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# An approach to the liar paradox

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**Abstract**

The liar paradox discloses such a property of the natural language which renders possible the formulation of a sentence expressing its own falsehood. Obviously the acceptance of such a sentence must result in contradiction. Thus, the so-called "solution of the liar paradox" cannot consist solely in the abolition of paradoxical sentences in question. In opposite, the "liar sentence" seems to require some simple and clear formalization which would allow to treat this sentence as another contradiction. Hence the desired language should contain an instrument to construct self-referential sentences.

In this paper, an approach to the liar paradox satisfies two, possibly controversial, conditions. The first one is the assumption that the liar paradox is only a sentential language problem. It implies that there is no need to employ predicates. According to the second assumption, the sentence expressing the paradox is free from such elements of the language which are not connectives, i.e. there is no place for such additional components like, for example, an agent. Thus, the liar paradox is here an "inner" problem of the almost ordinary sentential language.

The famous paradoxical sentence *this sentence is false* is here formalised in such a way that its interpretation is such a contrtautology as, for example, $p \wedge \neg p$. The only difference between both kinds of contradictory sentences is that the latter is expressed only by truthfunctional connectives, while the former one employs some intentional function.

Finally, we should notice that our approach is not limited to the classical case only but can be applied for many non-classical logics as well.

## 1 The liar sentence

Let us assume that some sentence $p$ is true and false simultaneously. In other words, we have

$$p \wedge \neg p. \tag{1}$$

Clearly, $\neg p$ follows from $p$ and $p$ follows from $\neg p$, which means that basing on our assumption we obtain $p \leftrightarrow \neg p$. No one, however, is astonished nor treats this situation as a paradoxical one. It is clear that if we assume some contradictory sentence, its truth is equivalent to its falsehood, which is a natural and well known fact.

Our approach to the liar paradox will be similar. We shall try to express the sentence

**this sentence is false**

in such a way, that it is a simple contradictory sentence. Then, a fact that the truth of this sentence follows from its falsehood, and vice versa, will be quite natural.

In the first parts of [7] (see [6] for original Polish version) Tarski writes:
"[...] For the sake of greater perspicuity we shall use the symbol "$c$" as a typographical abbreviation of the expression "*the sentence printed on this page, line 5 from the top*". Consider now the following sentence:

*c is not a true sentence.* [...]"

Of course, the last sentence above is writen in [7] in the fifth line from the top of the page. Following Leśniewski, Tarski infers a contradiction from this formalization.

Both of our assumptions stated in the abstract follow from Tarski's presentation. It is easy to see that neither a predicate nor an agent are necessary to express the liar sentence. Moreover, Tarski gives some suggestions for constructing a new connective satisfying our intuitions of self-referential sentences.

Let us consider an intentional connective "*... says that ...*". Probably the mark of colon ":" would here comply with our intuition the most as such is its reading in natural language practice. Now, our C-language (colon-language) is an algebra

$$\mathcal{L}_C = (L_C, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, :).$$

Thus, the sentence $p{:}q$ will be read as "*p says that q*". Its desired meaning is that every sentence in the form of the conjunction says about all of its components, i.e.

$$\alpha_1 \wedge ... \wedge \alpha_k \text{ says that } \alpha_i$$

for $i=1,...k$ and for any $\alpha_1,...,\alpha_k \in L_C$. If a formula is a conjunction, we can express this fact by means of colon. In other case, colon expresses only a trivial fact that $\alpha$ is of the form of $\alpha$.

This meaning will be rendered by the following axiom set for the colon connective.

Of course, each sentence says what it says
($A_1$) $\alpha{:}\alpha$.

If the first sentence says what the second sentence says and if this second sentence says what the third sentence says, then obviously the first sentence says what the third one says
($A_2$) $((\alpha{:}\beta)\wedge(\beta{:}\gamma))\rightarrow(\alpha{:}\gamma)$.

The next two axioms formalize some especially natural aspects of the main idea mentioned above:
($A_3$) $(\alpha\wedge\beta){:}\alpha$,
($A_4$) $(\alpha\wedge\beta){:}(\beta\wedge\alpha)$.

To repeat the same sentence is to say nothing new
($A_5$) $\alpha{:}(\alpha\wedge\alpha)$.

If one sentence says what the other sentence says and vice versa, it means that the sentences are identical. So-defined identity is an axiomatic extention of the Suszko's identity "$\equiv$" extended by two axioms: $\alpha\equiv(\alpha\wedge\alpha)$ and $(\alpha\wedge\beta)\equiv(\beta\wedge\alpha)$ (e.g. see [1], [5]).

After such a remark, the axioms of the invariance of the colon with respect to all other connectives seem to be natural

($A_6$) $((\alpha{:}\beta)\wedge(\beta{:}\alpha))\rightarrow((\neg\alpha{:}\neg\beta)\wedge(\neg\beta{:}\neg\alpha))$,

($A_8$) $((\alpha{:}\beta)\wedge(\beta{:}\alpha)\wedge(\gamma{:}\delta)\wedge(\delta{:}\gamma))\rightarrow(((\alpha\S\gamma){:}(\beta\S\delta))\wedge((\beta\S\delta){:}(\alpha\S\gamma)))$ for $\S\in\{\rightarrow,\leftrightarrow,:\}$.

The only exceptions are the conjunction and disjunction connectives. Because of the character of colon, we can assume even simpler a condition

($A_7$) $((\alpha{:}\beta)\wedge(\gamma{:}\delta))\rightarrow((\alpha\S\gamma){:}(\beta\S\delta))$ for $\S\in\{\wedge,\vee\}$.

The last axiom shows that the colon connective is another stronger implication

($A_9$) $(\alpha{:}\beta)\rightarrow(\alpha\rightarrow\beta)$.

So-defined connective can extend not only the classical but also, for example, intuitionistic logic. At first, there will be considered a semantics for the classical case. A modification of the semantics for the intuitionistic case is analogous to the one presented in [2].

Because of our philosophical intentions and some relation to Suszko's identity, the most natural semantics for logic with colon and for logic with identity connective should comprise *semantic correlates* as objects interpreting sentences. In the classical case, a matrix semantics is given by models being some modifications of Suszko's SCI-models (cf. [5]). Let $\mathcal{A}=(\mathrm{A},-,\cap,\cup,\rightarrow,\leftrightarrow,\rhd)$ be an algebra similar to C-language. For any D a nonempty proper subset of A, matrix $\mathcal{M}=(\mathcal{A},\mathrm{D})$ will be called Cc-model (a model for classical logic with colon given by ($A_1$)-($A_9$)), if for any $a,b\in\mathrm{A}$ $a=a\cap a$, $a\cap b=b\cap a$ and, moreover

| | | | |
|---|---|---|---|
| (1) | $-a\in\mathrm{D}$ | iff | $a\notin\mathrm{D}$; |
| (2) | $a\cap b\in\mathrm{D}$ | iff | $a\in\mathrm{D}$ and $b\in\mathrm{D}$; |
| (3) | $a\cup b\in\mathrm{D}$ | iff | $a\in\mathrm{D}$ or $b\in\mathrm{D}$; |
| (4) | $a\rightarrow b\in\mathrm{D}$ | iff | $a\notin\mathrm{D}$ or $b\in\mathrm{D}$; |
| (5) | $a\rhd b\in\mathrm{D}$ | iff | $a=b\cap c$ for some $c\in\mathrm{A}$. |

The last condition (5) realizes our earlier accepted assumptions.

The matrix consequence operation $C_{\mathcal{M}}$ is defined in a standard way, i.e. for any $B \subseteq L_C$ and for any $\alpha \in L_C$
$\alpha \in C_{\mathcal{M}}(B)$ iff for any $h \in Hom(\mathcal{L}_C, \mathcal{A})$
$h(\alpha) \in D$ provided $h(\beta) \in D$ for all $\beta \in B$.

An easy verification proves the following completeness theorem

$$\alpha \in C_{Cc}(B) \quad \text{iff} \quad \text{for any Cc-model } \mathcal{M} \ \ \alpha \in C_{\mathcal{M}}(B)$$

for any $B \subseteq L$ and for any $\alpha \in L$.

A proof of *soundness* is based upon the fact that $h((\alpha{:}\beta) \wedge (\beta{:}\alpha)) \in D$ implies $h(\alpha) = h(\beta)$. Let us assume an antecedent of this implication. Then, $h(\alpha) = h(\beta) \cap c$ and $h(\beta) = h(\alpha) \cap d$ for some $c, d \in A$. Hence, $h(\alpha) = h(\alpha) \cap d \cap c$ and $h(\beta) = h(\alpha) \cap d \cap c \cap d = h(\alpha)$.

For the *completeness* part of the proof there will be checked the last (fifth) condition, only. Obviously, for any Cc-theory T, the relation "$\sim$" such that $\alpha \sim \beta$ if and only if $(\alpha{:}\beta) \wedge (\beta{:}\alpha) \in T$, for any $\alpha, \beta \in L_C$, is a congruence of the matrix $(\mathcal{L}_C, T)$. The fact that $(\mathcal{L}_C/\sim, T/\sim)$ with a maximal Cc-theory T satisfies all "truthfunctional" conditions (1)-(4) is clear. Assume that $[\alpha]_\sim \rhd_\sim [\beta]_\sim \in T/\sim$. Then, $\alpha{:}\beta \in T$ and by $(A_1)$, $(A_2)$, $(A_5)$ and $(A_7)$ $\alpha{:}(\alpha \wedge \beta) \in T$. So, $[\alpha]_\sim = [\beta]_\sim \cap_\sim [\alpha]_\sim$. Conversely, suppose that $[\alpha]_\sim = [\beta]_\sim \cap_\sim [\gamma]_\sim$ for $\gamma \in L_C$. By $(A_2)$ and $(A_3)$ $\alpha{:}(\beta \wedge \gamma) \in T$ implies $\alpha{:}\beta \in T$. Thus, $[\alpha]_\sim \rhd_\sim [\beta]_\sim \in T/\sim$. □

Let us consider an additional connective of assertion Q, given by axiom
$(A_Q)$ $\alpha \leftrightarrow Q\alpha$.

Because $\alpha{:}Q\alpha$ is not a Cc-tautology, the assertion Q is not a trivial connective, here.

Now, following Tarski's formulation, we can express the liar paradox sentence as

$$p : \neg Qp. \tag{2}$$

Easy checking shows that (2) is a contrtautology. In this sense, both sentences (1) and (2) have the same status with the only difference being that (2) contains a non-truthfunctional connective. It is nothing unusual that the acceptance of (2) yields the situation in which the truth of the sentence $p$ follows from its falsehood and vice versa - the formula $(p{:}\neg Qp)\rightarrow(p\leftrightarrow\neg p)$ is a Cc-tautology. This means that in our approach, once the sentence $p\wedge\neg p$ is not treated as paradoxical, neither the liar sentence can be treated as such.

## 2 The Brouwerian counterpart of colon

Our approach is supposed to be independend from any concrete logic, therefore the colon is supposed to work on the base of many different logics. But unfortunately, our new connective cannot be applied in all, even very similar to the liar, cases. Let us consider two following sentences which, traditionally, should be treated as paradoxes

**this sentence says what it does not say,** (3)

**this sentence does not say what it says.** (4)

Clearly, sentences (3) and (4) have different meanings. Contrary to the liar sentence, both considered sentences do not deal with a problem of truth values. A formalization of (3) is obvious, $p{:}\neg p$. However, in the case of (4) some problem appears, especially when our logic is not classical. For example, in intuitionistic logic, $\neg(p{:}p)$ and $p \not{:} p$, where $\not{:}$ is an intuitionistic connective "...does not say, that ...", should not be equivalent.[1] Hence it is necessary to introduce the next "non-colon" connective by additional axiom set.[2] Neither intuitionistic non-identity nor intuitionistic non-colon can be defined by means of implication (c.f. [3]). A base for these two dual to identity and colon connectives is coimplication defined by Rauszer (e.g. [4]). Thus, let us suppose that our classical propositional language with colon and non-colon connectives

[1] It is analogous situation to, for example, the difference between the intuitionistic S4-possibility given by an independent from necessity set of axioms and possibility defined as $\neg\Box\neg$, with $\Box$ - an intuitionistic S4-necessity.

[2] A similar problem was considered in [3] for intuitionistic identity and intuitionistic non-identity. The method used there for these two connectives will be repeated here for intuitionistic colon and non-colon.

is enlarged by coimplication, $\leftharpoondown$, and weak negation, $\sim$, given by the following axiom formulas

$\alpha \to (\beta \vee (\alpha \leftharpoondown \beta))$
$(\alpha \leftharpoondown \beta) \to {\sim}(\alpha \to \beta)$
$((\alpha \leftharpoondown \beta) \leftharpoondown \gamma) \to (\alpha \leftharpoondown (\beta \vee \gamma))$
$\neg(\alpha \leftharpoondown \beta) \to (\alpha \to \beta)$
$(\alpha \to (\beta \leftharpoondown \beta)) \to \neg\alpha$
$\neg\alpha \to (\alpha \to (\beta \leftharpoondown \beta))$
$((\beta \to \beta) \leftharpoondown \alpha) \to {\sim}\alpha$
${\sim}\alpha \to ((\beta \to \beta) \leftharpoondown \alpha)$

An additional inference rule is $\alpha \vdash \neg{\sim}\alpha$. Still following [4], we will say that our both intentional connectives are defined on the Heyting-Brouwer logic (HB-logic). Thus, assume that our CNC-language (language with colon and non-colon) is an algebra

$$\mathcal{L}_{\mathrm{CNC}} = (\mathrm{L}_{\mathrm{CNC}}, \neg, \sim, \wedge, \vee, \to, \leftrightarrow, \leftharpoondown, \rightleftharpoons, :, \not:).$$

Analogously to the axiomatization of non-identity connective ([3]), our new non-colon connective $\not:$ is given by the following axioms

($\mathrm{B}_1$) $\neg(\alpha \not: \alpha)$.
($\mathrm{B}_2$) $(\alpha \not: \gamma) \to ((\alpha \not: \beta) \vee (\beta \not: \gamma))$,
($\mathrm{B}_3$) $\neg((\alpha \wedge \beta) \not: \alpha)$,
($\mathrm{B}_4$) $\neg((\alpha \wedge \beta) \not: (\beta \wedge \alpha))$,
($\mathrm{B}_5$) $\neg(\alpha \not: (\alpha \wedge \alpha))$,
($\mathrm{B}_6$) $(({\sim}\alpha \not: {\sim}\beta) \vee ({\sim}\beta \not: {\sim}\alpha)) \to ((\alpha \not: \beta) \vee (\beta \not: \alpha))$,
($\mathrm{B}_7$) $((\alpha \S \gamma) \not: (\beta \S \delta)) \to ((\alpha \not: \beta) \vee (\gamma \not: \delta))$ for $\S \in \{\wedge, \vee\}$,
($\mathrm{B}_8$) $(((\alpha \S \gamma) \not: (\beta \S \delta)) \vee ((\beta \S \delta) \not: (\alpha \S \gamma))) \to ((\alpha \not: \beta) \vee (\beta \not: \alpha) \vee (\gamma \not: \delta) \vee (\delta \not: \gamma))$
for $\S \in \{\leftharpoondown, \rightleftharpoons, \not:\}$,
($\mathrm{B}_9$) $(\alpha \leftharpoondown \beta) \to (\alpha \not: \beta)$.

A definition of coequivalence is as follows, $\alpha \rightleftharpoons \beta = (\alpha \leftharpoondown \beta) \vee (\beta \leftharpoondown \alpha)$ for any $\alpha, \beta \in \mathrm{L}_{\mathrm{CNC}}$. We shall not provide any further formal analysis of the meaning of these formulas considering them as simple ones.[3] The

[3]The interpretation of axiom formulas for non-colon is especially easy in the light of two facts: 1. $\alpha \to \beta$ is a HB-tautology, if and only if $\alpha \leftharpoondown \beta$ is a HB-contratautology and 2. if $\alpha \not: \beta$ is designated in the interpretation of the logic of falsehood, it means that $\alpha$ says that $\beta$.

only axioms which need some comments are ($B_8$) and partially ($B_6$). Even when the colon connective is defined on the HB-logic, ($A_3$) remains in the same form, it means that § can be neither coimplication, coequivalence nor non-colon. Indeed, any extention of the formula ($A_3$) for the cases of $\S \in \{\leftharpoondown, \rightleftharpoons, \not\because\}$ violates even the most basic intuitions. The same, ($A_6$) cannot be repeated for the weak negation. A similar situation is connected with axiom formulas for non-colon. Thus, the non-colon as well as non-identity can and must be invariant only with respect to all "Brouwerian" connectives (including them, of course), i.e. the connectives which can be defined on the base of coimplication thus, identity and colon are invariant only with respect to all connectives defined by implication, which are called "Heyting" connectives in this paper.

Because of such kind of separation between two groups of connectives, a semantics for logic with non-identity as well as for logic with colon and coimplication needs as a base some algebraic construction which would code this fact. In [3] there is presented such semantics for Heyting-Brouwer logic with identity and non-identity. Now again we can use that construction. Informally speaking, each model will be in the form of two cones. One of them can be understood as a possible space of the past. In a similar way, the other one will be connected with the future. Thus, a "past" interpetation will be typical for Brouwerian connectives (e.g. coimplication, coequivalence, weak negation, non-colon), while Heyting connectives (implication, equivalence, negation, colon) will be interpreted from the point of view of the future.

For formal presentation let us assume an algebra

$$\mathcal{A} = (A, \neg, \sim, \cap, \cup, \rightarrow, \leftharpoondown, \leftrightarrow, \rightleftharpoons, \rhd, \not\rhd)$$

similar to the CNC-language.

For every D, a nonempty subset of A, let $\simeq$ be the relation of the first kind i.e. a congruence of a matrix $\langle(A, \neg, \cap, \cup, \rightarrow, \leftrightarrow, \rhd). D\rangle$ and $\approx$ be the relation of the second kind i.e. a congruence of $\langle(A, \sim, \cap, \cup, \leftharpoondown, \rightleftharpoons, \not\rhd)\ D\rangle$.

Now we can consider two kinds of matrices

$$\mathbf{M}_1 = \{\langle \mathcal{A}/_{\simeq}, D/_{\simeq}\rangle : \ D \subset A \text{ and } \simeq \text{ is the relation of the first kind}\}$$

and

$\mathbf{M}_2 = \{\langle \mathcal{A}/_{\approx}, \mathrm{D}/_{\approx}\rangle :\ \mathrm{D}\subset\mathrm{A}$ and $\approx$ is the relation of the second kind$\}$.

A matrix of each kind can be rewriten as

$$\mathbf{M}_1 = \{\langle \mathcal{A}_x, \mathrm{D}_x\rangle :\ x\in\mathrm{X}\}, \qquad \mathbf{M}_2 = \{\langle \mathcal{A}_y, \mathrm{D}_y\rangle :\ y\in\mathrm{Y}\}$$

where X and Y are sets of indexes of elements of $\mathbf{M}_1$ and $\mathbf{M}_2$, respectively. The relations $\leq_{\mathrm{X}}$ and $\leq_{\mathrm{Y}}$ defined as
- for any $x_1, x_2\in\mathrm{X}$ $x_1\leq_{\mathrm{X}}x_2$ iff $\mathrm{D}_1\subseteq\mathrm{D}_2$, where $\mathrm{D}_{x_i}=\mathrm{D}_i/_{\simeq_i}$ for $i\in\{1,2\}$,

- for any $y_1, y_2\in\mathrm{Y}$ $y_1\leq_{\mathrm{Y}}y_2$ iff $\mathrm{D}_1\subseteq\mathrm{D}_2$, where $\mathrm{D}_{y_i}=\mathrm{D}_i/_{\approx_i}$ for $i\in\{1,2\}$
partially order sets X and Y, respectively.

Finally, we can define a HBcnc-model, i.e. a model for HBcnc-logic (HB-logic with colon and non-colon given by $(\mathrm{A}_1)$-$(\mathrm{A}_9)$ and $(\mathrm{B}_1)$-$(\mathrm{B}_9)$) as such subset $\mathcal{M}$ of an element of $\mathbf{M}_1\cup\mathbf{M}_2$ that for any $x\in\mathrm{X}$, $y\in\mathrm{Y}$ and $a,b\in\mathrm{A}$ $[a]_z\cap_z[b]_z=[b]_z\cap_z[a]_z$, $[a]_z\cap_z[a]_z=[a]_z$ and

(0) *if* $[a]_z\in\mathrm{D}_z$, *then* $\forall t\geq_{\mathrm{Z}}z\ [a]_t\in\mathrm{D}_t$;
(1) $\neg_x[a]_x\in\mathrm{D}_x$ *iff* $\forall t\geq_{\mathrm{X}}x\ [a]_t\notin\mathrm{D}_t$;
(2) $\sim_y[a]_y\in\mathrm{D}_y$ *iff* $\exists t\leq_{\mathrm{Y}}y\ [a]_t\notin\mathrm{D}_t$;
(3) $[a]_z\cap_z[b]_z\in\mathrm{D}_z$ *iff* $[a]_z\in\mathrm{D}_z$ *and* $[b]_z\in\mathrm{D}_z$;
(4) $[a]_z\cup_z[b]_z\in\mathrm{D}_z$ *iff* $[a]_z\in\mathrm{D}_z$ *or* $[b]_z\in\mathrm{D}_z$;
(5) $[a]_x\rightarrow_x[b]_x\in\mathrm{D}_x$ *iff* $\forall t\geq_{\mathrm{X}}x$ $([a]_t\notin\mathrm{D}_t$ *or* $[b]_t\in\mathrm{D}_t)$;
(6) $[a]_y\leftarrow_y[b]_y\in\mathrm{D}_y$ *iff* $\exists t\leq_{\mathrm{Y}}y$ $([a]_t\in\mathrm{D}_t$ *and* $[b]_t\notin\mathrm{D}_t)$;
(7) $[a]_x\rhd_x[b]_x\in\mathrm{D}_x$ *iff* $\forall t\geq_{\mathrm{X}}x\ \exists c\in\mathrm{A}_t\ [a]_t=[b]_t\cap c$;
(8) $[a]_y\not\rhd_y[b]_y\in\mathrm{D}_y$ *iff* $\exists t\leq_{\mathrm{Y}}y\ \forall c\in\mathrm{A}_t\ [a]_t\neq[b]_t\cap c$,

with $(z,\mathrm{Z}) = (x,\mathrm{X}),(y,\mathrm{Y})$.

The correctness of such construction bases on the fact that for any $a\in\mathrm{A}$ and for any $x\in\mathrm{X}$ ($y\in\mathrm{Y}$) there exists $y\in\mathrm{Y}$ ($x\in\mathrm{X}$) such that $[a]_x\in\mathrm{D}_x$ is equivalent to $[a]_y\in\mathrm{D}_y$.

A matrix-consequence is defined as follows: for any $\alpha\in\mathrm{L}_{\mathrm{CNC}}$ and for any $\mathrm{B}\subseteq\mathrm{L}_{\mathrm{CNC}}$, $\alpha\in\mathrm{C}_{\mathcal{M}}(\mathrm{B})$ if and only if for any $\mathrm{h}\in\mathrm{Hom}(\mathcal{L}_{\mathrm{CNC}},\mathcal{A})$ for any $z\in\mathrm{Z}$, $\mathrm{k}_z\mathrm{oh}(\alpha)\in\mathrm{D}_z$ provided $\mathrm{k}_z\mathrm{oh}(\beta)\in\mathrm{D}_z$ for all $\beta\in\mathrm{B}$; with $(z,\mathrm{Z})=(x,\mathrm{X})$, $(y,\mathrm{Y})$ and $\mathrm{k}_z$ - a canonical homomorphism $\mathcal{A}$ onto $\mathcal{A}_z$.
Now we can formulate a completeness theorem:

Let $\alpha \in L_{CNC}$, $B \subseteq L_{CNC}$. The following conditions are equivalent
(a) $\alpha \in C_{HBcnc}(B)$,
(b) $\alpha \in C_{\mathcal{M}}(B)$ for any HBcnc-model $\mathcal{M}$.

Similarly to the case of the colon connective, a *soundness* of the above theorem above bases on the same fact as in the classical case and, moreover, on the fact that $k_z oh((\alpha \not{:} \beta) \vee (\beta \not{:} \alpha)) \notin D_z$ implies $k_t oh(\alpha) = k_t oh(\beta)$ for any $t \leq z$. Indeed, assume that $k_z oh((\alpha \not{:} \beta) \vee (\beta \not{:} \alpha)) \notin D_z$, it means that $k_z oh(\alpha \not{:} \beta) \notin D_z$ and $k_z oh(\beta \not{:} \alpha) \notin D_z$. Thus for any $t \leq z$, $k_z oh(\alpha) = k_z oh(\beta) \cap c$ and $k_z oh(\beta) = k_z oh(\alpha) \cap d$ for some $c, d \in A_t$. Repeating an appropriate procedure as for the case of colon connective, finally we arrive at $k_t oh(\alpha) = k_t oh(\beta)$ for any $t \leq z$.

For the *completeness* let us construct a model using prime HBcnc-theories and their complementations. Obviously, T is a prime HBcnc-theory if and only if $\bar{T}$ is a prime $\bar{H}$Bcnc-theory (c.f. [4]).[4] From this moment "T" will be a symbol only for prime HBcnc-theory as well as "$\bar{T}$" for prime $\bar{H}$Bcnc-theory being a complementation of T. Relations of the first and second kind are defined, respectively, as follows

$$\alpha \simeq \beta \text{ iff } \alpha \leftrightarrow \beta \in T \qquad\qquad \alpha \approx \beta \text{ iff } \alpha \rightleftharpoons \beta \notin T.$$

The fact that both relations are appropriate partial congruences follows from $(A_6)$-$(A_9)$ and $(B_6)$-$(B_9)$. Now, let us construct two kinds of matrices

$$M_1 = \langle \mathcal{L}_{CNC} /_{\simeq}, T /_{\simeq} \rangle, \qquad M_2 = \langle \mathcal{L}_{CNC} /_{\approx}, T /_{\approx} \rangle.$$

Indexes X and Y as well as partial orders $\leq_X$ and $\leq_Y$ are introduced in the same way like for quotient algebras considered above. A verification that $M_1 \cup M_2$ satisfies the conditions (0)-(6) of the HBcnc-model is standard. (7) is checked similarly to the classical case. We will show the last condition, only. Assume that $[\alpha]_{\approx} \not{:}_{\approx} [\beta]_{\approx} \notin T_{\approx}$, i.e. $(\alpha \not{:} \beta) \in \bar{T}$. Because $\alpha \rightarrow \beta$ is a HBcnc-tautology if and only if $\alpha \leftharpoondown \beta$ is a $\bar{H}$Bcnc-tautology (see [4]), using Modus Tollens - an inference rule for $\bar{H}$Bcnc-logic $(\alpha \leftharpoondown \beta, \beta \vdash \alpha)$, by $(B_1)$ and $(B_7)$ $(\alpha \wedge \alpha) \not{:} (\alpha \wedge \beta) \in \bar{T}$. From $(B_2)$ and $(B_5)$, $\alpha \not{:} (\alpha \wedge \beta) \in \bar{T}$ and because of $(B_3)$, $[\alpha]_{\approx} = [\alpha]_{\approx} \cap [\beta]_{\approx}$. A proof

[4]According to the notation from [4], "bar" over some symbol means an object from the logic of falsehood dual to HBcnc-logic.

of this implication completes fact that $(\alpha \not: \beta) \in \bar{\mathrm{T}}'$ for any $\bar{\mathrm{T}}' \supseteq \bar{\mathrm{T}}$. Now, let for any $\mathrm{T}' \supseteq \mathrm{T}$ there exists $\gamma$ such that $[\alpha]_{\approx} = [\beta]_{\approx} \cap [\gamma]_{\approx}$. It means that $\alpha \not: (\beta \wedge \gamma) \in \bar{\mathrm{T}}$ for some $\gamma$. Finally, by ($\mathrm{B}_2$) and ($\mathrm{B}_3$), $(\alpha \not: \beta) \notin \mathrm{T}$. □

## 3 Modal associations

In [3] there are settled some relations between connectives formed in two pairs: identity with S4-necessity and non-identity with S4-possibility - all of them are intuitionistic. However, on the base of logic with colon and non-colon, it is possible to establish similar relations between these new connectives and the both S4-modalities. Thus, the "necessary" part of intuitionistic S4 can be embedded into intuitionistic logic with colon. Indeed, an easy verification shows that basing on the formulas ($\mathrm{A}_1$)-($\mathrm{A}_9$), by the translation: for any $\alpha \in \mathrm{L_C}$

$$\Box\alpha = \top{:}\alpha$$

for some tautology $\top$, there can be derived all axioms for S4-necessity

($\Box_1$) $\Box\alpha \to \alpha$
($\Box_2$) $\Box\alpha \to \Box\Box\alpha$
($\Box_3$) $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$

as well as the rule of necessity. Opposite definability, however, does not hold. Only appropriate extention of intuitionism with colon is equivalent to "necessary" part of intuitionistic S4. If C is an intuitionistic consequence operation, then desired extention is of the form

$$\mathrm{C_C}(\{\alpha{:}\beta \mid \alpha \to \beta \in \mathrm{C}(\emptyset)\}).$$

The second lacking translation for proving the mentioned equivalence is

$$\alpha{:}\beta = \Box(\alpha \to \beta)$$

for any $\alpha, \beta \in \mathrm{L_C}$.

Analogous relations between non-colon and S4-possibility given by

$(\Diamond_1)\quad \alpha \rightarrow \Diamond\alpha$
$(\Diamond_2)\quad \Diamond\Diamond\alpha \rightarrow \alpha$
$(\Diamond_3)\quad (\Diamond\alpha \leftharpoondown \Diamond\beta) \rightarrow \Diamond(\alpha \leftharpoondown \beta)$

and the rule of possibility (i.e. $\neg\alpha \vdash \neg\Diamond\alpha$), are settled by two translations

$$\alpha \not{:} \beta = \Diamond(\alpha \leftharpoondown \beta), \qquad\qquad \Diamond\alpha = \alpha \not{:} \bot, \text{ for any } \bot.$$

Similarly to the previous case, a "possibility" part of intuitionistic S4 can be embedded into the intuitionistic logic with non-colon, by the second given above translation. An axiomatic extention of $C_{\mathrm{HBnc}}$ (a HB-logic with non-colon) is as follows

$$C_{\mathrm{HBnc}}(\{\neg(\alpha \not{:} \beta) \mid \neg(\alpha \leftharpoondown \beta) \in C_{\mathrm{HB}}(\emptyset)\}).$$

As for idenitity, even replacement of $(\Diamond_3)$ by formula $(\Diamond\alpha \vee \Diamond\beta) \rightarrow \Diamond(\alpha \vee \beta)$[5] does not yield the equivalence between intuitionistic colon and intuitionistic S4-possibility. On the base of intuitionistic logic (also, on the HB-logic), a Brouwerian connective cannot be defined by Heyting one.

Of course, on the base of classical logic, every couple of translations can be mutually defined by one other.

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