the assumption*

$$w_k \in L^2(\Gamma), \quad \textit{and} \quad h_k \in H^{1/2}(\Gamma), \quad 1 \le k \le p. \tag{2.6}$$

Then the domain $\mathcal{D}(M_2)$ *is dense. There is a sector* $\overline{\sum}_{-\gamma} = \overline{\sum} - \gamma$, $\gamma > \alpha$, *such that* $\overline{\sum}_{-\gamma}$ *is contained in the resolvent set* $\rho(M_2)$ *and that the following estimate holds*:

$$\|(\lambda - M_2)^{-1}\| \le \frac{\text{const}}{1 + |\lambda|}, \quad \lambda \in \overline{\sum}_{-\gamma}. \tag{2.7}$$

From the control theoretic viewpoint, it is interesting to investigate the adjoint structures of M_1 and M_2. In fact, we have the following results:

Proposition 2.4. *We assume that the conditions* (2.4) *and* (2.6) *are satisfied in Case* I *and Case* II, *respectively.*

(i) *The adjoint operator of* M_1 *is described by*

$$\begin{gathered} M_1^* v = \mathcal{L}^* v + \sum_{k=1}^{p} \left\langle \frac{\partial v}{\partial \nu}, h_k \right\rangle_\Gamma w_k , \\ v \in \mathcal{D}(M_1^*) = H^2(\Omega) \cap H_0^1(\Omega) = \mathcal{D}(L_1), \end{gathered} \tag{2.8}$$

where $\mathcal{L}^*$ *denotes the formal adjoint of* $\mathcal{L}$:

$$\mathcal{L}^* u = - \sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial u}{\partial x_j}\right) - \sum_{i=1}^{m} \frac{\partial}{\partial x_j}\left(b_i(x) u\right) + c(x) u .$$

(ii) *Assume that w_k's in M_2 belong to $H^{1/2}(\Gamma)$ in addition. The adjoint operator of M_2 is then described by*

$$M_2^* v = \mathcal{L}^* v,$$
$$v \in \mathcal{D}(M_2^*) = \left\{ v \in H^2(\Omega);\ \tau_2^* v = \sum_{k=1}^{p} \langle v,\, h_k \rangle_\Gamma\, w_k \right\}, \tag{2.9}$$

where the pair $(\mathcal{L}^*, \tau_2^*)$ denotes the formal adjoint of $(\mathcal{L}, \tau_2)$.

Proof of Theorem 2.3.

Although Cases I and II look similar, we need different approaches. In fact, a sesquilinear form is available in Case II, while it is not in Case I.

Case I. Let us consider the boundary value problem

$$(\lambda - \mathcal{L})u = 0 \quad \text{in } \Omega \quad \text{and} \quad \tau_1 u = u|_\Gamma = f \quad \text{on } \Gamma$$

for any given $f \in H^{3/2}(\Gamma)$. There is a unique solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$ if λ is in $\rho(L_1)$, and the solution u is denoted by $N_1(\lambda)f$. The solution u is expressed, for example, as

$$u = N_1(\lambda)f = R_1 f - (\lambda - L_1)^{-1}(\lambda - \mathcal{L})R_1 f,$$

where R_1 denotes a linear operator belonging to $\mathcal{L}(H^{3/2}(\Gamma);\, H^2(\Omega))$ such that [8]

$$R_1 f\Big|_\Gamma = f, \quad \text{and} \quad \frac{\partial}{\partial \nu} R_1 f\Big|_\Gamma = 0. \tag{2.10}$$

The operator R_1 is not uniquely determined. We need the following lemma regarding the behavior of $N_1(\lambda)$, the proof of which is omitted:

Lemma 2.5. *Assumption* (2.4) *implies that*

$$\langle N_1(\lambda)h_j,\, w_k \rangle_\Omega \;\to 0 \quad as \ |\lambda| \;\to \infty, \quad \lambda \in \rho(L_1).$$

For a given $f \in L^2(\Omega)$, let us consider the problem

$$u = (\lambda - L_1)^{-1} f + \sum_{k=1}^{p} \langle u,\, w_k \rangle_\Omega\, N_1(\lambda) h_k. \tag{2.11}$$

If the problem has a solution $u \in H^2(\Omega)$, this solves the boundary value problem

$$(\lambda - M_1)u = f\,.$$

Suppose for a moment that (2.11) admits a solution u. Then it is immediately seen that, for a sufficiently large $|\lambda|$

$$\langle u,\, \mathrm{w} \rangle_\Omega = (1 - \Phi(\lambda))^{-1} \langle (\lambda - L_1)^{-1} f,\, \mathrm{w} \rangle_\Omega\,, \tag{2.12}$$

where $\langle \cdot, \mathrm{w} \rangle_\Omega$ denotes a $p \times 1$ column vector whose k-th component is given by $\langle \cdot, w_k \rangle_\Omega$, and $\Phi(\lambda)$ the $p \times p$ matrix given by

$$\Phi(\lambda) = \left[\langle N_1(\lambda) h_j, w_k \rangle_\Omega \; ; \; \begin{matrix} j \to 1, \dots, p \\ k \downarrow 1, \dots, p \end{matrix} \right].$$

Note that $(1 - \Phi(\lambda))^{-1}$ exists when $|\lambda|$ goes to ∞, due to the estimate in Lemma 2.5. By substituting this into (2.11), u must have the expression:

$$u = (\lambda - L_1)^{-1} f + \sum_{k=1}^{p} \left[(1 - \Phi(\lambda))^{-1} \langle (\lambda - L_1)^{-1} f, \mathrm{w} \rangle_\Omega \right]_k N_1(\lambda) h_k. \tag{2.13}$$

Conversely, it is easily seen that u given by (2.13) satisfies the relation (2.12), which immediately leads to the equation (2.11). Uniqueness of solutions to $(\lambda - M_1)u = f$ is almost immediate. The estimate (2.5) with some $\beta > 0$ is derived from the above expression (2.13) and Lemma 2.5.

Denseness of $\mathcal{D}(M_1)$. Let us choose a $\lambda \in \overline{\Sigma}_{-\beta}$. We only have to show that the relation

$$\langle (\lambda - M_1)^{-1} f, \varphi \rangle_\Omega = 0 \quad \text{for } {}^\forall f \in L^2(\Omega)$$

implies that $\varphi = 0$. We see from (2.13) that

$$\begin{aligned} 0 &= \langle (\lambda - L_1)^{-1} f, \varphi \rangle_\Omega + \sum_{k=1}^{p} \left[(1 - \Phi(\lambda))^{-1} \langle (\lambda - L_1)^{-1} f, \mathrm{w} \rangle_\Omega \right]_k \langle N_1(\lambda) h_k, \varphi \rangle_\Omega \\ &= \langle (\lambda - L_1)^{-1} f, \varphi \rangle_\Omega + \sum_{k=1}^{p} a_k \langle (\lambda - L_1)^{-1} f, w_k \rangle_\Omega \\ &= \left\langle f, (\overline{\lambda} - L_1^*)^{-1} \Big(\varphi + \sum_{k=1}^{p} \overline{a_k} w_k \Big) \right\rangle_\Omega, \end{aligned}$$

that is

$$(\overline{\lambda} - L_1^*)^{-1} \Big(\varphi + \sum_{k=1}^{p} \overline{a_k} w_k \Big) = 0, \quad \text{or} \quad \varphi + \sum_{k=1}^{p} \overline{a_k} w_k = 0,$$

where

$$(a_1 \cdots a_p) = \big(\langle N_1(\lambda) h_1, \varphi \rangle_\Omega \cdots \langle N_1(\lambda) h_p, \varphi \rangle_\Omega \big) (1 - \Phi(\lambda))^{-1}.$$

Thus we see that

$$\begin{aligned} 0 &= \left\langle N_1(\lambda) h_j, \varphi + \sum_{k=1}^{p} \overline{a_k} w_k \right\rangle_\Omega \\ &= \langle N_1(\lambda) h_j, \varphi \rangle_\Omega + \sum_{k=1}^{p} a_k \langle N_1(\lambda) h_j, w_k \rangle_\Omega, \quad 1 \le j \le p, \end{aligned}$$

or

$$\begin{aligned}(0 \cdots 0) &= \big(\langle N_1(\lambda)h_1, \varphi\rangle_\Omega \cdots \langle N_1(\lambda)h_p, \varphi\rangle_\Omega\big) + (a_1 \cdots a_p)\Phi(\lambda) \\ &= \big(\langle N_1(\lambda)h_1, \varphi\rangle_\Omega \cdots \langle N_1(\lambda)h_p, \varphi\rangle_\Omega\big)(1-\Phi(\lambda))^{-1} \\ &= (a_1 \cdots a_p).\end{aligned}$$

We have shown that $\varphi = 0$.

Case II. The domain $\mathcal{D}(M_2)$ is clearly dense, since $\mathcal{D}(\Omega)$ $(= C_0^\infty(\Omega))$ is contained in $\mathcal{D}(M_2)$. Let us consider the boundary value problem

$$(\lambda - \mathcal{L})u = 0 \quad \text{in } \Omega \quad \text{and} \quad \tau_2 u = \frac{\partial u}{\partial \nu} + \sigma(\xi)u = f \quad \text{on } \Gamma$$

for any given $f \in H^{1/2}(\Gamma)$. There is a unique solution $u \in H^2(\Omega)$ for $\lambda \in \rho(L_2)$, and the solution u is denoted by $N_2(\lambda)f$, where $N_2(\lambda) \in \mathcal{L}(H^{1/2}(\Gamma);\, H^2(\Omega))$. By introducing an operator R_2 such that [8]

$$R_2 f\Big|_\Gamma = 0, \quad \text{and} \quad \frac{\partial}{\partial \nu} R_2 f\Big|_\Gamma = f, \quad {}^\forall f \in H^{1/2}(\Gamma), \tag{2.14}$$

the solution $N_2(\lambda)f$ is expressed as

$$N_2(\lambda)f = R_2 f - (\lambda - L_2)^{-1}(\lambda - \mathcal{L})R_2 f.$$

In order to consider the boundary value problem

$$(\lambda - M_2)u = f, \tag{2.15}$$

a sesquilinear form is available in our case. The sesquilinear form associated with M_2 is the form on $H^1(\Omega)$ given by

$$\begin{aligned}B[u,\varphi] = &\sum_{i,j=1}^m \Big\langle a_{ij}(x)\frac{\partial u}{\partial x_j}, \frac{\partial \varphi}{\partial x_i}\Big\rangle_\Omega + \sum_{i=1}^m \Big\langle b_i(x)\frac{\partial u}{\partial x_i}, \varphi\Big\rangle_\Omega + \langle c(x)u, \varphi\rangle_\Omega \\ &+ \langle \sigma(\xi)u, \varphi\rangle_\Gamma - \sum_{k=1}^p \langle u, w_k\rangle_\Gamma \langle h_k, \varphi\rangle_\Gamma .\end{aligned}$$

By setting $B_c[u,\varphi] = B[u,\varphi] + c\langle u, \varphi\rangle_\Omega$ for a sufficiently large constant $c > 0$, a standard argument [9] shows that

$$\operatorname{Re} B_c[u,u] \geq \text{const}\, \|u\|^2_{H^1(\Omega)}, \quad u \in H^1(\Omega) \quad \text{and}$$

$$\big|B_c[u,\varphi]\big| \leq \text{const}\, \|u\|_{H^1(\Omega)}\, \|\varphi\|_{H^1(\Omega)}.$$

Thus, for any $f \in L^2(\Omega)$, there exists a unique $u \in H^1(\Omega)$ such that

$$B_c[u,\varphi] = \langle f, \varphi\rangle_\Omega, \quad {}^\forall \varphi \in H^1(\Omega).$$

Let $v \in H^2(\Omega)$ be the unique solution to the problem

$$\mathcal{L}_c v = f, \quad \tau_2 v = \sum_{k=1}^{p} \langle u, w_k \rangle_\Gamma h_k .$$

The solution v is expressed as

$$v = L_{2c}^{-1} f + \sum_{k=1}^{p} \langle u, w_k \rangle_\Gamma \, N_2(-c) h_k.$$

Green's formula implies that, for any $\varphi \in H^1(\Omega)$,

$$\langle f, \varphi \rangle_\Omega = \langle \mathcal{L}_c v, \varphi \rangle_\Omega = \Big\langle \sigma v - \sum_k \langle u, w_k \rangle_\Gamma h_k, \varphi \Big\rangle_\Gamma + \sum_{i,j} \Big\langle a_{ij} \frac{\partial v}{\partial x_j}, \frac{\partial \varphi}{\partial x_i} \Big\rangle_\Omega$$
$$+ \sum_i \Big\langle b_i \frac{\partial v}{\partial x_i}, \varphi \Big\rangle_\Omega + \langle (c(x) + c) v, \varphi \rangle_\Omega .$$

Thus we see that

$$\widetilde{B}_c \, [v - u, \varphi] = 0 \quad \text{for } {}^\forall \varphi \in H^1(\Omega),$$

where $\widetilde{B}_c$ denotes the sesquilinear form associated with L_{2c} ($\widetilde{B}_c$ is a special B_c in the case where $w_k = 0$ or $h_k = 0$, $1 \le k \le p$). Since $c > 0$ is large enough, we see that

$$\operatorname{Re} \widetilde{B}_c \, [g, g] \ge \text{const} \, \|g\|^2_{H^1(\Omega)}$$

for all $g \in H^1(\Omega)$. This shows that

$$u = v \in H^2(\Omega) \quad \text{and} \quad M_{2c} u = f.$$

Uniqueness of the solution u will be immediate, due to coerciveness of B_c. The operator M_{2c} is a continuous bijection from $\mathcal{D}(M_2)$ onto $L^2(\Omega)$. Thus the inverse M_{2c}^{-1} belongs to $\mathcal{L}(L^2(\Omega); \mathcal{D}(M_{2c}))$, or

$$\|u\|_{H^2(\Omega)} \le \text{const} \, \|M_{2c} u\|, \quad \text{for } {}^\forall u \in H^2(\Omega).$$

Let us go back to (2.15). The problem (2.15) is equivalent to the solvability of the problem

$$((\lambda + c) M_{2c}^{-1} - 1) u = M_{2c}^{-1} f$$

in $L^2(\Omega)$. Since $M_{2c}^{-1} \in \mathcal{L}(L^2(\Omega))$ is compact, we only have to seek the region of λ in which uniqueness of solutions to (2.15) holds (the Riesz-Schauder theory [9, 15]). Now it is straightforward to find out this region and to obtain the estimate (2.7) in some sector $\overline{\sum_{-\gamma}}$ (see, e.g., [1, 9]). Since the calculation is very elementary but tedious, we omit the rest of the proof. The proof of Theorem 2.3 is thereby completed. Q.E.D.

3. Main results

In Theorem 2.3, we have shown that, if $c > 0$ is chosen large enough, a sector obtained as a suitable right shift of $\overline{\sum}$ is contained in the resolvent sets $\rho(M_{1c})$ and $\rho(M_{2c})$, and the decay estimates for the resolvents $(\lambda - M_{1c})^{-1}$ and $(\lambda - M_{2c})^{-1}$ are guaranteed in that sector. Thus fractional powers for M_{1c} and M_{2c} are well defined. In this section, we extend Theorems 2.1 and 2.2 to the case of M_{1c} and M_{2c}, respectively. Our main results are Theorems 3.1 and 3.2 stated as follows:

Theorem 3.1 (Case I. The Dirichlet boundary condition). *Suppose that w_k, $1 \le k \le p$, belong to $H^{\epsilon}(\Omega)$ for an arbitrarily small $\epsilon > 0$. Then the domain of the fractional powers M_{1c}^{θ}, $0 \le \theta \le 1$, is characterized as follows:*

(i) $\mathcal{D}(M_{1c}^{\theta}) = H^{2\theta}(\Omega), \quad 0 \le \theta < \dfrac{1}{4};$

(ii) $\mathcal{D}(M_{1c}^{1/4}) = \left\{ u \in H^{1/2}(\Omega);\ \displaystyle\int_{\Omega} \frac{1}{\zeta(x)} \left| u - \sum_{k=1}^{p} \langle u, w_k \rangle_{\Omega} R_1 h_k \right|^2 dx < \infty \right\};$ *and*

(iii) $\mathcal{D}(M_{1c}^{\theta}) = H_{f1}^{2\theta}(\Omega), \quad \dfrac{1}{4} < \theta \le 1,$

where $H_{f1}^{2\theta}(\Omega)$ denotes the space defined by

$$H_{f1}^{2\theta}(\Omega) = \left\{ u \in H^{2\theta}(\Omega);\ u\Big|_{\Gamma} = \sum_{k=1}^{p} \langle u, w_k \rangle_{\Omega} h_k \ \text{ on } \Gamma \right\}, \qquad 2\theta > \frac{1}{2}.$$

Moreover, we have the interpolation relation

$$\mathcal{D}(M_{1c}^{\theta}) = [\mathcal{D}(M_1), L^2(\Omega)]_{1-\theta}, \quad 0 \le \theta \le 1,$$

where $[\cdot, \cdot]_{1-\theta}$ denotes an intermediate space lying between two spaces, one of which is densely embedded in the other.

Theorem 3.2 (Case II. The generalized Neumann boundary condition). *The domain of the fractional powers M_{2c}^{θ}, $0 \le \theta \le 1$, is characterized as follows:*

(i) $\mathcal{D}(M_{2c}^{\theta}) = H^{2\theta}(\Omega), \quad 0 \le \theta < \dfrac{3}{4};$

(ii) $\mathcal{D}(M_{2c}^{3/4}) = \left\{ u \in H^{3/2}(\Omega);\ \displaystyle\int_{\Omega} \frac{1}{\zeta(x)} \left| \tau_{\Omega} u - \sum_{k=1}^{p} \langle u, w_k \rangle_{\Gamma} \tau_{\Omega} R_2 h_k \right|^2 dx < \infty \right\};$

and

(iii) $\mathcal{D}(M_{2c}^{\theta}) = H_{f2}^{2\theta}(\Omega) = \left\{ u \in H^{2\theta}(\Omega);\ \tau_2 u = \displaystyle\sum_{k=1}^{p} \langle u, w_k \rangle_{\Gamma} h_k \ \text{ on } \Gamma \right\},$

$\dfrac{3}{4} < \theta \le 1.$

The following result discusses algebraic similarity of M_1 and M_2 to operators with homogeneous boundary conditions: $\tau_1 u = 0$ and $\tau_2 u = 0$, respectively. Originally it comes from a control theoretic study of M_1 and M_2:

Theorem 3.3. (i) *For any $\theta \in \mathbb{R}^1$, M_{1c}^{θ} is algebraically similar to $(L_{1c} - F_1)^{\theta}$ in the sense that*

$$M_{1c}^{\theta} = L_{1c}^{3/4+\epsilon}(L_{1c} - F_1)^{\theta} L_{1c}^{-3/4-\epsilon}, \quad \text{and} \quad \rho(M_{1c}^{\theta}) = \rho((L_{1c} - F_1)^{\theta}).$$

where $0 < \epsilon < 1/4$ and the operator F_1 is defined by

$$F_1 u = \sum_{k=1}^{p} \langle L_{1c}^{3/4+\epsilon} u,\, w_k \rangle_{\Omega}\, L_{1c}^{1/4-\epsilon} N_1(-c) h_k .$$

(ii) *For any $\theta \in \mathbb{R}^1$, M_{2c}^{θ} is algebraically similar to $(L_{2c} - F_2)^{\theta}$ in the sense that*

$$M_{2c}^{\theta} = L_{2c}^{1/4+\epsilon}(L_{2c} - F_2)^{\theta} L_{2c}^{-1/4-\epsilon}, \quad \text{and} \quad \rho(M_{2c}^{\theta}) = \rho((L_{2c} - F_2)^{\theta}).$$

where $0 < \epsilon < 1/2$ and the operator F_2 is defined by

$$F_2 u = \sum_{k=1}^{p} \langle L_{2c}^{1/4+\epsilon} u,\, w_k \rangle_{\Gamma}\, L_{1c}^{3/4-\epsilon} N_2(-c) h_k .$$

As we have seen in Section 2, the approach to M_1 in this section is also quite different from the one to M_2.

Proof of Theorem 3.1.

First Step (*Operator* T_1). A serious difficulty is that M_1 is *no more* an accretive operator. So, our strategy is to introduce, instead, another operator K defined below (Second Step) via T_1, where T_1 denotes an operator formally defined by

$$v = T_1 u = u - \sum_{k=1}^{p} \langle u,\, w_k \rangle_{\Omega}\, R_1 h_k . \tag{3.1}$$

It turns out that the operator K is accretive if an additional regularity assumption on w_k's is added (see Proposition 3.4, (ii)).

By definition, operator T_1 clearly belongs to $\mathcal{L}(L^2(\Omega)) \cap \mathcal{L}(\mathcal{D}(M_1)\,;\,\mathcal{D}(L_1))$, where both $\mathcal{D}(M_1)$ and $\mathcal{D}(L_1)$ are equipped with the topology of $H^2(\Omega)$. Let us examine its inverse. Set $T_1 u = 0$. Then

$$\langle u,\, w_j \rangle_{\Omega} = \sum_{k=1}^{p} \langle u,\, w_k \rangle_{\Omega}\, \langle R_1 h_k,\, w_j \rangle_{\Omega}, \quad 1 \le j \le p, \quad \text{or}$$

$$\langle u,\, \mathbf{w} \rangle_{\Omega} = \Psi \langle u,\, \mathbf{w} \rangle_{\Omega},$$

where Ψ means the $p \times p$ matrix defined by

$$\Psi = \left[\langle R_1 h_k,\, w_j \rangle_{\Omega}\,;\; \begin{matrix} k \to 1, \ldots, p \\ j \downarrow 1, \ldots, p \end{matrix} \right].$$

Since R_1 admits a great deal of freedom of choice, we first assume that $\det(1-\Psi) \neq 0$. In fact, we only have to make a slight modification of R_1, if necessary. A general R_1 assuming only (2.10) is considered later in the Fourth Step of the proof. Under this assumption, we see that

$$\langle u, \mathrm{w}\rangle_{\Omega} = \mathbf{0}, \quad \text{or} \quad u = 0.$$

Thus T_1 is injective (namely, its formal inverse T_1^{-1} exists), and the inverse T_1^{-1} is calculated as

$$u = T_1^{-1}v = v + \sum_{k=1}^{p} \Big[(1-\Psi)^{-1}\langle v, \mathrm{w}\rangle_{\Omega}\Big]_k R_1 h_k . \tag{3.2}$$

The operator T_1^{-1} belongs to $\mathcal{L}(L^2(\Omega))$. Moreover, $u = T_1^{-1}v$ satisfies the relation

$$\langle u, \mathrm{w}\rangle_{\Omega} = (1-\Psi)^{-1}\langle v, \mathrm{w}\rangle_{\Omega} .$$

Thus T_1^{-1} maps $\mathcal{D}(L_1)$ onto $\mathcal{D}(M_1)$ and belongs to $\mathcal{L}(\mathcal{D}(L_1)\,;\ \mathcal{D}(M_1))$. The well known interpolation theory [8] implies that

$$\begin{aligned} &T_1 \in \mathcal{L}([\mathcal{D}(M_1),\, L^2(\Omega)]_{1-\theta}\,;\ \mathcal{D}(L_{1c}^{\theta}))\,, \quad \text{and} \\ &T_1^{-1} \in \mathcal{L}(\mathcal{D}(L_{1c}^{\theta})\,;\ [\mathcal{D}(M_1),\, L^2(\Omega)]_{1-\theta})\,, \quad 0 \le \theta \le 1. \end{aligned} \tag{3.3}$$

Here we have used the fact that $[\mathcal{D}(L_1),\, L^2(\Omega)]_{1-\theta}$ is equal to $\mathcal{D}(L_{1c}^{\theta})$ due to the m-accretiveness of L_{1c}.

Second Step (*Operator* K). Owing to the First Step, we are able to introduce a new operator K by

$$K = T_1 M_1 T_1^{-1}, \quad \mathcal{D}(K) = \mathcal{D}(L_1) = H^2(\Omega) \cap H_0^1(\Omega). \tag{3.4}$$

The operator K plays a role of connecting M_1 with L_1 (see the diagram at the end of the Second Step). If λ is in $\rho(M_1)$, then $\lambda - K$ has a bounded inverse, and

$$(\lambda - K)^{-1} = T_1(\lambda - M_1)^{-1}T_1^{-1} \in \mathcal{L}(L^2(\Omega)).$$

In view of the decay estimate (2.5), the sector $\overline{\sum}_{-\beta}$ is contained in $\rho(K)$ and

$$\|(\lambda - K)^{-1}\| \le \frac{\text{const}}{1+|\lambda|}, \quad \lambda \in \overline{\textstyle\sum}_{-\beta}.$$

Thus, if c is larger than β, fractional powers of $K_c = K + c$ are well defined. The operator $K_c^{-\theta}$ is by definition calculated as follows:

$$\begin{aligned} K_c^{-\theta} &= \frac{-1}{2\pi \mathrm{i}}\int_C \lambda^{-\theta}(\lambda - K_c)^{-1}d\lambda = \frac{-1}{2\pi \mathrm{i}}\int_C \lambda^{-\theta}T_1(\lambda - M_{1c})^{-1}T_1^{-1}d\lambda \\ &= T_1 M_{1c}^{-\theta}T_1^{-1}\,, \quad \theta \ge 0, \end{aligned} \tag{3.5}$$

where i $= \sqrt{-1}$, and C denotes the boundary of a suitable right shift of the sector $\sum$, oriented according to increasing $\operatorname{Im}\lambda$. The operator K enjoys nice properties. For example, relation (3.5) immediately implies that

$$T_1 \in \mathcal{L}(\mathcal{D}(M_{1c}^{\theta})\,;\,\mathcal{D}(K_c^{\theta})) \quad \text{and} \quad T_1^{-1} \in \mathcal{L}(\mathcal{D}(K_c^{\theta})\,;\,\mathcal{D}(M_{1c}^{\theta}))\,, \quad 0 \le \theta \le 1. \tag{3.6}$$

The following proposition forms a key result of the theorem, the proof of which is stated in the Last Step:

Proposition 3.4. (i) *If c is large enough, the equivalence relation*

$$\mathcal{D}(K_c^{\theta}) = \mathcal{D}(L_{1c}^{\theta}), \quad 0 \le \theta \le 1 \tag{3.7}$$

holds algebraically and topologically.

(ii) *If w_k, $1 \le k \le p$, belong to $H_0^1(\Omega)$ in addition, then K_c is m-accretive, namely*

$$\operatorname{Re}\langle K_c u,\, u\rangle_{\Omega} \ge \text{const}\,\|u\|^2, \quad u \in \mathcal{D}(K). \tag{3.8}$$

Remark. The above (i) is proved independent of (ii). In the case where w_k's belong to $H_0^1(\Omega)$, however, (ii) immediately implies the assertion (i), once we observe the equivalence relation: $\mathcal{D}(K) = \mathcal{D}(L_1)$. In fact, now that both K_c and L_{1c} are m-accretive in this case, a generalization of the Heinz inequality [6] is now applied to show the equivalence relation (3.7).

According to this proposition, relation (3.6) is rewritten as

Lemma 3.5. *The operator T_1 is a continuous bijection from $\mathcal{D}(M_{1c}^{\theta})$ onto $\mathcal{D}(L_{1c}^{\theta})$ for each $0 \le \theta \le 1$, and thus,*

$$T_1 \in \mathcal{L}(\mathcal{D}(M_{1c}^{\theta})\,;\,\mathcal{D}(L_{1c}^{\theta})) \quad \textit{and} \quad T_1^{-1} \in \mathcal{L}(\mathcal{D}(L_{1c}^{\theta})\,;\,\mathcal{D}(M_{1c}^{\theta}))\,, \quad 0 \le \theta \le 1. \tag{3.9}$$

Although the m-accretiveness of M_{1c} is *never* expected and thus a generalization of the Heinz inequality [6] cannot be applied, relation (3.3) combined with Lemma 3.5 yields the last assertion of the theorem:

$$\mathcal{D}(M_{1c}^{\theta}) = [\mathcal{D}(M_1),\, L^2(\Omega)]_{1-\theta}\,, \quad 0 \le \theta \le 1.$$

The above relations are summarized as the following diagram:

$$[\mathcal{D}(M_1)\,;\,L^2(\Omega)]_{1-\theta} \;\underset{T_1^{-1}}{\overset{T_1}{\rightleftarrows}}\; \mathcal{D}(L_{1c}^{\theta}) = \mathcal{D}(K_c^{\theta}) \;\underset{T_1}{\overset{T_1^{-1}}{\rightleftarrows}}\; \mathcal{D}(M_{1c}^{\theta})$$

Third Step. We are in a position to prove the characterization : (i) to (iii) of the theorem.

Proof of (i) (the case where $0 \le 2\theta < 1/2$). Note that both T_1 and T_1^{-1} belong to $\mathcal{L}(H^{2\theta}(\Omega))$, $0 \le \theta < 1/4$. Since T_1 also belongs to $\mathcal{L}(\mathcal{D}(M_{1c}^{\theta})\,;\,H^{2\theta}(\Omega))$ and T_1^{-1} to

$\mathcal{L}(H^{2\theta}(\Omega)\,;\,\mathcal{D}(M_{1c}^{\omega}))$ by Lemma 3.5 and Theorem 2.1, (i), the assertion of (i) is now immediate.

Proof of (ii) (the case where $2\theta = 1/2$). Suppose that

$$u \in H^{1/2}(\Omega)\,, \quad \text{and} \quad \int_\Omega \frac{1}{\zeta(x)} |T_1 u|^2 dx < \infty. \tag{3.10}$$

Then $v = T_1 u$ belongs to $\mathcal{D}(L_{1c}^{1/2})$, due to Theorem 2.1, (ii). We see from Lemma 3.5 that $u = T_1^{-1} v$ belongs to $\mathcal{D}(M_{1c}^{1/2})$. Conversely, if u is in $\mathcal{D}(M_{1c}^{1/2})$, then $v = T_1 u$ belongs to $\mathcal{D}(L_{1c}^{1/2})$, again due to Lemma 3.5. Thus u satisfies (3.10).

Proof of (iii) (the case where $1/2 < 2\theta \le 2$). It is easily seen from Theorem 2.1, (iii) that

$$T_1 \in \mathcal{L}(H_{f1}^{2\theta}(\Omega)\,;\,\mathcal{D}(L_{1c}^{\theta})) \quad \text{and} \quad T_1^{-1} \in \mathcal{L}(\mathcal{D}(L_{1c}^{\theta})\,;\,H_{f1}^{2\theta}(\Omega))\,,$$

which, again combined with Lemma 3.5, shows that

$$\mathcal{D}(M_{1c}^{\theta}) = H_{f1}^{2\theta}(\Omega)\,, \quad 1/2 < 2\theta \le 2\,.$$

Fourth Step (*Operator* R_1). We have assumed so far that $\det(1 - \Psi) \neq 0$ in the First Step. Let us consider a general R_1, say $\widetilde{R}_1$ assuming only (2.10). Then

$$(\widetilde{R}_1 - R_1)h_k\big|_\Gamma = 0\,, \quad 1 \le k \le p\,.$$

It is enough to consider the behavior of these functions in a neighborhood of Γ. Introducing a partition of unity of Γ, we can move to the half space $\mathbb{R}^m_{y+} = \{y = (y_1, \cdots, y_m) \in \mathbb{R}^m;\ y_m > 0\}$. In a neighborhood of $\{y_m = 0\}$, the transformed $\zeta(x)$ behaves like y_m. The transformed $(\widetilde{R}_1 - R_1)h_k$, still denoted as the same symbol, belong to $H^2(\mathbb{R}^m_{y+})$; are absolutely continuous in y_m for almost all $y' = (y_1, \cdots, y_{m-1})$; and satisfy

$$(\widetilde{R}_1 - R_1)h_k(y', y_m) = \int_0^{y_m} \frac{\partial}{\partial y_m}(\widetilde{R}_1 - R_1)h_k(y', t)dt$$

for almost all $y' \in \mathbb{R}^{m-1}$. Thus, by going back to the original coordinates, it is immediately seen that

$$\int_\Omega \frac{1}{\zeta(x)} \left|(\widetilde{R}_1 - R_1)h_k\right|^2 dx < \infty\,,$$

which shows that the expression (ii) does not depend on a particular choice of R_1.

Last Step. In order to complete the theorem, let us turn to the proof of the auxiliary results mentioned above.

Proof of Proposition 3.4.

(i) Let us examine what the form of the operator $K = T_1 M_1 T_1^{-1}$ is. Note that w_k's belong to $H^{2\epsilon}(\Omega) = \mathcal{D}(L_{1c}^{*\epsilon})$, $0 < \epsilon < 1/4$. Then K is, by definition, written as

$$\begin{aligned} Ku &= L_1 u - \sum_{k=1}^{p} \langle L_1 u,\, w_k\rangle_\Omega \, R_1 h_k + \sum_{k=1}^{p} \Big[(1-\Psi)^{-1}\langle u,\, \mathrm{w}\rangle_\Omega\Big]_k T_1 \mathcal{L} R_1 h_k \\ &= L_1 u - \sum_{k=1}^{p} \langle L_{1c}^{1-\epsilon} u,\, L_{1c}^{*\epsilon} w_k\rangle_\Omega \, R_1 h_k \\ &\quad + \sum_{k=1}^{p} \Big[(1-\Psi)^{-1}\langle u,\, \mathrm{w}\rangle_\Omega\Big]_k T_1 \mathcal{L} R_1 h_k + c\sum_{k=1}^{p} K u w_k \, R_1 h_k \\ &= L_1 u + Du, \quad u \in \mathcal{D}(K). \end{aligned}$$

Here, D is an operator subordinate to $L_{1c}^{1-\epsilon}$, namely

$$\|Du\| \le \mathrm{const}\, \|L_{1c}^{1-\epsilon} u\|, \quad u \in \mathcal{D}(L_{1c}^{1-\epsilon}).$$

Since $\mathcal{D}(K_c)$ is equal to $\mathcal{D}(L_{1c})$ anyway (see the footnote below Proposition 3.4), we see that the relations

$$\mathcal{D}(K_c^{\beta}) \subset \mathcal{D}(L_{1c}^{\alpha}), \quad \text{and} \quad \mathcal{D}(L_{1c}^{\beta}) \subset \mathcal{D}(K_c^{\alpha}), \quad 0 \le \alpha < \beta \tag{3.11}$$

hold algebraically and topologically [7]. Note that

$$\begin{aligned} K_c^{-\omega} - L_{1c}^{-\omega} &= \frac{-1}{2\pi \mathrm{i}} \int_C \lambda^{-\omega} (\lambda - K_c)^{-1} D (\lambda - L_{1c})^{-1}\, d\lambda \\ &= \frac{-1}{2\pi \mathrm{i}} \int_C \lambda^{-\omega} (\lambda - L_{1c})^{-1} D (\lambda - K_c)^{-1}\, d\lambda, \quad 0 \le \omega \le 1, \end{aligned}$$

where C denotes a contour of a suitable right shift of $\partial\overline{\sum}$ oriented according to increasing $\mathrm{Im}\,\lambda$. For any given $u \in \mathcal{D}(K_c^{\omega})$, $0 \le \omega \le 1$, there is a unique $\varphi \in L^2(\Omega)$ such that $u = K_c^{-\omega}\varphi$. Thus,

$$K_c^{-\omega}\varphi = L_{1c}^{-\omega}\varphi - \frac{1}{2\pi \mathrm{i}} \int_C \lambda^{-\omega} (\lambda - L_{1c})^{-1} D (\lambda - K_c)^{-1} \varphi\, d\lambda .$$

According to (3.11) and the moment inequality for K_c, the integrand is estimated as folows:

$$\begin{aligned} \|D(\lambda - K_c)^{-1}\varphi\| &\le \mathrm{const}\, \|L_{1c}^{1-\epsilon}(\lambda - K_c)^{-1}\varphi\| \\ &\le \mathrm{const}\, \|L_{1c}^{1-\epsilon} K_c^{-\eta}\|\, \|K_c^{\eta}(\lambda - K_c)^{-1}\varphi\| \le \frac{\mathrm{const}}{(1+|\lambda|)^{1-\eta}} \|\varphi\|, \end{aligned}$$

where $1 - \epsilon < \eta < 1$. Thus we see that

$$\|\lambda^{-\omega} L_{1c}^{\omega} \lambda^{-\omega} (\lambda - L_{1c})^{-1} D (\lambda - K_c)^{-1}\varphi\| \le \frac{\mathrm{const}}{(1+|\lambda|)^{2-\eta}} \|\varphi\|,$$

the last term of which is integrable on C. This means that $\mathcal{D}(K_c^\omega)$ is contained in $\mathcal{D}(L_{1c}^\omega)$, and that

$$L_{1c}^\omega u = \varphi - \frac{1}{2\pi \mathrm{i}} \int_C \lambda^{-\omega} L_{1c}^\omega (\lambda - L_{1c})^{-1} D (\lambda - K_c)^{-1} \varphi \, d\lambda ,$$

and

$$\|L_{1c}^\omega u\| \le \text{const} \, \|\varphi\| = \text{const} \, \|K_c^\omega u\| .$$

The converse relation

$$\mathcal{D}(L_{1c}^\omega) \subset \mathcal{D}(K_c^\omega) , \quad \text{and} \quad \|K_c^\omega u\| \le \text{const} \, \|L_{1c}^\omega u\|$$

is similarly proved. This finishes the proof of (i).

(ii) We first note that the adjoint operator of T_1 in $L^2(\Omega)$ is given by

$$T_1^* u = u - \sum_{k=1}^{p} \langle u, \, R_1 h_k \rangle_\Omega \, w_k . \tag{3.12}$$

According to the assumption (2.4) on w_k, we see that

$$T_1^* u \big|_\Gamma = 0 , \quad \text{for } u \in H^\theta_{\gamma_0}(\Omega) , \quad \theta > \frac{1}{2} .$$

Applying Green's formula, we calculate as

$$\begin{aligned} \langle K_c u, \, u \rangle_\Omega &= \langle T_1 M_{1c} T_1^{-1} u, \, u \rangle_\Omega = \langle \mathcal{L}_c T_1^{-1} u, \, T_1^* u \rangle_\Omega \\ &= \Big\langle \frac{\partial}{\partial \nu} T_1^{-1} u, T_1^* u \Big\rangle_\Gamma + B_{1c}[T_1^{-1} u, T_1^* u] \\ &= B_{1c}[T_1^{-1} u, T_1^* u] , \qquad u \in \mathcal{D}(K_c) , \end{aligned}$$

where $B_{1c}[\cdot, \cdot] = B_1[\cdot, \cdot] + c \langle \cdot, \, \cdot \rangle_\Omega$ denotes the sesquilinear form on $H^1(\Omega)$, and

$$B_1[u, \varphi] = \sum_{i,j=1}^{m} \Big\langle a_{ij}(x) \frac{\partial u}{\partial x_j} , \frac{\partial \varphi}{\partial x_i} \Big\rangle_\Omega + \sum_{i=1}^{m} \Big\langle b_i(x) \frac{\partial u}{\partial x_i} , \varphi \Big\rangle_\Omega + \langle c(x) u, \, \varphi \rangle_\Omega .$$

Note that B_1 is a special case of the sesquilinear form B associated with M_2 (see, e.g., Section 2). Thus, if c is large enough, we have the inequalities

$$\operatorname{Re} B_{1c}[u, u] \ge \delta \, \|u\|^2_{H^1(\Omega)} , \quad u \in H^1(\Omega) \quad \text{and}$$

$$\big| B_{1c}[u, \varphi] \big| \le \gamma \, \|u\|_{H^1(\Omega)} \, \|\varphi\|_{H^1(\Omega)} ,$$

for some positive δ and γ. Thus we estimate as

$$\begin{aligned} \operatorname{Re} \langle K_c u, \, u \rangle_\Omega &= \operatorname{Re} B_{1c}[T_1^{-1} u, T_1^* u] \\ &= \operatorname{Re} B_{1c}[T_1^* u, T_1^* u] + \operatorname{Re} B_{1c}[T_1^{-1} u - T_1^* u, T_1^* u] \\ &\ge \|T_1^* u\|_{H^1(\Omega)} \left\{ \delta \, \|T_1^* u\|_{H^1(\Omega)} - \gamma \, \|T_1^{-1} u - T_1^* u\|_{H^1(\Omega)} \right\} . \end{aligned} \tag{3.13}$$

According to the expression (3.2) of T_1^{-1}, we note that

$$\begin{aligned}\|T_1^{-1}u - T_1^* u\|_{H^1(\Omega)} &= \left\| \sum_{k=1}^{p} \left[(1-\Psi)^{-1} \langle u,\, \mathrm{w} \rangle_\Omega \right]_k R_1 h_k + \sum_{k=1}^{p} \langle u,\, R_1 h_k \rangle_\Omega \, w_k \right\|_{H^1(\Omega)} \\ &\le C\,\|u\| .\end{aligned}$$

Here, $C > 0$ denotes some constant. It is significant that the above left-hand side is bounded from above by the $L^2(\Omega)$-*norm of* u. Substituting the above inequality into (3.13), we see that, for any $\epsilon > 0$,

$$\mathrm{Re}\,\langle K_c u,\, u\rangle_\Omega \ge \left(\delta - \frac{C\gamma\epsilon}{2}\right) \|T_1^* u\|^2_{H^1(\Omega)} - \frac{C\gamma}{2\epsilon}\,\|u\|^2 .$$

Choosing ϵ small enough and then $d > \dfrac{C\gamma}{2\epsilon}$, we obtain the desired estimate

$$\mathrm{Re}\,\langle K_{c+d} u,\, u\rangle_\Omega = \mathrm{Re}\,\langle K_c u,\, u\rangle_\Omega + d\,\langle u,\, u\rangle_\Omega \ge \mathrm{const}\,\|u\|^2 , \quad u \in \mathcal{D}(K) .$$

Thus, by replacing c by a larger constant $c+d$, the m-accretiveness of K_c has been proved. The proof of Theorem 3.1 is thereby complete. Q.E.D.

Proof of Theorem 3.2.

The proof is somewhat simpler than the proof of Theorem 3.1, since the operator M_{2c} is m-accretive in our case. An operator similar to T_1 appears later in the Third Step. In order to apply this operator, however, we must introduce the operator $L_{2c} - F_2$ similar to M_{2c} in the First Step.

First Step (*Operator* $L_2 - F_2$). We shall see that the operator F_2 in Theorem 3.3 naturally appears in the following context: Let us consider the following differential equation in $L^2(\Omega)$:

$$\frac{du}{dt} + M_2 u = 0, \quad u(0) = u_0 \in L^2(\Omega). \tag{3.14}$$

Problem (3.14) is well posed and generates an analytic semigroup $\exp(-tM_2)$, $t > 0$, due to Theorem 2.3, (ii), and a unique solution u is given by $u(t) = \exp(-tM_2)\,u_0$. For any fixed θ, $1/4 < \theta < 3/4$, set $v(t) = L_{2c}^{-\theta} u(t)$. According to Theorem 2.2, (i), $v(t)$ belongs to $\mathcal{D}(L_2)$ and satisfies the differential equation

$$\frac{dv}{dt} + (L_2 - F_2)v = 0, \quad v(0) = v_0 = L_{2c}^{-\theta} u_0, \tag{3.15}$$

where F_2 is defined in Theorem 3.3 as

$$F_2 v = \sum_{k=1}^{p} \langle L_{2c}^{\theta} v,\, w_k\rangle_\Gamma\, L_{2c}^{1-\theta} N_2(-c) h_k , \quad \mathcal{D}(F_2) \supset \mathcal{D}(L_2).$$

In fact, (3.14) is rewritten as

$$\begin{aligned} 0 &= \frac{du}{dt} - cu + \mathcal{L}_c\Big(u - \sum_{k=1}^{p} \langle u,\, w_k\rangle_\Gamma\, N_2(-c) h_k\Big) \\ &= \frac{du}{dt} - cu + L_{2c}\Big(u - \sum_{k=1}^{p} \langle u,\, w_k\rangle_\Gamma\, N_2(-c) h_k\Big). \end{aligned}$$

By applying $L_{2c}^{-\theta}$ to the both sides, equation (3.15) for v is obtained. Since θ is less than 3/4, the following lemma is immediate:

Lemma 3.6. *The operator $L_2 - F_2$ has a compact resolvent. There is a $\delta > 0$ such that $\overline{\sum}_{-\delta}$ is contained in $\rho(L_2 - F_2)$, and that*

$$\|(\lambda - L_2 + F_2)^{-1}\| \leq \frac{\text{const}}{1+|\lambda|}, \quad \lambda \in \overline{\sum}_{-\delta}.$$

Since the problem (3.15) generates an analytic semigroup $\exp(-t(L_2 - F_2))$, $t > 0$, we see that, for $u_0 \in L^2(\Omega)$ and $\operatorname{Re}\lambda < -\delta$

$$\begin{aligned}(\lambda - L_2 + F_2)^{-1} v_0 &= -\int_0^\infty e^{\lambda t}\, e^{-t(L_2 - F_2)} v_0\, dt \qquad (v_0 = L_{2c}^{-\theta} u_0)\\ &= -\int_0^\infty e^{\lambda t}\, L_{2c}^{-\theta} e^{-tM_2} u_0\, dt = L_{2c}^{-\theta}(\lambda - M_2)^{-1} u_0,\end{aligned}$$

or in other words

$$(\lambda - M_2)^{-1} = L_{2c}^{\theta}(\lambda - L_2 + F_2)^{-1} L_{2c}^{-\theta} \tag{3.16}$$

for $\operatorname{Re}\lambda < -\delta$. The right-hand side of eqn. (3.16) is analytic in $\lambda \in \rho(L_2 - F_2)$. Thus, $(\lambda - M_2)^{-1}$ has an extension to an operator analytic in $\lambda \in \rho(L_2 - F_2)$. The extension is, however, nothing but the resolvent of M_2 [2]. This shows that $\rho(L_2 - F_2)$ is contained in $\rho(M_2)$ and that eqn. (3.16) holds for $\lambda \in \rho(L_2 - F_2)$.

Second Step (Proof of (i)*).* Choose a constant $c \geq \max(\delta, \gamma)$ so that fractional powers for M_2 and $L_{2c} - F_2$ are well defined. According to (3.16), we calculate as

$$\begin{aligned}L_{2c}^{-\theta} M_{2c}^{-\theta} &= \frac{-1}{2\pi i}\int_C \lambda^{-\theta} L_{2c}^{-\theta}(\lambda - M_{2c})^{-1}\, d\lambda\\ &= \frac{-1}{2\pi i}\int_C \lambda^{-\theta}(\lambda - L_{2c} + F_2)^{-1} L_{2c}^{-\theta}\, d\lambda = (L_{2c} - F_2)^{-\theta} L_{2c}^{-\theta},\end{aligned}$$

where C denotes a contour of a suitable right shift of $\partial\overline{\sum}$. Thus,

$$M_{2c}^{-\theta} = L_{2c}^{\theta}(L_{2c} - F_2)^{-\theta} L_{2c}^{-\theta}. \tag{3.17}$$

We need to characterize the domain of $(L_{2c} - F_2)^{\theta}$. In view of the definition of the operator F_2, F_2 is subordinate to some power of L_{2c} with exponent *larger than* 1/2. So the m-accretiveness of $L_{2c} - F_2$ is not expected. Nevertheless, we have the following result, the proof of which is omitted.

Proposition 3.7. *The equivalence relation $\mathcal{D}((L_{2c} - F_2)^{\omega}) = \mathcal{D}(L_{2c}^{\omega})$, $0 \leq \omega < 3/4 + \theta$ holds algebraically and topologically.*

According to Proposition 3.7, we see that

$$L_{2c}^{\theta}(L_{2c} - F_2)^{\theta} L_{2c}^{-2\theta} = L_{2c}^{\theta}(L_{2c} - F_2)^{-\theta}(L_{2c} - F_2)^{2\theta} L_{2c}^{-2\theta} \in \mathcal{L}(L^2(\Omega)),$$

since 2θ is less than $3/4+\theta$. Thus the relation (3.17) implies that, for any $u \in \mathcal{D}(L_{2c}^{\theta})$,

$$M_{2c}^{-\theta}(L_{2c}^{\theta}(L_{2c}-F_2)^{\theta}L_{2c}^{-\theta}u) = u\,, \quad \text{or} \quad M_{2c}^{\theta}u = L_{2c}^{\theta}(L_{2c}-F_2)^{\theta}L_{2c}^{-\theta}u\,,$$

which shows that $\mathcal{D}(L_{2c}^{\theta})$ is contained in $\mathcal{D}(M_{2c}^{\theta})$, and that

$$\|M_{2c}^{\theta}u\| \le \text{const}\,\|L_{2c}^{\theta}u\|\,, \quad u \in \mathcal{D}(L_{2c}^{\theta})\,.$$

As to the converse relation, set $v = M_{2c}^{\theta}u$ for $u \in \mathcal{D}(M_{2c}^{\theta})$. Then,

$$u = L_{2c}^{\theta}(L_{2c}-F_2)^{-\theta}L_{2c}^{-\theta}v = L_{2c}^{-\theta}L_{2c}^{2\theta}(L_{2c}-F_2)^{-2\theta}(L_{2c}-F_2)^{\theta}L_{2c}^{-\theta}v \in \mathcal{D}(L_{2c}^{\theta})\,,$$

which shows that $\mathcal{D}(M_{2c}^{\theta})$ is contained in $\mathcal{D}(L_{2c}^{\theta})$, and that

$$\|L_{2c}^{\theta}u\| \le \text{const}\,\|M_{2c}^{\theta}u\|\,, \quad u \in \mathcal{D}(M_{2c}^{\theta})\,.$$

Therefore, we have shown that $\mathcal{D}(M_{2c}^{\theta}) = \mathcal{D}(L_{2c}^{\theta})$ with equivalent graph norms for any θ, $1/4 < \theta < 3/4$. We note that, since both M_{2c} and L_{2c} are m-accretive, the same is true for M_{2c}^{θ} and L_{2c}^{θ}. For a fixed θ, $1/4 < \theta < 3/4$, a generalization of the Heinz inequality [6] is applied to M_{2c}^{θ} and L_{2c}^{θ} to derive that

$$\mathcal{D}(M_{2c}^{\omega}) = \mathcal{D}((M_{2c}^{\theta})^{\omega/\theta}) = \mathcal{D}((L_{2c}^{\theta})^{\omega/\theta}) = \mathcal{D}(L_{2c}^{\omega})\,, \quad 0 \le \omega \le \theta$$

with equivalent graph norms, which proves (i) of the theorem.

Last Step (*Operator* T_2). The proof of (ii) and (iii) is carried out as follows: As we have shown in the First Step, note that $\mathcal{D}(M_{2c}^{1/2})$ is equal to $H^1(\Omega)$. Following T_1 in Theorem 3.1, let us define an operator T_2 formally by

$$v = T_2 u = u - \sum_{k=1}^{p} \langle u,\, w_k\rangle_{\Gamma}\, R_2 h_k\,. \tag{3.18}$$

We can consider $H^1(\Omega)$ as the basic space for T_2. According to the choice of the operator R_2, we note that

$$v\big|_{\Gamma} = u\big|_{\Gamma} - \sum_{k=1}^{p} \langle u,\, w_k\rangle_{\Gamma}\, R_2 h_k\big|_{\Gamma} = u\big|_{\Gamma}\,.$$

Thus it is clear that T_2 is injective, and T_2^{-1} is given by

$$u = T_2^{-1}v = v + \sum_{k=1}^{p} \langle v,\, w_k\rangle_{\Gamma}\, R_2 h_k\,.$$

It is easy to see that

$$T_2 \in \mathcal{L}(\mathcal{D}(M_2)\,;\,\mathcal{D}(L_2)) \cap \mathcal{L}(\mathcal{D}(M_{2c}^{1/2})\,;\,\mathcal{D}(L_{2c}^{1/2}))\,, \quad \text{and}$$
$$T_2^{-1} \in \mathcal{L}(\mathcal{D}(L_2)\,;\,\mathcal{D}(M_2)) \cap \mathcal{L}(\mathcal{D}(L_{2c}^{1/2})\,;\,\mathcal{D}(M_{2c}^{1/2}))\,.$$

Since both M_{2c} and L_{2c} are m-accretive, the interpolation theory implies that

$$\begin{aligned} T_2 \in \mathcal{L}([\mathcal{D}(M_2), \mathcal{D}(M_{2c}^{1/2})]_\theta ; [\mathcal{D}(L_2), \mathcal{D}(L_{2c}^{1/2})]_\theta) \\ = \mathcal{L}(\mathcal{D}(M_{2c}^{1-\theta/2}); \mathcal{D}(L_{2c}^{1-\theta/2})), \quad \text{and} \quad (3.19) \\ T_2^{-1} \in \mathcal{L}(\mathcal{D}(L_{2c}^{1-\theta/2}); \mathcal{D}(M_{2c}^{1-\theta/2})), \quad 0 \le \theta \le 1, \quad (3.20) \end{aligned}$$

(see, for example [8, Theorem 6.1]). Thus we see that, for any $u \in \mathcal{D}(M_{2c}^\theta)$, $3/4 < \theta \le 1$, $v = T_2 u$ belongs to $H^{2\theta}(\Omega)$ and $\tau_2 v = 0$ by Theorem 2.2, (iii), and that $\tau_2 u = \sum_{k=1}^p \langle u, w_k \rangle_\Gamma h_k$. Therefore u belongs to $H_{f2}^{2\theta}(\Omega)$. Conversely, for any $u \in H_{f2}^{2\theta}(\Omega)$, $v = T_2 u$ belongs to $H^{2\theta}(\Omega)$ and $\tau_2 v = 0$, that is, v belongs to $\mathcal{D}(L_{2c}^\theta)$. Thus $u = T_2^{-1} v$ belongs to $\mathcal{D}(M_{2c}^\theta)$ by (3.20), which proves (iii) of the theorem. Relation (ii) is similarly proved by means of the operator T_2 and thus omitted.

We note that the relation (ii) does not depend on a particular choice of R_2. In fact, the proof is essentially the same as the proof (the Fourth Step) of Theorem 3.1.

4. Application

In this section we apply one of the main results to robustness analysis of a boundary feedback control system. The boundary control system is described by

$$\frac{du}{dt} + M_2 u = 0, \quad u(0) = u_0. \tag{4.1}$$

When the coefficients in M_2 are perturbed, the perturbed system is then described by

$$\frac{du}{dt} + \widetilde{M}_2 u = 0, \quad u(0) = u_0, \tag{4.2}$$

where

$$\widetilde{M}_2 u = \widetilde{\mathcal{L}} u, \quad u \in \mathcal{D}(\widetilde{M}_2) = \left\{ u \in H^2(\Omega);\ \widetilde{\tau}_2 u = \sum_{k=1}^p \langle u, w_k \rangle_\Gamma h_k \ \text{on } \Gamma \right\},$$

and

$$\widetilde{\mathcal{L}} u = - \sum_{i,j=1}^m \frac{\partial}{\partial x_i} \left((1 + \kappa(x)) a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^m \widetilde{b}_i(x) \frac{\partial u}{\partial x_i} + \widetilde{c}(x) u,$$

$$\widetilde{\tau}_2 u = \frac{\partial u}{\partial \widetilde{\nu}} + \widetilde{\sigma}(\xi) u = \sum_{i,j=1}^m (1 + \kappa(\xi)) a_{ij}(\xi) \nu_i(\xi) \frac{\partial u}{\partial x_j}\bigg|_\Gamma + \widetilde{\sigma}(\xi) u \bigg|_\Gamma .$$

Thus the perturbation to the principal part of $\mathcal{L}$ is assumed uniform. Throughout the section we assume, in addition to (2.6), that w_k's belong to $H^{1/2}(\Gamma)$, so that the adjoint operator M_2^* enjoys the structure similar to that of M_2 (see, e.g., Proposition 2.4, (ii)). The index measuring the difference between M_2 and $\widetilde{M}_2$ is introduced as

$$\eta = \|\kappa\|_{C^2(\overline{\Omega})} + \sum_{i=1}^m \|\widetilde{b}_i - b_i\|_{C(\overline{\Omega})} + \|\widetilde{c} - c\|_{C(\overline{\Omega})} + \|\widetilde{\sigma} - \sigma\|_{C^1(\Gamma)} . \tag{4.3}$$

The domain $\mathcal{D}(\widetilde{M_2})$ differs a little bit from $\mathcal{D}(M_2)$, and the comparison of the resolvent set $\rho(\widetilde{M_2})$ with $\rho(M_2)$ seems not very simple. However, we assert the following:

Theorem 4.1. *If η is chosen small enough, there is an operator K_η subordinate to M_2 such that*

$$\rho(\widetilde{M_2}) = \rho(M_2 - K_\eta), \tag{4.4}$$

and

$$\begin{aligned} &(\lambda - \widetilde{M_2})^{-1} = M_{2c}^{\theta}(\lambda - M_2 + K_\eta)^{-1} M_{2c}^{-\theta}, \quad \lambda \in \rho(M_2 - K_\eta), \\ &\|K_\eta u\| \le c(\eta)\|M_2 u\|, \quad u \in \mathcal{D}(M_2), \quad c(\eta) \to 0 \quad as \quad \eta \to 0, \end{aligned} \tag{4.5}$$

where $\theta = 1/4 + \epsilon$, $0 < \epsilon < 1/4$.

The proof of Theorem 4.1 is carried out along the line of [13], and therefore omitted.

5. Concluding remarks

I. Theorem 3.1 has been proved on the assumption that w_k, $1 \le k \le p$, belong to $H^\epsilon(\Omega)$ for an arbitrarily small $\epsilon > 0$. If w_k's merely belong to $L^2(\Omega)$, what can we assert? It seems difficult at present to show that

$$\mathcal{D}(M_{1c}^{\theta}) = [\mathcal{D}(M_1), L^2(\Omega)]_{1-\theta}, \quad 0 \le \theta \le 1$$

for general w_k's in $L^2(\Omega)$. However, introducing the operator F_1 in Theorem 3.3, and applying the method in the First and the Second Steps in the proof of Theorem 3.2, we can show at least that

$$\mathcal{D}(M_{1c}^{\theta}) = H^{2\theta}(\Omega), \quad 0 \le \theta < \frac{1}{4}.$$

II. In our previous paper [13], we studied the operator $\widehat{M_2}$ and its fractional powers, where $\widehat{M_2}$ is defined by

$$\widehat{M_2} u = \mathcal{L} u, \quad \mathcal{D}(\widehat{M_2}) = \left\{ u \in H^2(\Omega);\ \tau_2 u = \sum_{k=1}^{p} \langle u, w_k \rangle_\Omega\, h_k \ \text{on}\ \Gamma \right\},$$

and w_k's belong to $L^2(\Omega)$. In Theorem 3.2, let us replace M_2 by $\widehat{M_2}$. Then $\widehat{M_{2c}} = \widehat{M_2} + c$ is m-accretive, too, if $c > 0$ is large enough, and fractional powers for $\widehat{M_{2c}}$ are well defined. Characterization of the domain $\mathcal{D}(\widehat{M_{2c}^{\theta}})$, $0 \le \theta \le 1$ is similar to Theorem 3.2:

(i) $\mathcal{D}(\widehat{M_{2c}^{\theta}}) = H^{2\theta}(\Omega), \quad 0 \le \theta < \frac{3}{4}$;

(ii) $\mathcal{D}(\widehat{M_{2c}^{3/4}}) = \left\{ u \in H^{3/2}(\Omega);\ \int_\Omega \frac{1}{\zeta(x)} \left| \tau_\Omega u - \sum_{k=1}^{p} \langle u, w_k \rangle_\Omega\, \tau_\Omega R_2 h_k \right|^2 dx < \infty \right\}$;

and

(iii) $\mathcal{D}(\widehat{M_{2c}^{\theta}}) = \widehat{H}_{f2}^{2\theta}(\Omega) = \left\{ u \in H^{2\theta}(\Omega);\ \tau_2 u = \sum_{k=1}^{p} \langle u, w_k \rangle_\Omega\, h_k \ \ on \ \ \Gamma \right\}$,

$$\frac{3}{4} < \theta \leq 1.$$

For the proof, we introduce the operator $\widehat{T}_2$ by

$$\widehat{T}_2 u = u - \sum_{k=1}^{p} \langle u, w_k \rangle_{\Omega} R_2 h_k .$$

The basic space for $\widehat{T}_2$ is simply $L^2(\Omega)$. We do not need the First Step in the proof of Theorem 3.2. It is easily seen that

$$\widehat{T}_2 \in \mathcal{L}(\mathcal{D}(\widehat{M}_{2c}^{\theta}) ; \mathcal{D}(L_{2c}^{\theta})) , \quad \text{and} \quad \widehat{T}_2^{-1} \in \mathcal{L}(\mathcal{D}(L_{2c}^{\theta}) ; \mathcal{D}(\widehat{M}_{2c}^{\theta})) , \quad 0 \leq \theta \leq 1 .$$

By applying this to the modified problem, the above characterization is obtained.

We note that, in [13], another approach is employed in the above (i) (it corresponds to the First and the Second Steps in the proof of Theorem 3.2). Thus the approach via $\widehat{T}_2$ gives a simpler alternative approach.

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