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LOCAL AND GLOBAL EXISTENCE IN TIME OF SMALL SOLUTIONS TO THE ELLIPTIC-HYPERBOLIC DAVEY-STEWARTSON SYSTEM

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§1. **Introduction.** We study the initial value problem for the Davey-Stewartson systems

$$\begin{cases} i\partial_t u + c_0\partial_{x_1}^2 u + \partial_{x_2}^2 u = c_1|u|^2u + c_2 u\partial_{x_1}\varphi, & (x,t)\in \mathbf{R}^3, \\ \partial_{x_1}^2\varphi + c_3\partial_{x_2}^2\varphi = \partial_{x_1}|u|^2, \\ u(x,0)=\phi(x), \end{cases} \tag{1.1}$$

where $c_0, c_3 \in \mathbf{R}$, $c_1, c_2 \in \mathbf{C}$, u is a complex valued function and φ is a real valued function. The systems (1.1) for $c_3 > 0$ were derived by Davey and Stewartson [7] and model the evolution equation of two-dimensional long waves over finite depth liquid. Djordjevic-Redekopp [8] showed that the parameter c_3 can become negative when capillary effects are significant. When $(c_0, c_1, c_2, c_3) = (1, -1, 2, -1)$, $(-1, -2, 1, 1)$ or $(-1, 2, -1, 1)$ the system (1.1) is referred as the DSI, DSII defocusing and DSII focusing respectively in the inverse scattering literature. In [10], Ghidaglia and Saut classified (1.1) as elliptic-elliptic, elliptic- hyperbolic, hyperbolic-elliptic and hyperbolic-hyperbolic according to the respective sign of $(c_0, c_3) : (+,+), (+,-), (-,+)$ and $(-,-)$. For the elliptic-elliptic and hyperbolic-elliptic cases, local and global properties of solutions were studied in [10] in the usual Sobolev spaces L^2, H^1 and H^2. In this paper we consider the elliptic-hyperbolic case. In this case after a rotation in the x_1x_2-plane and rescaling, the system (1.1) can be written as

$$\begin{cases} i\partial_t u + \Delta u = d_1|u|^2u + d_2 u\partial_{x_1}\varphi + d_3 u\partial_{x_2}\varphi, \\ \partial_{x_1}\partial_{x_2}\varphi = d_4\partial_{x_1}|u|^2 + d_5\partial_{x_2}|u|^2, \end{cases} \tag{1.2}$$

where $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$, $d_1, \cdots, d_5$ are arbitrary constants. In order to solve the system of equations, one has to assume that $\varphi(\cdot)$ satisfies the radiation condition, namely, we assume that for given functions φ_1 and φ_2

$$\lim_{x_2\to\infty} \varphi(x_1, x_2, t) = \varphi_1(x_1, t) \quad \text{and} \quad \lim_{x_1\to\infty} \varphi(x_1, x_2, t) = \varphi_2(x_2, t). \tag{1.3}$$

Under the radiation condition (1.3), the system (1.2) can be written as

$$\begin{aligned} i\partial_t u + \Delta u =& d_1|u|^2 u + d_2 u \int_{x_2}^{\infty} \partial_{x_1}|u|^2(x_1, x_2', t)dx_2' \\ &+ d_3 u \int_{x_1}^{\infty} \partial_{x_2}|u|^2(x_1', x_2, t)dx_1' + d_4 u\partial_{x_1}\varphi_1 + d_5 u \partial_{x_2}\varphi_2 \end{aligned} \tag{1.4}$$

with the initial condition $u(x, 0) = \phi(x)$. In what follows we consider the equation (1.4).

In order to state the local existence result, we define several notations. We let $\partial = (\partial_{x_1}, \partial_{x_2})$, $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| = \alpha_1 + \alpha_2$ and $\alpha_1, \alpha_2 \in \mathbf{R} \cup \{0\}$. We define the weighted Sobolev space as follows :

$$H^{m,l} = \{f \in L^2; \|(1 - \partial_{x_1}^2 - \partial_{x_2}^2)^{m/2}(1 + |x_1|^2 + |x_2|^2)^{l/2} f\| < \infty\},$$

$$H^{m,l}(\mathbf{R}_{x_j}) = \{f \in L^2(\mathbf{R}_{x_j}); \|(1 - \partial_{x_j}^2)^{m/2}(1 + |x_j|^2)^{l/2} f\|_{L^2(\mathbf{R}_{x_j})} < \infty\},$$

where $\|\cdot\|$ denotes the usual L^2 norm. We denote the usual L^p norm by $\|\cdot\|_p$. For any Banach space E, $L^p(A; E)$ means the set of E valued L^p functions on A and $C([0,T]; E)$ means the set of E valued continuous functions on $[0,T]$, where $A = [0,T]$, $A = \mathbf{R}^2$ or $A = \mathbf{R}_{x_j}$, We write $L^p([0,T]; E) = L_T^p E, L^p(\mathbf{R}_{x_j}; E) = L_{x_j}^p E$ which make the notation simple. For example $L^{p_1}(\mathbf{R}_{x_1}; L^{p_2}([0,T]; L^{p_3}(\mathbf{R}_{x_3})))$ can be denoted as $L_{x_1}^{p_1} L_T^{p_2} L_{x_3}^{p_3}$. We also write $H^{s,0} = H^s$ and $H^{s,0}(\mathbf{R}_{x_j}) = H^s(\mathbf{R}_{x_j}) = H_{x_j}^s$ for simplicity.

Our first theorem says the local existence of small solution to (1.4) in usual Sobolev spaces.

Theorem 1.1. *We assume that $\phi \in H^s$,where $s \geq 5/2$, $\partial_{x_1}\varphi_1 \in C(\mathbf{R}; H_{x_1}^s)$, $\partial_{x_2}\varphi_2 \in C(\mathbf{R}; H_{x_2}^s)$, and $\|\phi\|_{L^2} < 1/\sqrt{\max\{|d_2|, |d_3|\}}$. Then there exists a positive constant $T > 0$ and a unique solution u of (1.4) such that $u \in C([0,T]; H^s)$.*

Theorem 1.1 is considered as an improvment of the previous papers by Chihara [4] and Linares and Ponce [19]. We only prove Theorem 1.1 in the case of $s = 5/2$ since in the case of $s \geq 5/2$, Theorem 1.1 can be proved in the same way. To obtain our result we introduce the function space.

$$X_T = \{f \in C([0,T]; L^2); \|f\|_{X_T} < \infty\}, Y_T = \{f \in C([0,T]; L^2); \|f\|_{Y_T} < \infty\},$$

where

$$\|f\|_{X_T} = \|f\|_{Y_T} + \|\partial_{x_1}^3 f\|_{L^\infty_{x_1} L^2_T L^2_{x_2}} + \|\partial_{x_2}^3 f\|_{L^\infty_{x_2} L^2_T L^2_{x_1}}$$

$$\|f\|_{Y_T} = \Big\{ \sum_{|\alpha|\le 2} \|\partial^\alpha f\|^2_{L^\infty_T L^2} + \sum_{|\alpha|=2} \big(\|D^{1/2}_{x_1}\partial^\alpha f\|^2_{L^\infty_T L^2} + \|D^{1/2}_{x_2}\partial^\alpha f\|^2_{L^\infty_T L^2}\big)\Big\}^{1/2},$$

$D^a_{x_j} = \mathcal{F}^{-1}|\xi_j|^a \mathcal{F}$, $\partial^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}$, and $|\alpha| = \alpha_1 + \alpha_2$.

The function space Y_T is the natural Sobolev space when we use the classical energy method with the data $\phi \in H^{5/2}$. The use of the function space X_T suggests that we make use of smoothing properties of solutions to the linear Schrödinger equation (see Section 2). As mentioned in [19], it seems that the classical energy method is not sufficient to yield a existence result. In this paper we use the two dimensional version of the smoothing effect of Kenig-Ponce-Vega type (see, e.g., [15]). We note that the method used in this paper does not work to remove the decay condition on the data in the hyperbolic-hyperbolic case which was assumed in [19],[11] to obtain local existence results. A smallness assumption on the data can be removed in real analytic data [12], however we do not know whether it can be removed or not in the usual Sobolev space.

To state the global existence results, we use the following notations moreover.

$J = (J_{x_1}, J_{x_2})$, $J_{x_j} = x_j + 2it\partial_{x_j}$. $\|\cdot\|_{X^{m,l}(t)} = \sum_{|\alpha|\le m}\|\partial^\alpha \cdot\| + \sum_{|\alpha|\le l}\|J^\alpha\cdot\|$, where $\alpha = (\alpha_1,\alpha_2)$, $|\alpha| = \alpha_1+\alpha_2$, $\alpha_1, \alpha_2 \in \mathbf{N}\cup\{0\}$.

Our second theorem shows the global existence of small solutions to (1.4) in the usual weighted Sobolev spaces $H^{3,0}\cap H^{0,3}$, which is considered as lower order Sobolev class compared to one used in [4], by the calculus of commutator of operators. We shall prove

Theorem 1.2. *Let $\phi \in H^{3,0}\cap H^{0,3}$, $\partial_{x_1}^{j+1}\varphi_1 \in C(\mathbf{R}; L^\infty_{x_1})$, $\partial_{x_2}^{j+1}\varphi_2 \in C(\mathbf{R}; L^\infty_{x_2})$, $(0 \le j \le 3)$, ϵ_3 and δ_3 be sufficiently small, where*

$$\epsilon_m = \sup_{t\in\mathbf{R}} \sum_{0\le j\le m} (1+t)^{1+a}\Big(\|(t\partial_{x_1})^j\partial_{x_1}\varphi_1(t)\|_{L^\infty_{x_1}} + \|\partial_{x_1}^{j+1}\varphi_1(t)\|_{L^\infty_{x_1}}$$

$$+ \|(t\partial_{x_2})^j\partial_{x_2}\varphi_2(t)\|_{L^\infty_{x_2}} + \|\partial_{x_2}^{j+1}\varphi_2(t)\|_{L^\infty_{x_2}}\Big), \quad a>0,$$

$$\delta_m \ge \Big(\sum_{|\alpha|+|\beta|\le m} \|\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2} {x_1}^{\beta_1}{x_2}^{\beta_2}\phi\|^2\Big)^{1/2}.$$

Then there exists a unique global solution u of (1.4) such that

$$u \in L^\infty_{local}(\mathbf{R}; H^{3,0}\cap H^{0,3}) \cap C(\mathbf{R}; H^{2,0}\cap H^{0,2}), \tag{1.5}$$

$$\sup_{t\in\mathbf{R}}\Big(\sum_{|\alpha|+|\beta|\le 2} \|\partial^\alpha J^\beta u(t)\| + \sum_{|\alpha|+|\beta|\le 3} (1+t)^{-C\delta_3}\|\partial^\alpha J^\beta u(t)\|\Big) \le 4\delta_3. \tag{1.6}$$

Corollary 1.3. *Let u be the solution constructed in Theorem 1.2. Then we have*

$$\|u(t)\|_{L^\infty} \le C(1+|t|)^{-1}(\|\phi\|_{H^{3,0}} + \|\phi\|_{H^{0,3}}).$$

Moreover, for any $\phi \in H^{3,0} \cap H^{0,3}$ there exist $u^\pm$ such that

$$\|u(t) - U(t)u^\pm\|_{H^{2,0}} \to 0 \quad as \quad t \to \pm\infty,$$

where $U(t) = e^{it(\partial_{x_1}^2 + \partial_{x_2}^2)}$.

The rate of decay obtained in Corollary 1.3 is the same as that of solutions to linear Schrödinger equations. Time decay of solutions for the Davey-Stewartson systems (1.1) was obtained in [6],[10] when $(c_0, c_3) = (+,+)$ and $(c_0, c_3) = (-,+)$ and in [12] when $(c_0, c_3) = (+,-)$ and $(c_0, c_3) = (-,-)$ under exponential decay conditions on the data.

By using inverse scattering methods several results were obtained for DSI system ($d_1 = 0, d_2 = d_3 = 1/2$, and $d_4 = d_5 = 1$ in (1.4)). In [9] A.S.Fokas and L.Y.Sung showed that if the initial function ϕ is in the Schwartz class and if $\partial_{x_1}\varphi_1(t, x_1)$ and $\partial_{x_2}\varphi_2(t, x_2)$ are also in the Schwartz class with respect to the spatial variables and continuous in t, then DSI system has a unique solution global in t which, for each fixed t, belongs to the Schwartz class in the spatial variables. Furthermore it is known that DSI system has the localized soliton type exact solutions which called dromion (for the study of the dromion solutions, see , e.g., [13],[20]).

§**2. Linear Schrödinger equations.** In this section we state smoothing properties of the inhomogeneous Schrödinger equations

$$\begin{cases} i\partial_t u + \Delta u = f, & (x,t) \in \mathbf{R}^2 \times \mathbf{R}, \\ u(0,x) = \phi(x). \end{cases} \tag{2.1}$$

We let U and S be $U(t) = \exp(it\Delta)$ and $(Sf)(t) = \int_0^t U(t-s)f(s)\,ds$ as defined in Section 1.

Following estimates were obtained by Strichartz [21], Kenig-Ponce-Vega [15],[16], Bekiranov-Ogawa-Ponce [3] and Hirata[14] e.t.c.

Lemma 2.1. *For the linear operator U and S, we have following estimates.*

$$\|U\phi\|_{L_T^\infty L^2} + \|D_{x_1}^{1/2} U\phi\|_{L_{x_1}^\infty L_T^2 L_{x_2}^2} + \|D_{x_2}^{1/2} U\phi\|_{L_{x_2}^\infty L_T^2 L_{x_1}^2} \le C_0 \|\phi\|_2, \tag{2.2}$$

$$\|\partial_{x_1} Sf\|_{L_{x_1}^\infty L_T^2 L_{x_2}^2} \le \begin{cases} \frac{1}{2}\|f\|_{L_{x_1}^1 L_T^2 L_{x_2}^2}, \\ C_1 \|D_{x_1}^{1/2} f\|_{L_T^1 L^2}, \end{cases} \tag{2.3}$$

$$\|\partial_{x_2} Sf\|_{L_{x_2}^\infty L_T^2 L_{x_1}^2} \le \begin{cases} \frac{1}{2}\|f\|_{L_{x_2}^1 L_T^2 L_{x_1}^2}, \\ C_1 \|D_{x_2}^{1/2} f\|_{L_T^1 L^2}, \end{cases} \tag{2.4}$$

$$\|Sf\|_{L_T^\infty L^2} \le \|f\|_{L_T^1 L^2}. \tag{2.5}$$

Next lemma is Hölder type estimate of Leibniz rule for fractional order derivative.

Lemma 2.2. *Let* $0 < \alpha < 1$ *and* $1 < p < \infty$. *Then*

$$\|D_x^\alpha(fg) - fD_x^\alpha g - gD_x^\alpha f\|_p \le C\|g\|_\infty \|D_x^\alpha f\|_p.$$

Let $p, p_1, p_2 \in (1, \infty)$ *such that* $1/p = 1/p_1 + 1/p_2$. *Then*

$$\|D_x^\alpha(fg) - fD_x^\alpha g - gD_x^\alpha f\|_p \le C\|g\|_{p_1} \|D_x^\alpha f\|_{p_2}.$$

For the proof of this lemma, see Appendix of [17;Theorem A.1].

§**3. The estimates for the nonlinear terms.** In what follows, we use following notations.

$$F(v) = \sum_{j=1}^{3} f_j(v),$$

where

$$f_1(v) = d_1|v|^2 v, \quad f_2(v) = d_2 v \int_{x_2}^{\infty} \partial_{x_1} |v(x_1, x_2')|^2 \, dx_2',$$
$$\text{and} \quad f_3(v) = d_3 v \int_{x_1}^{\infty} \partial_{x_2} |v(x_1', x_2)|^2 \, dx_1'.$$

By direct culculations and using Lemma 2.1 and 2.2, we have following estimate.

Lemma 3.1. *We have*

$$\begin{aligned} &\|\partial_{x_1}^3 SF(v)\|_{L_{x_1}^\infty L_T^2 L_{x_2}^2} \\ \le & CT\|v\|_{Y_T}^3 + (2|d_2|T\|v\|_{L_T^\infty L^2}\|\partial_t v\|_{L_T^\infty L^2} + |d_2|\|v(0)\|^2)\|\partial_{x_1}^3 v\|_{L_{x_1}^\infty L_T^2 L_{x_2}^2}, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} &\|\partial_{x_2}^3 SF(v)\|_{L_{x_2}^\infty L_T^2 L_{x_1}^2} \\ \le & CT\|v\|_{Y_T}^3 + (2|d_3|T\|v\|_{L_T^\infty L^2}\|\partial_t v\|_{L_T^\infty L^2} + |d_3|\|v(0)\|^2)\|\partial_{x_2}^3 v\|_{L_{x_2}^\infty L_T^2 L_{x_1}^2}. \end{aligned} \tag{3.2}$$

Lemma 3.2. *We have*

$$\begin{aligned} &\int_0^T |\operatorname{Im}(D_{x_1}^{1/2}\partial_{x_1}^2 F(v), D_{x_1}^{1/2}\partial_{x_1}^2 u)|dt \le CT\|v\|_{Y_T}^3 \|u\|_{Y_T} \\ &+ (4T\|v\|_{L_T^\infty L^2}\|\partial_t v\|_{L_T^\infty L^2} + 2|d_2|\|v(0)\|^2 \|\partial_{x_1}^3 v\|_{L_{x_1}^\infty L_T^2 L_{x_2}^2})\|\partial_{x_1}^3 u\|_{L_{x_1}^\infty L_T^2 L_{x_2}^2}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &\int_0^T |\operatorname{Im}(D_{x_2}^{1/2}\partial_{x_2}^2 F(v), D_{x_2}^{1/2}\partial_{x_2}^2 u)|dt \le CT\|v\|_{Y_T}^3 \|u\|_{Y_T} \\ &+ (4T\|v\|_{L_T^\infty L^2}\|\partial_t v\|_{L_T^\infty L^2} + 2|d_3|\|v(0)\|^2 \|\partial_{x_2}^3 v\|_{L_{x_2}^\infty L_T^2 L_{x_1}^2})\|\partial_{x_2}^3 u\|_{L_{x_2}^\infty L_T^2 L_{x_1}^2}, \end{aligned} \tag{3.4}$$

and

$$(3.5)\qquad \begin{aligned}&\int_0^T |\operatorname{Im}(D_{x_2}^{1/2}\partial_{x_1}\partial_{x_2}F(v), D_{x_2}^{1/2}\partial_{x_1}\partial_{x_2}u)|\,dt\\&+\int_0^T |\operatorname{Im}(D_{x_1}^{1/2}\partial_{x_1}\partial_{x_2}F(v), D_{x_1}^{1/2}\partial_{x_1}\partial_{x_2}u)|\,dt\\&\le CT\|v\|_{Y_T}^3\|u\|_{Y_T}.\end{aligned}$$

Lemma 3.3. *We have*

$$(3.6)\qquad \sum_{|\alpha|\le 2}\int_0^T |\operatorname{Im}(\partial^\alpha F(v), \partial^\alpha u)|\,dt \le CT\|v\|_{Y_T}^3\|u\|_{Y_T}.$$

We next consider the term

$$G(v;\varphi) = d_4 v\partial_{x_1}\varphi_1 + d_5 v\partial_{x_2}\varphi_2.$$

Using the similar way to above Lemmas, we have following.

Lemma 3.4. *We have*

$$\begin{aligned}\|\partial_{x_1}^3 SG(v;\varphi)\|_{L^\infty_{x_1}L^2_TL^2_{x_2}} &\le C_\varphi T\|v\|_{Y_T},\\ \|\partial_{x_2}^3 SG(v;\varphi)\|_{L^\infty_{x_2}L^2_TL^2_{x_1}} &\le C_\varphi T\|v\|_{Y_T},\\ \sum_{|\alpha|\le 2}\int_0^T |\operatorname{Im}(D_{x_1}^{1/2}\partial^\alpha G(v;\varphi), D_{x_1}^{1/2}\partial^\alpha u)|dt &\le C_\varphi T\|v\|_{Y_T}\|u\|_{Y_T},\\ \textit{and}\quad \sum_{|\alpha|\le 2}\int_0^T |\operatorname{Im}(D_{x_2}^{1/2}\partial^\alpha G(v;\varphi), D_{x_2}^{1/2}\partial^\alpha u)|dt &\le C_\varphi T\|v\|_{Y_T}\|u\|_{Y_T},\end{aligned}$$

where

$$C_\varphi = C(\|\partial_{x_1}\varphi_1\|_{H^{5/2}_{x_1}} + \|\partial_{x_2}\varphi_2\|_{H^{5/2}_{x_2}}).$$

§4. **Proof of Theorem 1.1.** We define the sequence $\{u_n(t)\}_{n\in\mathbf{N}\cup\{0\}}$ as follows:

$$(4.1)\qquad \begin{cases} u_0 = U\phi,\\ u_n = u_0 - iS(F(u_{n-1}) + G(u_{n-1};\varphi)),\end{cases}$$

where F and G are the same ones defined in Section 3. We first remark $u_0 \in X_T$ for some $\rho > 0$ by virtue of the first estimate in Lemma 2.1. From now on we will prove that $\{u_n(t)\}_{n\in\mathbf{N}}$ is a Cauchy sequence in $X_{T,\rho}$ for some time T, where

$$X_{T,\rho} = \{f \in X_T; \|f\|_{Y_T} \le \rho/2, \|\partial_{x_1}^3 f\|_{L^\infty_{x_1}L^2_TL^2_{x_2}} \le \rho/4, \|\partial_{x_2}^3 f\|_{L^\infty_{x_2}L^2_TL^2_{x_1}} \le \rho/4\}.$$

We assume that $u_j(t) \in X_{T,\rho}$ for all $0 \le j \le n-1$. By Lemma 3.1 and Lemma 3.4, we have

$$
\begin{aligned}
&\|\partial_{x_1}^3 u_n\|_{L^\infty_{x_1} L^2_T L^2_{x_2}} \\
\le & C_0 \|D_{x_1}^{1/2} \partial_{x_1}^2 \phi\| + CT \|u_{n-1}\|_{Y_T}^3 \\
& + |d_2| (2T \|u_{n-1}\|_{L^\infty_T L^2} \|\partial_t u_{n-1}\|_{L^\infty_T L^2} + \|\phi\|^2) \|\partial_{x_1}^3 u_{n-1}\|_{L^\infty_{x_2} L^2_T L^2_{x_1}} \\
& + C_\varphi T \|u_{n-1}\|_{Y_T}^3
\end{aligned} \tag{4.2}
$$

and

$$
\begin{aligned}
&\|\partial_{x_2}^3 u_n\|_{L^\infty_{x_2} L^2_T L^2_{x_1}} \\
\le & C_0 \|D_{x_2}^{1/2} \partial_{x_2}^2 \phi\| + CT \|u_{n-1}\|_{Y_T}^3 \\
& + |d_3| (2T \|u_{n-1}\|_{L^\infty_T L^2} \|\partial_t u_{n-1}\|_{L^\infty_T L^2} + \|\phi\|^2) \|\partial_{x_2}^3 u_{n-1}\|_{L^\infty_{x_2} L^2_T L^2_{x_1}} \\
& + C_\varphi T \|u_{n-1}\|_{Y_T}^3 .
\end{aligned} \tag{4.3}
$$

Here u_{n-1} satisties the differential equality

$$
\begin{cases} i\partial_t u_{n-1} = -\Delta u_{n-1} + F(u_{n-2}) + G(u_{n-2}), \\ u_{n-1}(0) = \phi, \end{cases}
$$

where we define $u_{-1} = 0$. So, by virtue of usual Sobolev's inequalities, we have

$$
\begin{aligned}
\|\partial_t u_{n-1}\|_{L^\infty_T L^2} \le & \|\Delta u_{n-1}\|_{L^\infty_T L^2} + \|F(u_{n-2})\|_{L^\infty_T L^2} + \|G(u_{n-2})\|_{L^\infty_T L^2} \\
\le & \|\Delta u_{n-1}\|_{L^\infty_T L^2} + d_1 \|u_{n-2}\|_{L^\infty_T L^6}^3 \\
& + d_2 \|u_{n-2}\|_{L^\infty_T L^\infty_{x_1} L^2_{x_2}} \|\partial_{x_1} |u_{n-2}|^2\|_{L^\infty_T L^2_{x_1} L^1_{x_2}} \\
& + d_3 \|u_{n-2}\|_{L^\infty_T L^\infty_{x_2} L^2_{x_1}} \|\partial_{x_2} |u_{n-2}|^2\|_{L^\infty_T L^2_{x_2} L^1_{x_1}} \\
& + d_4 \|u_{n-2}\|_{L^\infty_T L^2} \|\varphi_1\|_{L^\infty_T L^\infty_{x_1}} + d_5 \|u_{n-2}\|_{L^\infty_T L^2} \|\varphi_2\|_{L^\infty_T L^\infty_{x_2}} \\
\le & \|\Delta u_{n-1}\|_{L^\infty_T L^2} + C \|u_{n-2}\|_{L^\infty_T H^1}^3 + C_\varphi \|u_{n-2}\|_{L^\infty_T L^2} .
\end{aligned}
$$

Applying this estimate to (4.2) and (4.3), we have

$$
\begin{aligned}
&\|\partial_{x_1}^3 u_n\|_{L^\infty_{x_1} L^2_T L^2_{x_2}} \\
\le & C_0 \|\phi\|_{H^{5/2}} + CT \|u_{n-1}\|_{Y_T}^3 + C_\varphi T \|u_{n-1}\|_{Y_T}^3 \\
& + |d_2| \big(2T \|u_{n-1}\|_{L^\infty_T L^2} (\|\Delta u_{n-1}\|_{L^\infty_T L^2} + C \|u_{n-2}\|_{L^\infty_T H^1}^3) + \|\phi\|^2\big) \|\partial_{x_1}^3 u_{n-1}\|_{L^\infty_{x_1} L^2_T L^2_{x_2}} \\
\le & C_0 \|\phi\|_{H^{5/2}} + |d_2| \|\phi\|^2 \|\partial_{x_1}^3 u_{n-1}\|_{L^\infty_{x_1} L^2_T L^2_{x_2}} \\
& + C_\varphi T \|u_{n-1}\|_{Y_T} \big(\|u_{n-1}\|_{Y_T}^2 + (\|u_{n-1}\|_{Y_T} + \|u_{n-2}\|_{Y_T}^3) \|\partial_{x_1}^3 u_{n-1}\|_{L^\infty_{x_1} L^2_T L^2_{x_2}}\big) \\
\le & C_0 \|\phi\|_{H^{5/2}} + \tfrac{1}{4}\rho |d_2| \|\phi\|^2 + C_\varphi T \tfrac{1}{2}\rho \big(\tfrac{1}{4}\rho^2 + \tfrac{1}{4}\rho(\tfrac{1}{2}\rho + \tfrac{1}{8}\rho^3)\big) \\
= & C_0 \|\phi\|_{H^{5/2}} + \tfrac{1}{4}\rho |d_2| \|\phi\|^2 + \tfrac{1}{64} C_\varphi T \rho^3 (12 + \rho^3)
\end{aligned} \tag{4.4}
$$

and

$$\|\partial_{x_2}^3 u_n\|_{L^\infty_{x_2}L^2_TL^2_{x_1}} \le C_0\|\phi\|_{H^{5/2}} + \tfrac{1}{4}\rho|d_3|\|\phi\|^2 + \tfrac{1}{64}C_\varphi T\rho^3(12+\rho^3). \tag{4.5}$$

Now, by the assumptions on ϕ, we can define small positive constant δ such that $\max(|d_2|,|d_3|)\|\phi\|^2 \le 1-8\delta$. For this δ, we put ρ such that $C_0\|\phi\|_{H^{5/2}} \le \delta\rho$ and T such that $\frac{1}{64}C_\varphi T\rho^2(12+\rho^3) \le \delta$. Under these conditions, we see that

$$\|\partial_{x_1}^3 u_n\|_{L^\infty_{x_1}L^2_TL^2_{x_2}} \le \rho/4, \text{ and } \|\partial_{x_2}^3 u_n\|_{L^\infty_{x_2}L^2_TL^2_{x_1}} \le \rho/4. \tag{4.6}$$

Next, to estimate $D_x^{1/2}\partial^2 u$, we note that (4.1) is equivalent to

$$i\partial_t u_0(t) + \Delta u_0(t) = 0, \quad u_0(0) = \phi, \tag{4.7}$$

and

$$i\partial_t u_n + \Delta u_n = F(u_{n-1}) + G(u_{n-1}), \quad u_n(0) = \phi. \tag{4.8}$$

Applying both sides of (4.7) and (4.8) by $D_{x_1}^{1/2}\partial_{x_1}^2$, multiplying both sides of the resulting equations by $D_{x_1}^{1/2}\partial_{x_1}^2\bar{u}_0(t)$ and $D_{x_1}^{1/2}\partial_{x_2}^2\bar{u}_n(t)$, respectively, integrating over $\mathbf{R}^2$, and taking the imaginary part, we obtain

$$\frac{d}{dt}\|D_{x_1}^{1/2}\partial_{x_1}^2 u_0(t)\|^2 = 0, \tag{4.9}$$

and

$$\frac{d}{dt}\|D_{x_1}^{1/2}\partial_{x_1}^2 u_n(t)\|^2 = 2\,\mathrm{Im}(D_{x_1}^{1/2}\partial_{x_1}^2(F(u_{n-1}(t)) + G(u_{n-1}(t))), D_{x_1}^{1/2}\partial_{x_1}^2 u_n(t)). \tag{4.10}$$

Integrating (4.9) and (4.10) in t and using Lemma 3.2, we find that

$$\|D_{x_1}^{1/2}\partial_{x_1}^2 u_0\|^2_{L^\infty_T L^2} = \|D_{x_1}^{1/2}\partial_{x_1}^2\phi\|^2, \tag{4.11}$$

and

$$\begin{aligned}\|D_{x_1}^{1/2}\partial_{x_1}^2 u_n\|^2_{L^\infty_T L^2} \le& \|D_{x_1}^{1/2}\partial_{x_1}^2\phi\|^2 + 2CT\|u_{n-1}\|^3_{Y_T}\|u_n\|_{Y_T} \\ &+ \big(8T\|u_{n-1}\|_{L^\infty_T L^2}\|\partial_t u_{n-1}\|_{L^\infty_T L^2} \\ &\quad + 4|d_2|\|\phi\|^2\|\partial_{x_1}^3 u_{n-1}\|_{L^\infty_{x_1}L^2_TL^2_{x_2}}\big)\|\partial_{x_1}^3 u_n\|_{L^\infty_{x_1}L^2_TL^2_{x_2}} \\ &+ C_\varphi T\|u_{n-1}\|_{Y_T}\|u_n\|_{Y_T}.\end{aligned} \tag{4.12}$$

In the same way as in the proofs of (4.11) and (4.12) we have

$$\|D_{x_2}^{1/2}\partial_{x_2}^2 u_0\|^2_{L^\infty_T L^2} = \|D_{x_2}^{1/2}\partial_{x_2}^2\phi\|^2, \tag{4.13}$$

$$\begin{aligned}\|D_{x_2}^{1/2}\partial_{x_2}^2 u_n\|_{L_T^\infty L^2}^2 \le&\|D_{x_2}^{1/2}\partial_{x_2}^2\phi\|^2 + 2CT(\rho+\rho^3)\|u_n\|_{Y_T}\\ &+\big(8T\|u_{n-1}\|_{L_T^\infty L^2}\|\partial_t u_{n-1}\|_{L_T^\infty L^2}\\ &\quad+4|d_3|\|\phi\|^2\|\partial_{x_2}^3 u_{n-1}\|_{L_{x_2}^\infty L_T^2 L_{x_1}^2}\big)\|\partial_{x_2}^3 u_n\|_{L_{x_2}^\infty L_T^2 L_{x_1}^2}\\ &+C_\varphi T\|u_{n-1}\|_{Y_T}\|u_n\|_{Y_T},\end{aligned} \tag{4.14}$$

$$\begin{aligned}&\|D_{x_1}^{1/2}\partial_{x_1}\partial_{x_2}u_0\|_{L_T^\infty L^2}^2+\|D_{x_2}^{1/2}\partial_{x_1}\partial_{x_2}u_0\|_{L_T^\infty L^2}^2\\ =&\|D_{x_1}^{1/2}\partial_{x_1}\partial_{x_2}\phi\|^2+\|D_{x_2}^{1/2}\partial_{x_1}\partial_{x_2}\phi\|^2,\end{aligned} \tag{4.15}$$

and

(4.16)

$$\begin{aligned}&\|D_{x_1}^{1/2}\partial_{x_1}\partial_{x_2}u_n\|_{L_T^\infty L^2}^2+\|D_{x_2}^{1/2}\partial_{x_1}\partial_{x_2}u_n\|_{L_T^\infty L^2}^2\\ \le&\|D_{x_1}^{1/2}\partial_{x_1}\partial_{x_2}\phi\|^2+\|D_{x_2}^{1/2}\partial_{x_1}\partial_{x_2}\phi\|^2+CT\|u_{n-1}\|_{Y_T}^3\|u_n\|_{Y_T}+C_\varphi T\|u_{n-1}\|_{Y_T}\|u_n\|_{Y_T}.\end{aligned}$$

Integration by parts shows that

$$\begin{aligned}\|D_{x_1}^{1/2}\partial_{x_2}^2 u_n\|^2 \le&\|D_{x_2}^{1/2}\partial_{x_2}^2 u_n\|\|D_{x_2}^{1/2}\partial_{x_1}\partial_{x_2}u_n\|\\ \le&\varepsilon\|D_{x_2}^{1/2}\partial_{x_2}^2 u\|^2+\tfrac{1}{4\varepsilon}\|D_{x_2}^{1/2}\partial_{x_1}\partial_{x_2}u\|^2,\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}\|D_{x_2}^{1/2}\partial_{x_1}^2 u_n\|^2 \le&\|D_{x_1}^{1/2}\partial_{x_1}^2 u_n\|\|D_{x_1}^{1/2}\partial_{x_1}\partial_{x_2}u\|\\ \le&\varepsilon\|D_{x_1}^{1/2}\partial_{x_1}^2 u\|^2+\tfrac{1}{4\varepsilon}\|D_{x_1}^{1/2}\partial_{x_1}\partial_{x_2}u\|^2,\end{aligned} \tag{4.18}$$

where $\varepsilon>0$ is determined later. By the usual energy method and Lemma 3.3 we have

$$\sum_{|\alpha|\le 2}\|\partial^\alpha u_n\|_{L_T^\infty L^2}^2 \le \sum_{|\alpha|\le 2}\|\partial^\alpha\phi\|^2+CT(\rho+\rho^3)\|u_n\|_{Y_T}. \tag{4.19}$$

From (4.11)-(4.19) and the Schwarz inequality it follows that

$$\begin{aligned}\|u_n\|_{Y_T}^2 \le&C\|\phi\|_{H^{5/2}}^2+\tfrac{1}{16}C_\varphi T\rho^2(4+\rho^2)+\tfrac{1}{32}(8+\varepsilon)T\rho^3(4+\rho^2)\\ &+\tfrac{1}{32}(4+\varepsilon)(|d_2|+|d_3|)\|\phi\|^2\rho^2.\end{aligned}$$

Hence, if necessary, we retake ρ and T such that

$$\begin{cases}\tfrac{1}{16}(8+\varepsilon)T\rho(4+\rho)\le\delta,\\ C\|\phi\|_{H^{5/2}}^2\le\tfrac12\delta\rho^2,\\ \tfrac18 C_\varphi T(4+\rho^2)\le\delta,\end{cases}$$

we find that

$$\|u_n\|_{Y_T} \le \frac{\rho}{2}. \tag{4.20}$$

From (4.6) and (4.20), we see that $\{u_n\}$ is well-defined sequence in $X_{T,\rho}$. For $u_0(t) = U(t)\phi$ we have the following estimate by Lemma 2.1

$$\|u_0\|_{X_T} \le \|\phi\|_{H^{5/2}}. \tag{4.21}$$

The induction argument and (4.20)-(4.21) show that (4.21) holds for any $n \in \mathbf{N} \cup \{0\}$. A similar calculation shows $\{u_n\}$ is a Cauchy sequence which implies Theorem 1.1. □

§**5. Some comutator estimates.** Before starting the proof of Theorem 1.2, we state some lemmas.

Lemma 5.1. *We have*

$$\|f\|_{L^\infty_{x_1}} \le C(1+|t|)^{-1/2}(\|\langle D_{x_1}\rangle f\|_{L^\infty_{x_1}} + \|J_{x_1} f\|_{L^2_{x_1}}).$$

Proof. We apply to Sobolev's inequality to $\exp(-i|x_1|^2/2t)f$ to get

$$\|f\|_{L^\infty_{x_1}} \le C|t|^{-1/2}\|J_{x_1} f\|^{1/2}_{L^2_{x_1}}\|f\|^{1/2}_{L^2_{x_1}} \le C|t|^{-1/2}(\|J_{x_1} f\|_{L^2_{x_1}} + \|f\|_{L^2_{x_1}}),$$

which with the usual Sobolev's inequality yields the lemma.

Lemma 5.2. *We have*

$$\|[\langle D_{x_1}\rangle^{1/2}, f]g\|_{L^2_{x_1}} + \|[\langle D_{x_1}\rangle, f]g\|_{L^2_{x_1}} \le C\|\langle D_{x_1}\rangle f\|_{L^\infty_{x_1}}\|g\|_{L^2_{x_1}}.$$

The proof of the lemma is obtained by the following result due to R.R.Coifman and Y.Meyer (see [5], pp. 154).

Lemma.5.3. *Let $\sigma \in C^\infty(\mathbf{R}^m \times \mathbf{R}^m \setminus (0,0))$ satisfy*

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi,\eta)| \le C_{\alpha,\beta}(|\xi| + |\eta|)^{-|\alpha|-|\beta|}$$

for $(\xi,\eta) \ne (0,0)$ and any $\alpha, \beta \in (\mathbf{N})^m$. If $\sigma(D)$ denotes the bilinear operator

$$\sigma(D)(a,h)(x) = \int\int e^{i(x,\xi+\eta)}\sigma(\xi,\eta)\hat{a}(\xi)\hat{h}(\eta)d\xi d\eta,$$

then

$$\|\sigma(D)(a,h)\|_{L^2(\mathbf{R}^m)} \le C\|a\|_{L^\infty(\mathbf{R}^m)}\|h\|_{L^2(\mathbf{R}^m)}.$$

Proof of Lemma 5.2 We have

$$[\langle D_{x_1}\rangle^{1/2}, f]g(x_1) = (\langle D_{x_1}\rangle^{1/2}(fg) - f(\langle D_{x_1}\rangle^{1/2}g))(x_1)$$
$$= \left(\frac{1}{2\pi}\right)^2 \int\int e^{ix_1(\xi_1+\eta_1)}\{(1+|\xi_1+\eta_1|^2)^{1/4} - (1+|\eta_1|^2)^{1/4}\}\hat{f}(\xi_1)\hat{g}(\eta_1)d\xi_1 d\eta_1,$$

where

$$\hat{f}(\xi_1) = \int e^{-ix\xi_1} f(x)dx.$$

We easily see that

$$(1+|\xi_1+\eta_1|^2)^{1/4} - (1+|\eta_1|^2)^{1/4}$$
$$= \frac{\xi_1(\xi_1+2\eta_1)}{\left((1+|\xi_1+\eta_1|^2)^{1/4} + (1+|\eta_1|^2)^{1/4}\right)\left((1+|\xi_1+\eta_1|^2)^{1/2} + (1+|\eta_1|^2)^{1/2}\right)}$$

Therefore Lemma 5.3 gives

$$\|[\langle D_{x_1}\rangle^{1/2}, f]g\|_{L^2_{x_1}} \le C\|\langle D_{x_1}\rangle f\|_{L^\infty_{x_1}}\|g\|_{L^2_{x_1}}.$$

In the same way we have

$$\|[\langle D_{x_1}\rangle, f]g\|_{L^2_{x_1}} \le C\|\langle D_{x_1}\rangle f\|_{L^\infty_{x_1}}\|g\|_{L^2_{x_1}}.$$

This completes the proof of the lemma. □

Lemma 5.4. *We have*

$$\left|\int\int\int |v|^2(x_1,x_2')\bar{h}(x_1,x_2)(\langle D_{x_1}\rangle h(x_1,x_2))dx_1dx_2dx_2'\right|$$
$$\ge -C\|\langle D_{x_1}\rangle v\|_{L^2_{x_2}L^\infty_{x_1}}(\|\langle D_{x_1}\rangle v\|_{L^2_{x_2}L^\infty_{x_1}} + \|v\|_{L^2_{x_2}L^\infty_{x_1}})\|h\|^2_{L^2_{x_2}L^2_{x_1}}$$
$$+\frac{1}{2}\|\|v\|_{L^2_{x_2}}\|\langle D_{x_1}\rangle^{1/2}h\|_{L^2_{x_2}}\|^2_{L^2_{x_1}}.$$

Proof. We denote the left hand side of the inequality in the lemma by

$$I = |(|v|^2\langle D_{x_1}\rangle h, h)_{L^2_{x_2}L^2_{x_2'},L^2_{x_1}}|.$$

We find that by the Hölder inequality and the Plancherel theorem

$$
\begin{aligned}
I &\geq -|(h,[\langle D_{x_1}\rangle,\bar{v}]vh)_{L^2_{x_2}L^2_{x_2'},L^2_{x_1}}| + \|\langle D_{x_1}\rangle^{1/2}vh\|^2_{L^2_{x_2}L^2_{x_2'},L^2_{x_1}} \\
&\geq -|(h,[\langle D_{x_1}\rangle,\bar{v}]vh)_{L^2_{x_2}L^2_{x_2'},L^2_{x_1}}| + \|[\langle D_{x_1}\rangle^{1/2},v]h + v\langle D_{x_1}\rangle^{1/2}h\|^2_{L^2_{x_2}L^2_{x_2'},L^2_{x_1}} \\
&\geq -|(h,[\langle D_{x_1}\rangle,\bar{v}]vh)_{L^2_{x_2}L^2_{x_2'},L^2_{x_1}}| + \|[\langle D_{x_1}\rangle^{1/2},v]h\|^2_{L^2_{x_2}L^2_{x_2'},L^2_{x_1}} \\
&+ \|v\langle D_{x_1}\rangle^{1/2}h\|^2_{L^2_{x_2}L^2_{x_2'},L^2_{x_1}} + 2\,\mathrm{Re}([\langle D_{x_1}\rangle^{1/2},v]h, v\langle D_{x_1}\rangle^{1/2}h)_{L^2_{x_2}L^2_{x_2'},L^2_{x_1}} \\
&\geq -\|h\|_{L^\infty_{x_2'},L^2_{x_2}L^2_{x_1}}\|[\langle D_{x_1}\rangle,\bar{v}]vh\|_{L^1_{x_2'},L^2_{x_2}L^2_{x_1}} \\
&- 2\|[\langle D_{x_1}\rangle^{1/2},v]h\|^2_{L^2_{x_2}L^2_{x_2'},L^2_{x_1}} + \frac{1}{2}\|v\langle D_{x_1}\rangle^{1/2}h\|^2_{L^2_{x_2}L^2_{x_2'},L^2_{x_1}}.
\end{aligned}
$$

We now apply Lemma 5.2 to the above to get the desired result. □

§**6. Outline of the proof of Theorem 1.2.** Since the proof of theorem is so complicated, we consider following equation:

$$
i\partial_t u + \Delta u = u\int_{x_2}^{\infty} \partial_{x_1}|u|^2 dx_2', \tag{6.1}
$$

which have only one nonlinear term. The estimates of other terms are similar or easier, so the essential part of the proof is not lost.

We define the operator K_{x_1} and K_{x_2} as

$$
K_{x_1} = K_{x_1}(v) = \sum_{m=0}^{\infty} \frac{A^m}{m!}\left(\int_{-\infty}^{x_1} \|v(t,x_1')\|^2_{L^2_{x_2}} dx_1' \frac{D_{x_1}}{\langle D_{x_1}\rangle}\right)^m,
$$

$$
K_{x_2} = K_{x_2}(v) = \sum_{m=0}^{\infty} \frac{A^m}{m!}\left(\int_{-\infty}^{x_2} \|v(t,x_2')\|^2_{L^2_{x_1}} dx_2' \frac{D_{x_2}}{\langle D_{x_2}\rangle}\right)^m,
$$

and $A^2 = 1/\delta_3$ (for the definition of δ_3, see Theorem 1.2). Then operating $K_{x_j}\partial^\alpha J^\beta$ to (6.1) and taking L^2-inner product with $K_{x_j}\partial^\alpha J^\beta u$ ($|\alpha|+|\beta|\leq 3$), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|+|\beta|\leq 3}(\|K_{x_1}\partial^\alpha J^\beta u(t)\|^2 + \|K_{x_2}\partial^\alpha J^\beta u(t)\|^2) \\
&+ \frac{1}{4\delta_3^{1/2}}\sum_{|\alpha|+|\beta|\leq 3}\left(\big\|\|u(t)\|_{L^2_{x_2}}\|\langle D_{x_1}\rangle^{1/2}K_{x_1}\partial^\alpha J^\beta u(t)\|_{L^2_{x_2}}\big\|^2_{L^2_{x_1}}\right. \\
&\left.+ \big\|\|u(t)\|_{L^2_{x_1}}\|\langle D_{x_2}\rangle^{1/2}K_{x_2}\partial^\alpha J^\beta u(t)\|_{L^2_{x_1}}\big\|^2_{L^2_{x_2}}\right) \\
&\leq C(1+A)^2(1+t)^{-1}\|u(t)\|^2_{X^{2,2}(t)}(1+\|u(t)\|^2_{X^{2,2}(t)})\|u(t)\|^2_{X^{3,3}(t)} \\
&+ \sum_{|\alpha|+|\beta|\leq 3}\left(\left|\mathrm{Im}(K_{x_1}\partial^\alpha J^\beta u\int_{x_2}^{\infty}\partial_{x_1}|u|^2dx_2', K_{x_1}\partial^\alpha J^\beta u)\right|\right. \\
&\left.+\left|\mathrm{Im}(K_{x_2}\partial^\alpha J^\beta u\int_{x_2}^{\infty}\partial_{x_1}|u|^2dx_2', K_{x_2}\partial^\alpha J^\beta u)\right|\right).
\end{aligned} \tag{6.2}
$$

The second term of the left hand side of (6.2) means smoothing properties of solutions to the equation. By virtue of Lemma5.1-5.4 and the explection:

$$u\int_{x_2}^{\infty}\partial_{x_1}|u|^2dx_2' = u\frac{1}{2it}\int_{x_2}^{\infty}\bar{u}J_{x_1}u - u\overline{J_{x_1}u}dx_2', \tag{6.3}$$

we have

$$\begin{aligned}&\sum_{|\alpha|+|\beta|\le 2}\|\partial^\alpha J^\beta F_{x_2}(u(t))\| + \|\partial_{x_2}^3 F_{x_2}(u(t))\| + \|J_{x_2}^3 F_{x_2}(u(t))\|\\ &\le C(1+|t|)^{-2}\|u(t)\|_{X^{2,2}(t)}^2\|u(t)\|_{X^{3,3}(t)},\end{aligned} \tag{6.4}$$

and

$$\begin{aligned}&|(K_{x_1}\partial_{x_1}^3 F_{x_2}(u(t)), K_{x_1}\partial_{x_1}^3 u(t))| + |(K_{x_1}J_{x_1}^3 F_{x_2}(u(t)), K_{x_1}J_{x_1}^3 u(t))|\\ &\le Ce^{CA\|u(t)\|^2}(1+\|u(t)\|_{X^{2,2}(t)}^2)\{(1+A)^2(1+|t|)^{-1}\|u(t)\|_{X^{2,2}(t)}^2\|u(t)\|_{X^{3,3}(t)}^2\\ &+ \big\|\|u(t)\|_{L^2_{x_2}}\|\langle D_{x_1}\rangle^{1/2}K_{x_1}\partial_{x_1}^3 u(t)\|_{L^2_{x_2}}\big\|_{L^2_{x_1}}^2\\ &+ \big\|\|u(t)\|_{L^2_{x_2}}\|\langle D_{x_1}\rangle^{1/2}K_{x_1}J_{x_1}^3 u(t)\|_{L^2_{x_2}}\big\|_{L^2_{x_1}}^2\},\end{aligned} \tag{6.5}$$

where $K_{x_1} = K_{x_1}(u)$ and $F_{x_2}(u(t)) = u\int_{x_2}^{\infty}\partial_{x_1}|u|^2dx_2'$. Applying (6.4) and (6.5) to the right hand side of (6.2), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|+|\beta|\le 3}(\|K_{x_1}\partial^\alpha J^\beta u(t)\|^2 + \|K_{x_2}\partial^\alpha J^\beta u(t)\|^2)\\ &+\left(\frac{1}{4\delta_3^{1/2}} - Ce^{C\delta_3}\right)\sum_{|\alpha|+|\beta|\le 3}\Big(\big\|\|u(t)\|_{L^2_{x_2}}\|\langle D_{x_1}\rangle^{1/2}K_{x_1}\partial^\alpha J^\beta u(t)\|_{L^2_{x_2}}\big\|_{L^2_{x_1}}^2\\ &+\big\|\|u(t)\|_{L^2_{x_1}}\|\langle D_{x_2}\rangle^{1/2}K_{x_2}\partial^\alpha J^\beta u(t)\|_{L^2_{x_1}}\big\|_{L^2_{x_2}}^2\Big) \le C(1+t)^{-1}\delta_3\|u(t)\|_{X^{3,3}(t)}^2\end{aligned} \tag{6.6}$$

provided that δ_3 is sufficiently small and

$$\sup_{-T\le t\le T}\|u(t)\|_{X^{2,2}(t)}^2 \le 4\delta_3^2, \tag{6.7}$$

$$\sup_{-T\le t\le T}(1+|t|)^{-C\delta_3}\|u(t)\|_{X^{3,3}(t)}^2 \le 4\delta_3^2 \tag{6.8}$$

for some time $T > 0$. We choose δ_3 satisfying

$$\frac{1}{4\delta_3^{1/2}} - Ce^{C\delta_3} \ge 0.$$

Then we have

$$\|u(t)\|^2_{X^{3,3}(t)} \le e^{C\delta_3}\delta_3^2 + C\delta_3 \int_0^t (1+s)^{-1}\|u(s)\|^2_{X^{3,3}(t)}ds. \tag{6.9}$$

Thus (6.6) shows that the nonliear term is controlled by the second term of the left hand side of (6.2) and the right hand side of (6.6). Global existence theorem is obtained by showing that (6.7) and (6.8) hold for any T. In order to prove (1.9) and (1.10) for any $T > 0$ we need (6.9) and the following inequality

$$\|u(t)\|^2_{X^{2,2}(t)} \le e^{C\delta_3}\delta_3^2 + C\delta_3 \int_0^t (1+s)^{-1-2C\delta_3}\|u(s)\|^2_{X^{3,3}(t)}ds. \tag{6.10}$$

The inequality (6.10) is obtained by making use of the structure of nonlinear term (6.3). Theorem 1.2 is obtained by applying the Gronwall inequality to (6.9) and (6.10). It seems to be difficult to get the inequality (6.9) through the methods used in Theorem 1.1 because nonlinear terms are not taken into account to derive smoothing properties. On the other hand the operators K_{x_1} and K_{x_2} are made based on the nonlocal nonlinear terms (the second and the third terms on the right hand side of (1.4)). The similar operators as those of K_{x_1} and K_{x_2} have been used in [4].

Remark. Our method does not work for the hyperbolic-hyperbolic Davey- Stewartson system. If we apply the similar methods to the local solutions of

$$i\partial_t u + 2\partial_{x_1}\partial_{x_2}u = u\int_{x_2}^{\infty} \partial_{x_1}|u|^2 dx_2{}',$$

we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|+|\beta|\le 3}(\|K_{x_1}\partial^\alpha J^\beta u(t)\|^2 + \|K_{x_2}\partial^\alpha J^\beta u(t)\|^2)\\
&+\frac{1}{4\delta_3^{1/2}}\sum_{|\alpha|+|\beta|\le 3}(\| \|u(t)\|_{L^2_{x_2}} \|\langle D_{x_1}\rangle^{1/2}\tilde{K}_{x_1}\partial^\alpha J^\beta u(t)\|_{L^2_{x_2}}\|^2_{L^2_{x_1}}\\
&+\| \|u(t)\|_{L^2_{x_1}} \|\langle D_{x_2}\rangle^{1/2}\tilde{K}_{x_2}\partial^\alpha J^\beta u(t)\|_{L^2_{x_1}}\|^2_{L^2_{x_2}})\\
&\le C(1+A)^2(1+t)^{-1}\|u(t)\|^2_{X^{2,2}(t)}(1+\|u(t)\|^2_{X^{2,2}(t)})\|u(t)\|^2_{X^{3,3}(t)}\\
&+\sum_{|\alpha|+|\beta|\le 3}(|\operatorname{Im}(\tilde{K}_{x_1}\partial^\alpha J^\beta u\int_{x_2}^{\infty}\partial_{x_1}|u|^2dx_2{}', \tilde{K}_{x_1}\partial^\alpha J^\beta u)|\\
&+|\operatorname{Im}(\tilde{K}_{x_2}\partial^\alpha J^\beta u\int_{x_2}^{\infty}\partial_{x_1}|u|^2dx_2{}', \tilde{K}_{x_2}\partial^\alpha J^\beta u)|)
\end{aligned}$$

where

$$\tilde{K}_{x_1} = \sum_{m=0}^{\infty}\frac{A^m}{m!}\left(\int_{-\infty}^{x_2}\|v(t,x_2{}')\|^2_{L^2_{x_1}}dx_2{}'\frac{D_{x_1}}{\langle D_{x_1}\rangle}\right)^m = e^{A\int_{-\infty}^{x_2}\|v(t,x_2{}')\|^2_{L^2_{x_1}}dx_2{}'\frac{D_{x_1}}{\langle D_{x_1}\rangle}},$$

$$\tilde{K}_{x_2} = \sum_{m=0}^{\infty} \frac{A^m}{m!}\left(\int_{-\infty}^{x_1} \|v(t,x_1')\|_{L^2_{x_2}}^2 \, dx_1' \frac{D_{x_2}}{\langle D_{x_2}\rangle}\right)^m = e^{A\int_{-\infty}^{x_1} \|v(t,x_1')\|_{L^2_{x_2}}^2 \, dx_1' \frac{D_{x_2}}{\langle D_{x_2}\rangle}}.$$

We apply (6.4) and (6.5) to the right hand side of the above inequality to get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|+|\beta|\le 3}(\|K_{x_1}\partial^\alpha J^\beta u(t)\|^2 + \|K_{x_2}\partial^\alpha J^\beta u(t)\|^2)\\
&+\frac{1}{4\delta_3^{1/2}}\sum_{|\alpha|+|\beta|\le 3}\left(\Big\|\|u(t)\|_{L^2_{x_2}}\|\langle D_{x_1}\rangle^{1/2}\tilde{K}_{x_1}\partial^\alpha J^\beta u(t)\|_{L^2_{x_2}}\Big\|_{L^2_{x_1}}^2\right.\\
(6.11)\qquad &\left.+\Big\|\|u(t)\|_{L^2_{x_1}}\|\langle D_{x_2}\rangle^{1/2}\tilde{K}_{x_2}\partial^\alpha J^\beta u(t)\|_{L^2_{x_1}}\Big\|_{L^2_{x_2}}^2\right)\\
&\le C(1+A)^2(1+t)^{-1}\|u(t)\|_{X^{2,2}(t)}^2(1+\|u(t)\|_{X^{2,2}(t)}^2)\|u(t)\|_{X^{3,3}(t)}^2\\
&+Ce^{C\delta_3}\sum_{|\alpha|+|\beta|\le 3}\Big\|\|u(t)\|_{L^2_{x_2}}\|\langle D_{x_1}\rangle^{1/2}\tilde{K}_{x_1}\partial^\alpha J^\beta u(t)\|_{L^2_{x_2}}\Big\|_{L^2_{x_1}}^2
\end{aligned}$$

under the conditions (6.7) and (6.8). It is easy to see that the last term of the right hand side of (6.11) can not be controlled by the second term of the left hand side of (6.11). This is the reason why our method does not work for the hyperbolic-hyperbolic system.

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