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A RESTRICTION THEOREM AND WEIGHTED L^2 ESTIMATES FOR WAVE EQUATIONS

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There are two aims in this note. The first one is to give a review of recent developments of this subject. The second one is to state my new results. Especially, in the first part I would like to descirbe the relation among three kinds of the weighted estimates: the estimates of restriction theorem , the estimates of limiting absorption principle and the estimates of smoothing effect.

Part 1.

In this paper, we will deal with restriction theorem , smoothing effect and limiting absorption principle. They connect each others, and seem as trinity. Here we explain the relationship. The first subject is restriction theorem. Let $f(x)$ be a function in $\mathbb{R}^n$ and $\hat{f}(\xi)$ be its Fourier transform. If $f \in L^1(\mathbb{R}^n)$ then $\hat{f}(\xi) \in L^\infty(\mathbb{R}^n)$, hence $\hat{f}(\xi)$ can be restricted any subset of the Euclidean space. Restriction theorem is analog of such trivial fact. Namely it tells us that the restriction to some subset S of $\mathbb{R}^n$ can be extended to a bounded operator from some function space of $\mathbb{R}^n$ to the other function space of S. The second subject is smoothing effect. It tells us that the solutions to dispersive equations are a little bit smoother than the Cauchy data in some function space of space-time variables. Recently there is much activity in this subject. The third subject is limiting absorption principle. It tells us that the resolvent operator $R(\zeta)$ has limit as a bounded operator in some function spaces even if ζ approaches the spectrum of the Laplacian. The purpose of the present part is to explain the relation among such subjects by describe a review of recent developments of the second subject.

The first result of smoothing effect is due to Prof. Kato [8] . Concerning linearized KdV (i.e., Airy) equation, he obtained the following result: Let u be a solution of

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} & = 0, \\ u|_{t=0} & = \phi. \end{cases}$$

If $\phi \in L^2(\mathbb{R})$ then u enjoys that $|D|u(t,x) \in L^2_{loc}(\mathbb{R}^2)$. He also proved that the similar conclusion holds even if we add the nonlinear term to the equation. After such epoc making work, P. Constantin and J.C. Saut [4] have found new approach. Also they remark that such phenomenon is common in dispersive equations. They proved the following result. Let u be a solution of the initial value problem

$$\begin{cases} \frac{\partial u}{\partial t} + i\,P(D)u &= 0, \\ u|_{t=0} &= \phi, \end{cases}$$

where it is assumed that $P(\xi)$ is real valued and has asymptotic behavior like $|\xi|^m$ as $|\xi| \to \infty$. Then it holds that $|D|^{\frac{m-1}{2}} u \in L^2_{loc}(\mathbb{R}^{n+1})$, if $\phi \in L^2(\mathbb{R}^n)$. Their proof is based on harmonic analysis. More precisely, they proved the following inequality, which is adjoint of the previous result: Let $\chi(t,x) \in C_0^\infty(\mathbb{R}^{n+1})$. Then there exists a constant C such that

$$\int_{\xi\in\mathbb{R}^n} (1+|\xi|)^{m-1} |(\widehat{\chi f})(P(\xi),\xi)|^2 d\xi \le C\|f\|^2_{L^2(\mathbb{R}^{n+1})},$$

where $\hat{f}(\tau,\xi)$ is the Fourier transform of f with respect to space-time variables. Such procedure is not due to them, but essentially due to R. Strichartz [16]. Indeed R. Strichartz [16] proved that, if $\phi \in L^2(\mathbb{R}^n)$ and $n \ge 3$, then the solution to Schrödinger equation $u = e^{-it\Delta}\phi$ enjoys $u(t,x) \in L^q(\mathbb{R}^{n+1})$ where $q = \frac{2n+4}{n}$.Note that $q > 2$, so it tells us that the solution is smoother than the initial datum. To prove it, he showed the following restriction theorem to paraboloid:

$$\int_{\mathbb{R}^n} |\hat{f}(|\xi|^2,\xi)|^2 d\xi \le C\|f\|^2_{L^p(\mathbb{R}^{n+1})},$$

where $p = \frac{2n+4}{n+4}$(i.e., dual exponent of q). Such kind of restriction theorem has been an important problem in harmonic analysis from 1960's. Especially P. Tomas and E.M. Stein (see [17]) proved that, concerning the restriction to sphere, it holds

$$\int_{|\xi|=1} |\hat{f}(\xi)|^2 dS \le C\|f\|^2_{L^p(\mathbb{R}^n)}$$

if $n \ge 3$ and $p = \frac{2n+2}{n+3}$.

After the work by P. Constantin and J.C. Saut, it is natural to seek a global version of such result as the one obtained by R. Strichartz [16]. T. Kato-K. Yajima [9] and M.Ben Artzi-S. Klainerman [2] independently succeeded to obtain it. Supposing $n \geq 3$, they gave the following result. If $\phi \in L^2(\mathbb{R}^n)$ then the solution $u(t,x) = e^{-it\Delta}\phi$ enjoys

$$|x|^{\alpha-1}|D|^{\alpha}u \in L^2(\mathbb{R}^{n+1}) \quad (0 \leq \alpha \leq \frac{1}{2}) \tag{1}$$

and

$$(1+|x|)^{-(\frac{1}{2}+\epsilon)}|D|^{1/2}u \in L^2(\mathbb{R}^{n+1}).$$

M. Ben-Artzi and S. Klainerman begin the proof by introducing the restriction theorem as follows:

$$\int_{|\xi|=\lambda} |\hat{f}(\xi)|^2 dS_\lambda \leq C \min(\lambda^{2s-1},\, 1)\|f\|^2_{L^{2,s}(\mathbb{R}^n)}, \tag{2}$$

where dS_λ is surface element of the sphere $\{\xi \in \mathbb{R}^n \| \xi| = \lambda\}$ and

$$\|f\|^2_{L^{2,s}(\mathbb{R}^n)} = \int (1+|x|^2)^s |f(x)|^2 dx.$$

Moreover the inequality (2) relys on the one of limiting absorption principle, which was proved by S. Agmon [1]. Let me explain about it. Precisely S. Agmon proved the following estimate: If $s > \frac{1}{2}$ it holds

$$|\zeta|\|(1+|x|)^{-s}u\|_{L^2(\mathbb{R}^n)} \leq C\|(1+|x|)^s(-\Delta-\zeta)u\|_{L^2(\mathbb{R}^n)}, \tag{3}$$

where the constant C is independent of u and $\zeta \in \mathbb{C}$.

Let $R(\zeta) = (-\Delta - \zeta)^{-1}$ and notice that

$$(2\pi)^n \lim_{\zeta \to \lambda^2, \mathrm{Im}\zeta > 0} \mathrm{Im}\,(R(\zeta)f,\, f) = \frac{\pi}{2\lambda}\int_{|\xi|=\lambda} |\hat{f}(\xi)|^2 dS_\lambda.$$

(We can easily see the above inequality if we recall that $\lim_{\eta\downarrow 0}\frac{1}{2\pi i}\{\frac{1}{\xi-i\eta}-\frac{1}{\xi+i\eta}\}=\delta(\xi)$.) Since $\zeta\sim\lambda^2$ in the above limit, it follows that

$$\begin{aligned}\int_{|\xi|=\lambda}|\hat{f}(\xi)|^2 dS_\lambda &= \left|\frac{1}{c_n}\lambda \lim_{\zeta\to\lambda^2,\,\mathrm{Im}\,\zeta>0}\mathrm{Im}\,(R(\zeta)f,\,f)\right|\\ &\le C\sup_{\zeta}|\zeta|\|(1+|x|)^{-s}(-\Delta-\zeta)^{-1}f\|\|(1+|x|)^s f\|\\ &\le C'\|f\|^2_{L^{2,s}}.\end{aligned}$$

Thus the inequality (2) for $\lambda\ge 1$ follows and we can see that limiting absorption principle implies restriction theorem. The inequality (2) for $\lambda\le 1$ can be proved by another investigation (see [2]).

The procedure to prove (1) from the inequality (2) is very simple. Let us follow the argument by M. Ben-Artzi and S. Klainerman [2]. Set now

$$(A(\lambda)f,\,g)=\int_{|\xi|=\lambda}\hat{f}(\xi)\overline{\hat{g}(\xi)}\,dS_\lambda.$$

The interior product $(|D|^\alpha u,\,v)_{L^2(\mathbb{R}^{n+1})}$ with $u=e^{-it\Delta}\phi$ and $v=v(t,x)$ can be represented by the operator $A(\lambda)$. Indeed we have

$$\begin{aligned}(|D|^\alpha u,\,v)_{L^2(\mathbb{R}^{n+1})} &= \int dt\int d\xi|\xi|^\alpha e^{it|\xi|^2}\hat{\phi}(\xi)\,\overline{\hat{v}(t,\xi)}\\ &= \int_0^\infty \lambda^\alpha(A(\lambda)\phi,\int e^{it\lambda^2}v(t,\cdot)\,dt)\,d\lambda.\end{aligned}$$

Hence

$$\begin{aligned}&(|D|^\alpha u,\,v)\\ &\le\left(\int_0^\infty(A(\lambda)\phi,\,\phi)\,d\lambda\right)^{1/2}\left(\int_0^\infty\lambda^{2\alpha}(A(\lambda)\tilde{v}(\lambda^2,\cdot),\,\tilde{v}(\lambda^2,\cdot))\,d\lambda\right)^{1/2}\\ &=(2\pi)^{n/2}\|\phi\|\left(\int_0^\infty\lambda^{2\alpha}(A(\lambda)\tilde{v}(\lambda^2,\cdot),\,\tilde{v}(\lambda^2,\cdot))\,d\lambda\right)^{1/2},\end{aligned}$$

where $\tilde{v}(\tau, x) = \int e^{it\tau} v(t,x)\, dt$. Now let us put $2\alpha + 2s - 1 = 1$, then (2) implies

$$\begin{aligned}
&\int_0^\infty \lambda^{2\alpha}(A(\lambda)\tilde{v}(\lambda^2, \cdot),\, \tilde{v}(\lambda^2, \cdot))\, d\lambda \\
&\leq C \int_0^\infty \lambda^{2\alpha} \cdot \lambda^{2s-1} \|\tilde{v}(\lambda^2 . \cdot)\|^2_{L^{2,s}(\mathbb{R}^n)} d\lambda \\
&= \frac{C}{2} \int_0^\infty \|\tilde{v}(\tau, \cdot)\|^2_{L^{2,s}} d\tau \\
&\leq C' \int_{-\infty}^\infty \|v(t, \cdot)\|^2_{L^{2,s}} dt.
\end{aligned}$$

Thus it follows that

$$(|D|^\alpha u,\, v)_{L^2(\mathbb{R}^{n+1})} \leq C\|\phi\|_{L^2(\mathbb{R}^n)} \left(\int_{-\infty}^\infty \|v(t, \cdot)\|^2_{L^{2,s}} dt\right)^{1/2} .$$

Moreover (1) holds by scaling, and thus inequality (2), which is a kind of restriction theorem, implies the estimate of smoothing effect.

To terminate the first part of this note, let us summarize the relationships among three kinds of the estimates. At first it is obvious that the estimate of limiting absorption principle implies the one of restriction theorem. Second, by previous argument, restriction theorem implies the smoothing effect for homogeneous equations. Let us remark that there are another relationships. Indeed the estimates of limiting absorption principle imply the ones of smoothing effect for inhomogeneous equations. For this, see C. Kenig, G. Ponce and L. Vega [10]. Also with the additional assumption of Hölder continuity, restriction theorem implies the limiting absorption principle. For this, see T. Kato and K. Yajima [9].

Part 2

1. Results

In the previous part, we saw that new restriction theorem or new estimate of limiting absorption principle yields new estimate for the smoothing effect. We begin this part by giving a restriction theorem.

At first let Λ be Laplace-Beltrami operator of the unit sphere S^{n-1}. Namely,

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\Lambda u,$$

where Δ is the Laplace operator in $\mathbb{R}^n$. The following facts are well known:

(i) The eigenvalues of $-\Lambda$ are

$$\lambda_k = k(k+n-2), \quad k = 0, 1, 2, \ldots$$

(ii) The projection H_k to the eigen space (the space if spherical harmonics of degree k) in $L^2(\mathbb{R}^n)$ can be represented in the following way:

$$H_k f(\omega) = \frac{\nu + k}{\nu |S^{n-1}|} \int_{|\tilde{\omega}|=1} C_k^\nu(\omega \cdot \tilde{\omega}) f(\tilde{\omega})\, d\tilde{\omega},$$

where $\nu = (n-2)/2$, $d\tilde{\omega}$: the unit surface measure, $|S^{n-1}|$: area of S^{n-1} (namely is equal to $2\pi^{n/2}/\Gamma(n/2)$), $C_k^\nu(z)$: the Gegenbauer polynomial of degree k and $\omega \cdot \tilde{\omega} = \sum_{j=1}^n \omega_j \tilde{\omega}_j$.

From these facts, the fractional power $(I - \Lambda)^\alpha$ $(\alpha \in \mathbb{R})$ can be defined by

$$(I - \Lambda)^\alpha = \sum_{k=0}^{\infty} (1 + \lambda_k)^\alpha H_k.$$

Our restriction theorem is as follows:

Theorem 1. *Let s be a positive number satisfying $\frac{1}{2} < s < \frac{n}{2}$. Then we have*

$$(4) \qquad \int_{|\xi|=\lambda} |\hat{f}(\xi)|^2 dS_\lambda \leq C\,\lambda^{2s-1} \int |x|^{2s} \left|(I-\Lambda)^{-(2s-1)/4} f(x)\right|^2 dx,$$

with a constant C independent of $\lambda > 0$ and $f \in C_0^\infty(\mathbb{R}^n)$. (Here dS_λ is the Lebesgue surface measure, namely $dS_\lambda = \lambda^{n-1} d\omega$.)

We remark that there are some advances in (4) compared with (2). Indeed, concerning the exponent of $I-\Lambda$, $-\frac{2s-1}{4} > 0$ if $s > \frac{1}{2}$. By the argument of Ben-Arzti and Klainerman we obtain the following result:

Theorem 2. *Suppose $n \geq 3$ and $0 \leq \alpha < \frac{1}{2}$. Let $u(t,x) = e^{-it\Delta}\phi$. Then there exists a constant $C = C(\alpha, n)$ such that*

$$(5) \quad \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |x|^{2\alpha-2} \left||D|^\alpha (I-\Lambda)^{(1-2\alpha)/4} u(t,x)\right|^2 dx dt \leq C\|\phi\|^2, \quad \phi \in L^2(\mathbb{R}^n).$$

($|x|^2 = x_1^2 + \cdots + x_n^2$, $\|u\| = \|u\|_{L^2(\mathbb{R}^n)}$.)

The last theorem says that the solution $u(t,x)$ enjoys the better property concerning the smoothness of the angular direction. Note that the operator $|D|^\alpha (I-\Lambda)^{(1-2\alpha)/4}$ is of order $\frac{1}{2}$. Also taking $\alpha = 0$ in (4), we can see by Sobolev's imbedding theorem that

$$\int_{-\infty}^{\infty} \int_0^{\infty} r^{n-3} \left(\int_{|\omega|=1} |u(t, r\omega)|^q d\omega \right)^{2/q} dr dt \leq C\|\phi\|^2,$$

with $q = 2(n-1)/(n-2)$.

Next we turn to the result of inhomogeneous initial value problem.

Theorem 3. *Let $u(t,x)$ be a solution of the following initial value problem:*

$$\partial_t u + i\Delta u = f,$$

$$u|_{t=0} = \phi.$$

If $\phi \in L^2(\mathbb{R}^n)$ *and* $|x|f \in L^2(\mathbb{R}^{n+1})$ *, then we have* $|x|^{-1}u \in L^2(\mathbb{R}^{n+1})$.

This theorem does not seem excellent. However in the process to prove it, I obtained the following result, which is closely related to the one of C. Kenig, A.Ruiz and C.D. Sogge [11].

Theorem 4. *Suppose* $n \geq 3$. *Let* $\theta, p, q, \alpha, \beta$ *be real numbers satisfying the following relations:* $0 < \theta \leq 1$,

$$\frac{2(n+1)(n-1)}{(n+1)^2\theta + (n-1)(n+3)(1-\theta)} < p \leq q < \frac{2(n+1)(n-1)}{(n+1)(n-3)\theta + (n-1)^2(1-\theta)}$$

, $\alpha + \beta \geq 0$,

$$\alpha < n\left\{\frac{1}{p'} - \frac{n-1-2\theta}{2(n+1-2\theta)}\right\},$$
$$\beta < n\left\{\frac{1}{q} - \frac{n-1-2\theta}{2(n+1-2\theta)}\right\},$$

and

$$\frac{1}{q} = \frac{1}{p} + \frac{\alpha + \beta - \frac{2(n-\theta)}{n+1-2\theta}}{n}.$$

Put $\kappa = \frac{1-\theta}{n+1-2\theta}$. *Then it holds that, for every* $u \in C_0^\infty(\mathbb{R}^n)$ *and* $\lambda \in \mathbb{C} \setminus (\mathbb{R}_+ \cup \{0\})$,

$$\left\| |x|^{-\beta} u \right\|_{L^q(\mathbb{R}^n)} \leq C|\lambda|^{-\kappa} \left\| |x|^{\alpha}(-\Delta - \lambda)u \right\|_{L^p(\mathbb{R}^n)},$$

where C *is a constant independent of* u *and* λ.

2. Sketch of the proofs

Here we shall sketch the outline of proofs of Theorem 1 and Theorem 4.

In order to show Theorem 1, the following expression is essential:

$$(6) \qquad \int_{|\xi|=\lambda} |\hat{f}(\xi)|^2 dS_\lambda = (2\pi)^n \lambda \sum_{k=0}^{\infty} \int_{|\omega|=1} \left| \int_0^\infty J_{\nu+k}(\lambda r) r^{n/2} H_k f(r,\omega)\, dr \right|^2 d\omega,$$

where $\nu = (n-2)/2$ and $J_{\nu+k}(r)$ is the Bessel function of order $\nu + k$. Moreover the inequality (6) is due to the following classical formulas (see G. Watson [19]):

$$\int_{|\xi|=\lambda} e^{i(y-x)\xi} dS_\lambda = \lambda^{n-1} \int_{|\omega|=1} e^{i(y-x)\lambda\omega} d\omega$$
$$= \lambda^{n-1}(2\pi)^{n/2} \frac{J_\nu(\lambda|x-y|)}{|\lambda|x-y||^\nu},$$

$$\frac{J_\nu(|x-y|)}{|x-y|^\nu} = 2^\nu \Gamma(\nu) \sum_{k=0}^{\infty} (\nu+k) \frac{J_{\nu+k}(r)}{r^\nu} \frac{J_{\nu+k}(\rho)}{\rho^\nu} C_k^\nu(\omega_1 \cdot \omega_2),$$
$$(x = r\omega_1, \quad y = \rho\omega_2)$$

and

$$\frac{\nu+k}{\nu|S^{n-1}|} \int_{|\omega|=1} C_k^\nu(\omega_1 \cdot \omega) C_\ell^\nu(\omega \cdot \omega_2)\, d\omega = \delta_{k\ell} C_k^\nu(\omega_1 \cdot \omega_2).$$

From (6) it immediately follows

$$\int_{|\xi|=\lambda} |\hat{f}(\xi)|^2 dS_\lambda$$
$$\leq (2\pi)^n \lambda \sum_{k=0}^{\infty} \left(\int_0^\infty J_{\nu+k}(\lambda r)^2 r^{1-2s} dr \right) \left(\int_{|\omega|=1} \int_0^\infty |r^s H_k f(r,\omega)|^2 r^{n-1} dr d\omega \right)$$
$$= (2\pi)^n \lambda^{2s-1} \sum_{k=0}^{\infty} \left(\int_0^\infty J_{\nu+k}(r)^2 r^{1-2s} dr \right) \left(\int_{|\omega|=1} \int_0^\infty |r^s H_k f(r,\omega)|^2 r^{n-1} dr d\omega \right).$$

Now let us remark that it holds

$$\int_0^\infty J_{\nu+k}(r)^2 r^{1-2s} dr = \frac{\Gamma(2s-1)\Gamma(\nu+k-s+1)}{2^{2s-1}\Gamma(s)^2\Gamma(\nu+k+s)}$$

and moreover the right hand side is asymptotically $0((1+k)^{1-2s})$ as $k \to \infty$. Hence we obtain

$$\int_{|\xi|=\lambda} |\hat{f}(\xi)|^2 dS_\lambda \leq C\lambda^{2s-1} \sum_{k=0}^{\infty} (1+k)^{1-2s} \int_0^\infty \int_{|\omega|=1} |r^s H_k f(r,\omega)|^2 r^{n-1} d\omega dr$$
$$\leq C'\lambda^{2s-1} \sum_{k=0}^{\infty} (1+\lambda_k)^{(1-2s)/2} \int_0^\infty \int_{|\omega|=1} |r^s H_k f(r,\omega)|^2 r^{n-1} d\omega dr$$
$$= C'\lambda^{2s-1} \int |x|^{2s} \left| (I-\Lambda)^{-(2s-1)/4} f(x) \right|^2 dx,$$

which proves Theorem 1.

Let us turn to the proof of Theorem 4. The proof of Theorem 4 is due to the complex interpolation between operators $(-\Delta-\lambda)^{-z}$ ($z \in \mathbb{C}$) of the cases Re $z = 0$ and Re $= \frac{n+1}{2} - \theta$. First note that the following inequality immediately follows from Parceval's inequality:

$$\|u\| \leq e^{\pi|\gamma|}\|(-\Delta-\lambda)^{i\gamma}u\|. \tag{7}$$

Moreover it is known that the kernel of the operator $(-\Delta-\lambda)^{-z}$ is expressed in the following form:

$$K_z(\lambda, x-y) = \frac{2^{1-z}(\sqrt{2\pi})^n}{\Gamma(z)}\left|\frac{\sqrt{\lambda}}{|x-y|}\right|^{\frac{n}{2}+z} K_{\frac{n}{2}+z}(\sqrt{\lambda}|x-y|),$$

where $K_{\frac{n}{2}+z}$ is the modified Bessel function of the second kind. Let us recall that the asymptotic behavior of $K_{\frac{n}{2}+z}$ is known. Especially we have

$$K_{\frac{n+1}{2}-\theta+i\gamma}(\lambda, x-y) \leq \frac{C_\theta e^{\pi|\gamma|}}{|x-y|^\theta|\lambda|^{\frac{1-\theta}{2}}},$$

for $0 < \theta \leq 1$ (see T. Hoshiro [0]). Now we quote a fractional integral estimate by E.M. Stein and G. Weiss [15].

Proposition. *(E.M. Stein and G. Weiss) Let T_μ be the Riesz potential, namely*

$$T_\mu f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^\mu}\,dy, \quad 0 < \mu < n.$$

Suppose $1 < p \leq q < \infty$, $\alpha < n/p'$, $\beta < n/q$, $\alpha+\beta \geq 0$ and $1/q = 1/p + (\mu+\alpha+\beta)/n$. ($p'$ is defined by $\frac{1}{p}+\frac{1}{p'} = 1$.) Then it holds

$$\left(\int_{\mathbb{R}^n} \{|T_\mu f||x|^{-\beta}\}^q\,dx\right)^{1/q} \leq C\left(\int_{\mathbb{R}^n} \{|f(x)||x|^\alpha\}^p\,dx\right)^{1/p},$$

where C is independent of $f(x)$, but may depends on p, q, α, β, μ and n.

Theorem 4 follows from (6) and some estimates, which are induced by the inequality (7).

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