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On radial and non-radial positive steady-states for Lotka-Volterra competition model on two dimensional annulus

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1 Introduction

Consider the following Lotka-Volterra competition model:

$$\begin{cases} u_t = D\Delta u + u(a-u-bv) & \text{in } \Omega\times[0,\infty), \\ v_t = D\Delta v + v(d-v-cu) & \text{in } \Omega\times[0,\infty), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega\times[0,\infty), \\ u(x,0)=u_0\geq 0, v(x,0)=v_0\geq 0, \end{cases} \tag{1}$$

where $\Omega = \{x\in\mathbf{R}^2; R\leq |x|\leq R+1\}$ and D,a,b,c,d are positive constants. We will discuss the bistable case, i.e, $bd-a>0$ and $ac-d>0$. In this case, there exist four nonnegative constant solutions;
$(0,0)$, $(a,0)$, $(0,d)$ and $(u^*,v^*) = (\dfrac{bd-a}{bc-1}, \dfrac{ac-d}{bc-1})$.
They are represented in (u,v)−plane in Fig.1, where $(a,0)$, $(0,d)$ are stable steady-states and $(0,0)$, (u^*,v^*) are unstable.

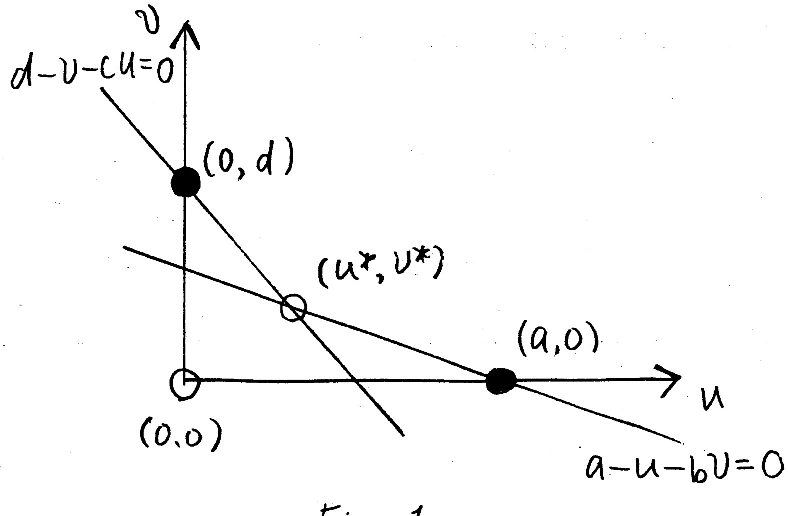

Fig. 1.

We are interested in the steady-state problem associated with (1);

$$\begin{cases} D\Delta u + u(a-u-bv) = 0 & \text{in } \Omega, \\ D\Delta v + v(d-v-cu) = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ u \geq 0, v \geq 0, & \text{in } \Omega, \end{cases} \tag{2}$$

especially, the multiplicity of nonconstant solutions for (2).

We shall study radial solutions in section 1 and non-radial solutions in section 2.

2 Radial solutions

In this section we will study radial solutions for (2). Let (u, v) be a radial solution for (2). We take polar coordinates such as $|x| = R + s$, then (u, v) satisfies

$$\begin{cases} Du_{ss} + \dfrac{D}{R+s}u_s + u(a-u-bv) = 0 & \text{in } (0,1), \\ Dv_{ss} + \dfrac{D}{R+s}v_s + v(d-v-cu) = 0 & \text{in } (0,1), \\ u_s(0) = u_s(1) = v_s(0) = v_s(1) = 0, & \\ u \geq 0, v \geq 0, & \text{in } (0,1). \end{cases} \tag{3}$$

From now on we study (3). First we will make some definitions.

Definition 1 *Let* $(u, v) = (u(s), v(s))$ *be a solution for* (3). *Then* (u, v) *is called an* $n-$*mode radial solution if and only if*

$$\#\{s \in (0,1); u_s(s) = 0\} = \#\{s \in (0,1); v_s(s) = 0\} = n - 1.$$

Here $\#A$ denote the number of elements of A.

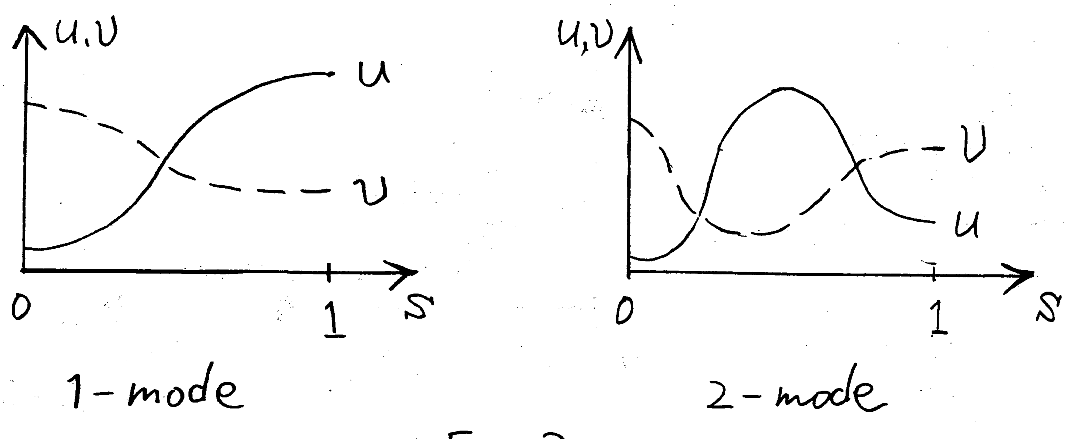

Fig. 2

Before stating results we will prepare some notation. Let α^* be the positive solution of $\mu^2 + (u^* + v^*)\mu - (bc-1)u^*v^* = 0$.

Denote the eigenvalues of $-\dfrac{\partial^2}{\partial s^2} - \dfrac{1}{R+s}\dfrac{\partial}{\partial s}$ ($= -\Delta$ in the space of radial functions) with Neumann zero boundary condition by λ_n, $(0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots)$. In discussing radial solutions we regard D^{-1} as a parameter and consider branches of $n-$mode solutions for (3). Set

$S := \{(u,v,\tau) \in C^1[0,1] \times C^1[0,1] \times \mathbf{R}^+;$ (u,v) is a nonconstant positive radial solution for (3) with $D^{-1} = \tau\}$.

$S_n := \{(u,v,\tau) \in C^1[0,1] \times C^1[0,1] \times \mathbf{R}^+;$ (u,v) is an $n-$mode positive radial solution for (3) with $D^{-1} = \tau\}$.

Theorem 1 *(i)* $S = \bigcup_{n=1}^{\infty} S_n$.

(ii) S_n *contains a connected component B such that* $(u^*, v^*, \dfrac{\lambda_n}{\alpha^*}) \in \overline{B}$ *and B is unbounded in* $\{(u,v,\tau) \in C^1[0,1] \times C^1[0,1] \times \mathbf{R}^+; 0 \le u \le a, 0 \le v \le d\}$.

For the proof see Nakashima[11]

Remark 1 Every positive solution for (2) has a priori estimate such that $0 \le u \le a$ and $0 \le v \le d$. Moreover, Theorem 1 implies that every solution for (3) becomes an $n-$mode solution for some n.

Remark 2 When D is large (or a parameter D^{-1} is small), there exists no nonconstant solution for (3). (Conway-Hoff-Smoller[1].)

From the above remarks B keeps the property of $n-$mode and continues up to $D^{-1} = \infty$.

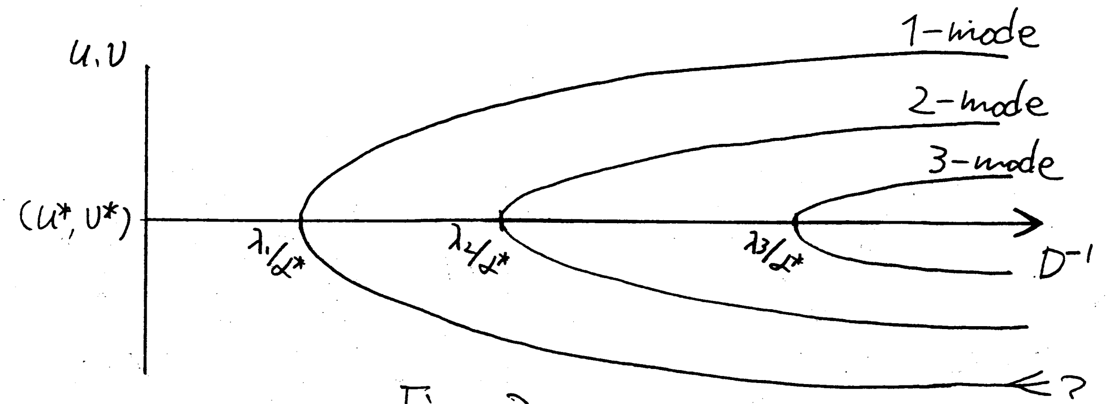

Fig. 3

3 non-radial solutions

In this section we will study the existence and multiplicity of non-radial positive solutions for (2). Here we restrict ourselves to the case $N = 2$ and we show that the following result holds for (2).

For any $k \in \mathbf{N}$ there exists $R_0 = R_0(k) > 0$ such that if $R > R_0(k)$ then (2) has at least $k-$ non-radial positive solutions, which are not equivalent with respect to rotation.

Such an existence result is well-studied for a single equation like

$$\Delta u + u^p = 0 \qquad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega, \qquad u \geq 0, \tag{4}$$

when $1 < p < \infty$. We refer to ([2], [4], [6], [8], [9], [10]).

Taking polar coordinates (s, θ) such as

$$x = (R+s)\cos\theta, \qquad y = (R+s)\sin\theta,$$

the steady-state problem for (2) is written as

$$\begin{cases} Du_{ss} + \dfrac{D}{R+s}u_s + \dfrac{D}{(R+s)^2}u_{\theta\theta} + u(a-u-bv) = 0 & \text{in } [0,1]\times[0,2\pi), \\ Dv_{ss} + \dfrac{D}{R+s}v_s + \dfrac{D}{(R+s)^2}v_{\theta\theta} + v(d-v-cu) = 0 & \text{in } [0,1]\times[0,2\pi), \\ u_s(0,\theta) = u_s(1,\theta) = v_s(0,\theta) = v_s(1,\theta) = 0, & \\ u \geq 0, v \geq 0. & \text{in } [0,1]\times[0,2\pi). \end{cases} \tag{5}$$

From now on we will study nonconstant solutions for (5). Define a cone such as

$$C_k := \{(u,v) \in C^1([0,1]\times[0,2\pi))\times C^1([0,1]\times[0,2\pi));\ u \geq 0,\ v \geq 0,\ u_\theta \geq 0,\ v_\theta \leq 0 \ \text{ on } [0, \tfrac{\pi}{k}].$$
$$u, v \text{ is symmetric with respect to } \theta = 0, \tfrac{\pi}{k}, \tfrac{2\pi}{k}, \cdot\cdot, \tfrac{(k-1)\pi}{k}\}$$

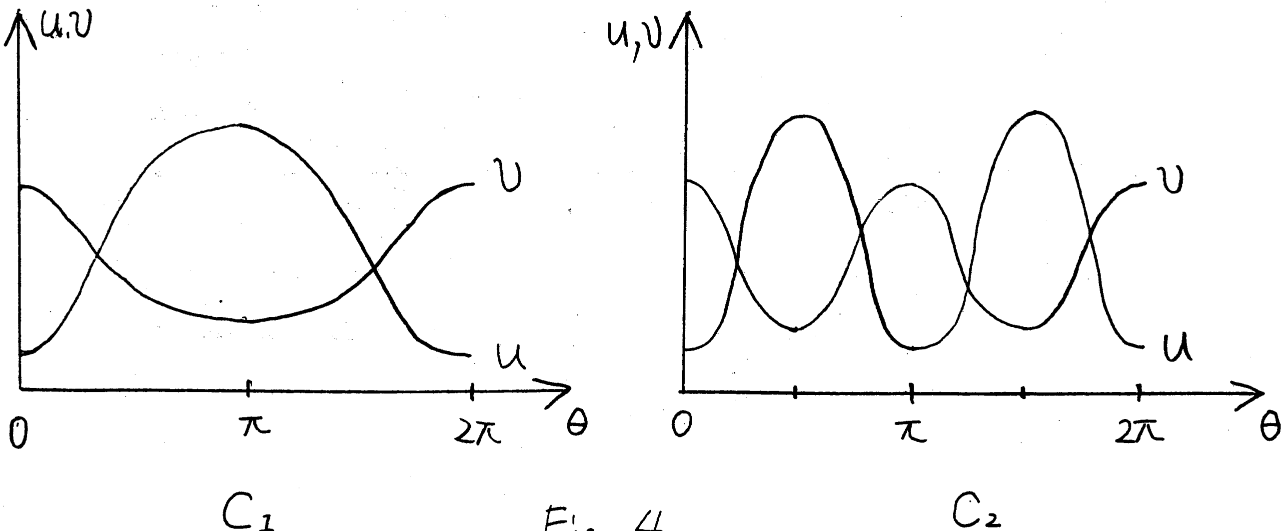

Fig. 4

Here and henceforce we assume the following assumption (N).

(N) Every positive solution (ϕ_1, ϕ_2) for

$$\begin{cases} D\frac{d^2}{dx^2}u + u(a-u-bv) = 0 & \text{in } [0,1], \\ D\frac{d^2}{dx^2}v + v(d-v-cu) = 0 & \text{in } [0,1], \\ u_x(0) = u_x(1) = v_x(0) = v_x(1) = 0, \end{cases} \tag{6}$$

is nondegenerate, i.e. zero is not an eigenvalue for the linearized problem for (6) at (ϕ_1, ϕ_2).
Remark 3 From the results of Kan-on, we know (N) holds if $a = d$, $b = c$. Using the fixed point index theory on C_k, we can get the following result.

Theorem 2 *Assume (N). For each $k \in \mathbf{N}$ there exists $R_k(D,a,b,c,d) > 0$, such that* (2) *has a non-radial solution in C_k for every $R \geq R_k$.*

Remark 4 Observe that $C_k \cap C_l$ is identical with the set of radial solutions Phi if $k \neq l$, $k, l \in \mathbf{N}$. Theorem 2 implies that there exist two solutions $(u_k, v_k) \in C_k - \Phi$ and $(u_l, v_l) \in C_l - \Phi$ if we set $R \geq \max\{R_k, R_l\}$. Since (u_k, v_k) and (u_l, v_l) are different with respect to rotation, we can get the multiplicity.

In this way we can find any number of nonradial positive solutions for (5) if R is sufficiently large.

4 Proof

In this section we show the outline of the proof of Theorem 2. We will find a non-radial solution in each C_k. $(k = 1, 2, \cdot\cdot, m.)$

We will remark a priori estimate for the solutions for (5). Every solution (u, v) has a priori estimate such that

$$0 \leq u \leq a \qquad \text{and} \qquad 0 \leq v \leq d.$$

To show this, let u has its maximum at $x_0 \in \Omega$. We have $a - u(x_0) - bv(x_0) = -\Delta u(x_0) \geq 0$. From the positivity of v, $u(x_0) \leq a - bv(x_0) \leq a$. We can show $v \leq d$ in the same way.

Owing to Shaudar estimates for elliptic equations, there exist sufficiently large M_1, M_2 such that $\|u\| \leq M_1$ and $\|v\| \leq M_2$ for every solution for (5), where $\|\cdot\|$ denote the C^1-norm.

Set a bounded set

$$T_k := \{(u,v) \in C_k;\ \|u\| < M_1 + 1, \|v\| < M_2 + 1\}$$

Note that the solutions for (6) is not on the boundary of T_k. Here we use the word "boundary" in the meaning of the relative topology with respect to C_k.

We define a compact operator A from $C^1([0,1]\times[0,2\pi))\times C^1([0,1]\times[0,2\pi))$ into itself by

$$A\begin{pmatrix} u \\ v \end{pmatrix} = \left(-\frac{d^2}{dx^2}+p\right)^{-1}\begin{pmatrix} pu + D^{-1}(a-u-bv)u \\ pv + D^{-1}(d-v-cu)v \end{pmatrix}. \tag{7}$$

$$\text{where} \quad p = \max\{D^{-1}(2M_1+bM_2), D^{-1}(2M_2+cM_1)\}$$

Note that there is one to one correspondence between a fixed point of A and a solution for (5). So A has no fixed point on the boundary of T_k. Moreover the standard regularity theory of elliptic equations tells us that A is completely continuous.

From the above fact and the following Lemma 1, we can define degree of $I-A$ on C_k, which is denoted by $\deg_{Ck}(I-A,\cdot\,)$. For the definition of $\deg_{Ck}(I-A,\cdot\,)$, see Dancer[3].

Lemma 1 *A maps T_k into C_k.*

Proof. Let $(u,v)\in T_k$. First we will show the positivity of $A(u,v)$. Note that $u\geq 0$ and $v\geq 0$ and that p is sufficiently large. Using the maximum principle in (7), we see that each element of $A(u,v)$ is positive.

Next we will show the monotonicity of $A(u,v)$. Defferentiating with respect to θ,

$$\frac{d}{d\theta}\left\{A\begin{pmatrix} u \\ v \end{pmatrix}\right\} =$$

$$\left(-\frac{d^2}{dx^2}+p\right)^{-1}\begin{pmatrix} p+D^{-1}(a-2u-bv) & -D^{-1}bu \\ -D^{-1}cv & p+D^{-1}(d-2v-cu) \end{pmatrix}\begin{pmatrix} u_\theta \\ v_\theta \end{pmatrix}.$$

Since $u_\theta\geq 0$ and $v_\theta\leq 0$, it follows fron the maximum principle that the first element of the above equality is nonnegative and the second is nonpositive. □

Now we can define $\deg_{Ck}(I-A,T_k)$.

Lemma 2 *$deg_{Ck}(I-A,T_k)=1$.*

Proof. We use the homotopy invariance and excision property of the fixed point index theory. When we regard D as a parameter, we sometimes write A_D to emphasize D dependence of A.

From the result of Conway-Hoff-Smoller[1], it is well known that there is a sufficiently large $D_0 = D_0(a,b,c,d,\Omega) > 0$ such that (5) has no nonconstant solution for $D > D_0(a,b,c,d,\Omega)$; so that the excision property gives

$$\deg_{C_k}(I-A_D,T_k) = \mathrm{index}_{C_k}(A_D,(a,0)) + \mathrm{index}_{C_k}(A_D,(0,d))$$

$$+\mathrm{index}_{C_k}(A_D,(0,0)) + \mathrm{index}_{C_k}(A_D,(u^*,v^*)) \qquad \text{for} \quad D>D_0. \tag{8}$$

To calculate the righthand side of (8), we will give the value of fixed point indices of constant solutions in the following lemma, whose proof is omitted.

Lemma 3 $(i)index_{C_k}(A,(a,0)) = index_{C_k}(A,(0,d)) = 1.$
$(ii)index_{C_k}(A,(0,0)) = 0.$
$(iii)index_{C_k}(A,(u^*,v^*)) = -1 \quad if \quad D\lambda_1 > \alpha^*.$

We will continue the proof of Lemma 2. Since we can make D_0 sufficiently large, $D_0\lambda_1 > \alpha^*$ holds. So the righthand side of (8) is 1 from Lemma 3. Remember that A_D has no fixed point on the boundary of T_k. Using the homotopy invariance property, it follows that

$$\deg_{C_k}(I - A_D, T_k) = 1 \quad \text{for every } D > 0.$$

□

In the rest of the proof we restrict ourselves to the case when the parameter R is sufficiently large.

Our stratgy is as follows. In Lemma 4 we get all the positive radial solutions, and in Lemma 5 we study the fixed point indices of these radial solutions. Finally combining Lemmas 2, 4 and 5, we conclude the existence of a non-radial fixed point of A in T_k by contradiction.

Set $\epsilon := \frac{1}{R}$, then (5) is equivallent to

$$\begin{cases} Du_{ss} + \dfrac{D\epsilon}{1+\epsilon s}u_s + \dfrac{D^2\epsilon^2}{(1+\epsilon s)^2}u_{\theta\theta} + u(a-u-bv) = 0 & \text{in } [0,1]\times[0,2\pi), \\ Dv_{ss} + \dfrac{D\epsilon}{1+\epsilon s}v_s + \dfrac{D^2\epsilon^2}{(1+\epsilon s)^2}v_{\theta\theta} + v(d-v-cu) = 0 & \text{in } [0,1]\times[0,2\pi), \\ u_s(0,\theta) = u_s(1,\theta) = v_s(0,\theta) = v_s(1,\theta) = 0, & \\ u \geq 0, v \geq 0. & \text{in } [0,1]\times[0,2\pi). \end{cases} \tag{9}$$

Note that (9) with $\epsilon = 0$ is equivalent to the one dimensional system (6). Denote by $\{(u_0^i, v_0^i)\}_{i=1,2,\cdot\cdot,m}$ all the nonconstant solutions for (6). Observe that the number of such solutions is finite because of the nondigeneracy assumption.

Lemma 4 *Assume (N). For small $\epsilon > 0$, (9) has a nonconstant radial solution $(u_\epsilon^i, v_\epsilon^i)$ near (u_0^i, v_0^i) for $i = 1, 2, \cdot\cdot, m$.*

Moreover, if (u,v) is a nonconstant radial solution, then $(u,v) = (u_\epsilon^i, v_\epsilon^i)$ for some $i \in \{1, 2, \cdot\cdot, m\}$.

The proof can be accomplished with use of the implicit function theorem.

We can calculate the fixed point index of each radial solution using Dancer's index formula.

Lemma 5 *Let (u,v) be a positive radial solution for (7) including (u^*,v^*). If the eigenvalue problem*

$$\begin{pmatrix} \Delta\bar{u} \\ \Delta\bar{v} \end{pmatrix} + \frac{1}{D}\begin{pmatrix} a-2u-bv, & -bu \\ -cv & d-2v-cu \end{pmatrix}\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = \lambda\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}, \tag{10}$$

$$\bar{u}_s(0) = \bar{u}_s(1) = \bar{v}_s(0) = \bar{v}_s(1) = 0,$$

$$\text{where } \Delta = \frac{\partial^2}{\partial s^2} + \frac{\epsilon}{1+\epsilon}\frac{\partial}{\partial s} + \frac{\epsilon^2}{1+\epsilon^2}\frac{\partial^2}{\partial\theta^2},$$

has a positive real eigenvalue, then

$$index_{C_k}(A,(u,v)) = 0.$$

Lemma 5 is useful to get the fixed point index of $(u_\epsilon^i, v_\epsilon^i)$; we study the eigenvalue problem (10) with
$(u,v) = (u_\epsilon^i, v_\epsilon^i)$.

First, we consider the case $\epsilon = 0$;

$$\begin{pmatrix} \bar{u}_{xx} \\ \bar{v}_{xx} \end{pmatrix} + \frac{1}{d}\begin{pmatrix} a-2u_0^i-bv_0^i, & -bu_0^i \\ -cv_0^i & d-2v_0^i-cu_0^i \end{pmatrix}\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = \lambda\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}, \tag{11}$$

$$\bar{u}_x(0) = \bar{u}_x(1) = \bar{v}_x(0) = \bar{v}_x(1) = 0.$$

For $(u,v) = (u^*,v^*)$, (11) has a positive real eigenvalue. (Recall that we are discussing the bistable case.) For nonconstant solutions, we will introduce the result of Kishimoto-Weinberger[7].

Theorem 3 (Kishimoto-Weinberger)
For any nonconstant solution (u_0^i, v_0^i), (11) has a positive real simple eigenvalue λ_0.

Taking account of these results, the implicit function theorem shows the following lemma.

Lemma 6 *For sufficiently small $\epsilon > 0$, (10) with $(u,v) = (u_\epsilon^i, v_\epsilon^i)$ has a simple real eigenvalue λ_ϵ near λ_0. Therefore, λ_ϵ is also positive since λ_0 is positive.*

We are ready to complete the proof of Theorem 2. It follows from Lemmas 5 and 6 that

$$\text{index}_{Ck}(A,(u^*,v^*)) = 0, \tag{12}$$

and

$$\text{index}_{Ck}(A,(u_\epsilon^i, v_\epsilon^i)) = 0 \quad \text{for every} \quad i = 1,2,\cdot\cdot,m. \tag{13}$$

Assume that there exists no non-radial solution. From the excision property,

$$\deg_{C_k}(I-A,T_k) = \text{index}_{C_k}(A,(a,0)) + \text{index}_{C_k}(A,(0,d)) +$$
$$\text{index}_{C_k}(A,(0,0)) + \text{index}_{C_k}(A,(u^*,v^*)) + \sum_{i=1}^{m}\text{index}_{C_k}(A,u^i\epsilon, v^i\epsilon).$$

We see from (12),(13) and (i), (ii) in Lemma 3 that the righthand side is equal to 2. This contradicts to Lemma 2. Thus we can obtain a non-radial fixed point in C_k.

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