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ON BLOCH FUNCTIONS AND THE CONTRACTION OF TEICHMÜLLER METRICS

HUANG XINZHONG AND SHIGEYOSHI OWA

ABSTRACT. In this note, we consider the properties of Bloch functions determined by Beltrami coefficient. A sufficient condition for extremal quasiconformal mapping with nonexistence of degenerating sequence is obtained. As a result, we consider the contraction or preserved of Teichmüller metrics for the related Beltrami lines under the projection mapping π.

1. INTRODUCTION

Let Q_I be the class of quasiconformal mappings f of the unit disk $D = \{z||z| \leq 1\}$ onto itself with $f(0) = f(1) - 1 = 0$, μ_f be the complex dilatation of f, $k_f = \|\mu_f\|_\infty = \text{esssup}_{z\in D}|\mu_f|$, $k_0(f) = \inf_g k_g$, where $g \in Q_I$ with $g|_{\partial D} = f|_{\partial D}$. We say that $f(z)$ is extremal if $k_f = k_0(f)$, and the corresponding μ_f is called extremal.

We know that the universal Teichmüller space $T(1)$ can be represented as a quotient space of QS by the Möbius group $PSL(2, R)$, where QS is the group of all quasi-symmetric homeomorphisms of a circle, and the Teichmüller distance $d([f], [g])$, from a point $[g]$ to another point $[f]$ in $T(1)$, is equal to

$$d([f],[g]) = \frac{1}{2}\log\frac{1+k_0(g\circ f^{-1})}{1-k_0(g\circ f^{-1})}. \tag{1.1}$$

QS contains another topological subgroup, which is much larger than $PSL(2, R)$, the subgroup S of symmetric homeomorphisms. Gardiner-Sullivan [1] showed that $QSmodS$ also has a natural complex Banach manifold structure and a natural quotient metric $\bar{d}$, called the Teichmüller metric in $QSmodS$. Let $\bar{k}_f = \inf_U \text{esssup}_{z\in U}|\mu_f(z)|$, where U moves all neighborhoods of ∂D in D, $\bar{k}_f$ is called the boundary dilatation of f. Set $\bar{k}_0(f) = \inf_g \bar{k}_g$, where g moves all quasiconformal mappings of D with the same boundary values as f. If $\bar{k}_0(f) = \bar{k}_f$, then $f(z)$ is called extremal in $QSmodS$. The distance between two points $\pi[f]$ and $\pi[g]$ in $QSmodS$ is equal to

$$\bar{d}(\pi[f],\pi[g]) = \frac{1}{2}\log\frac{1+\bar{k}_0(g\circ f^{-1})}{1-\bar{k}_0(g\circ f^{-1})}. \tag{1.2}$$

Suppose $\mu(z)$ is a given Beltrami coefficient, we consider the Beltrami line $C_\mu = \{[f^t]| -1 \leq t \leq 1\}$ or $\pi C_\mu = \{\pi[f^t]| -1 \leq t \leq 1\}$, where $\mu_{f^t} = t\frac{\mu}{\|\mu\|_\infty}$. If μ is

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extremal in $T(1)$ or in $QSmodS$, then the natural mapping $t \mapsto t\frac{\mu}{\|\mu\|_\infty}$ from the open interval $(-1,1)$ with the Poincaré metric onto C_μ or πC_μ with the Teichmüller metric is an isometry. Whether μ is extremal or not, such mapping is weakly contracting. The following problem is very interesting and considered by many authors(cf. [2],[3]):

For which points $[f] \in T(1)$, does the Teichmüller distance from 0 to $[f]$ in QS strictly greater than the distance from 0 to $\pi[f]$ in $QSmodS$?

In this note, we will investigate some properties for Bloch functions determined by μ and partially solve the above problem.

2. Main results and their proofs

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in D, $f(z)$ is called a Bloch function if

$$\|f\|_B = \sup_{z\in D}(1-|z|^2)|f'(z)| < \infty. \tag{2.1}$$

The Bloch functions will be denoted by B. B_0 will be the subset of B with

$$\|f\|_{B_0} = \lim_{|z|\to 1} \sup(1-|z|^2)|f'(z)| = 0. \tag{2.2}$$

$A(D) = \{f(z)|\text{f(z) is analytic in D}, \|f(z)\|_1 = \frac{1}{\pi}\iint_D |f(z)|\,dxdy < \infty\}$. The quasiconformal mapping f from D onto itself is called a Teichmüller mapping of finite type, if $\mu_f = \|\mu(z)\|_\infty \frac{\bar\varphi_0}{|\varphi_0|}$, $\varphi_0 \in A(D)$. From Reich's example(cf.[4]), we know that even the point $[f]$ corresponds to a Teichmüller mapping of finite type, the distance from 0 to $[f]$ under the projection π may not contract. However, if $[f] \in T(1)$, and $\bar d(0,\pi[f]) < d(0,[f])$, then $[f]$ contains a Teichmüller mapping of finite type. This makes the above problem more complicated.

Suppose $\kappa(z) \in L^\infty(D)$, the space of complex-valued bounded measurable functions in D with $\|\kappa\|_\infty = \operatorname{esssup}_{z\in D}|\kappa(z)|$, we consider a linear functional L_κ on $A(D)$

$$L_\kappa(f) = \frac{1}{\pi}\iint_D \kappa(z)f(z)\,dxdy, \qquad f(z)\in A(D), \tag{2.3}$$

then

$$\|L_\kappa\| \le \|\kappa\|_\infty. \tag{2.4}$$

Hamilton, Reich and Streble [5, 6] showed that

Theorem A. *A Beltrami coefficient μ is extremal if and only if one of the following statements holds:*

1) There exist $\varphi \in A(D)$ and $k \in [0,1)$ such that $\mu = k\bar\varphi/|\varphi|$ for almost everywhere on D.

2) There is a degeneration sequence $\{\varphi_n\} \in A(D)$, $\|\varphi_n\|_1 = 1$, converging to 0 locally uniformly in D, such that

(2.5) $$\lim_{n\to\infty} \left| \iint_D \varphi_n \mu \, dxdy \right| = \|\mu\|_\infty.$$

For a given Beltrami coefficient $\mu(z)$, let

(2.6) $$b_n = \frac{n+2}{\pi} \iint_D \mu(z) z^n \, dxdy, \qquad g(\zeta) = \sum_{n=0}^{\infty} b_n \zeta^n,$$

it is clearly that $|b_n| \leq 2\|\mu(z)\|_\infty$ and $g(\zeta)$ is analytic in D. We call that the analytic function $g(\zeta)$ is determined by $\mu(z)$.

Let $G(\zeta) = \zeta g(\zeta)$, Anderson proved in [7] the following

Theorem B. *For a given $\mu(z) \in L^\infty(D)$, then*

(2.7) $$\|L_\mu\| \leq \|G(\zeta)\|_B \leq 4\|L_\mu\|,$$

where $G'(\zeta) = \frac{2}{\pi} \iint_D \frac{\mu(z)}{(1-\zeta z)^3} \, dxdy$.

Theorem C. *If $\mu(z)$ possesses a degenerating sequence, then*

(2.8) $$\|L_\mu\| \leq \lim_{|z|\to 1} \sup (1 - |z|^2)|G'(z)|,$$

where $G'(\zeta) = \frac{2}{\pi} \iint_D \frac{\mu(z)}{(1-\zeta z)^3} \, dxdy$. In particular, if

(2.9) $$\iint_D \frac{\mu(z)}{(1-\zeta z)^3} \, dxdy = o(1 - |\zeta|^2)^{-1} \qquad (|\zeta| \to 1^-),$$

then $\mu(z) = \|\mu\|_\infty \frac{\bar{\varphi}_0(z)}{|\varphi_0(z)|}$, $\varphi_0 \in A(D)$, for almost all $z \in D$.

Theorem C means that if $\mu(z)$ is extremal and $\lim_{|z|\to 1} \sup (1 - |z|^2)|G'(z)| = 0$, then

$$\mu(z) = \|\mu\|_\infty \bar{\varphi}_0 / |\varphi_0|, \qquad \varphi_0(z) \in A(D),$$

for almost everywhere $z \in D$. For an extremal quasiconformal mapping $f^{\mu(z)} \in Q_I$, in what case, is it a finite type Teichmüller mapping or even has it no degenerating sequence? This problem is very interesting itself(cf. [8, 9] and the references cited there). First, we will prove the following

Theorem 1. *Suppose $\mu(z)$ is extremal, let $g(z)$ be defined in (2.6), if there exists a ρ_0, $0 < \rho_0 < 1$, such that*

(2.10) $$\sup_{\rho_0 < |z| < 1} (1 - |z|^2)|g'(z)| < 1,$$

then there exists a $\varphi_0 \in A(D)$ with $\mu(z) = \|\mu(z)\|_\infty \frac{\bar{\varphi}_0}{|\varphi_0|}$ for almost all $z \in D$. In particular, $\mu(z)$ possesses no degenerating sequence.

The proof of Theorem 1. If $\mu(z)$ is an extremal Beltrami coefficient, let $g(\zeta)$ be defined in (2.6), if $f(z) = \sum_{n=0}^{\infty} a_n z^n \in A(D)$, $0 < \rho < 1$, we have

$$L_\mu(f(\rho z)) = \sum_{n=0}^{\infty} a_n \rho^n L_\mu(z^n) = \sum_{n=0}^{\infty} \frac{a_n b_n}{n+2} \rho^n.$$

Since $\|f(\rho z)-f(z)\|_1 \to 0$, when $\rho \to 1^-$, then we have

$$L_\mu(f) = \lim_{\rho\to 1^-} \sum_{n=0}^{\infty} \frac{a_n b_n}{n+2}\rho^n.$$

On the other hand, if $G(\zeta) = \zeta g(\zeta)$, then

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})G'(\zeta re^{-i\theta})\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} (\sum_{n=0}^{\infty} a_n r^n e^{in\theta})(\sum_{n=0}^{\infty}(n+1)b_n\zeta^n r^n e^{-in\theta})\,d\theta \\
&= \sum_{n=0}^{\infty}(n+1)a_n b_n \zeta^n r^{2n}.
\end{aligned}$$

Thus, we have

$$\lim_{\rho\to 1^-}\sum_{n=0}^{\infty}\frac{a_n b_n}{n+2}\rho^n = \frac{1}{\pi}\int_0^1\int_0^{2\pi} f(re^{i\theta})G'(\zeta re^{-i\theta})(1-r^2)r,\,dr\,d\theta, \tag{2.11}$$

for any $f(z)\in A(D)$. Since

$$\begin{aligned}
g(\zeta) = \sum_{n=0}^{\infty} b_n\zeta^n &= \sum_{n=0}^{\infty}(\frac{n+2}{\pi}\iint_D z^n\mu(z)\,dxdy)\zeta^n \\
&= \frac{1}{\pi}\iint_D(\sum_{n=0}^{\infty}(n+2)z^n\zeta^n\mu(z))\,dxdy \\
&= \frac{1}{\pi}\iint_D[\frac{2-z\zeta}{(1-z\zeta)^2}]\mu(z)\,dxdy,
\end{aligned}$$

then,

$$|g(\zeta)| \le \frac{3\|\mu\|_\infty}{\pi|\zeta|}\log\frac{1+|\zeta|}{1-|\zeta|} = o((1-|\zeta|^2)^{-1}), \qquad |\zeta|\to 1^-. \tag{2.11}$$

If $\{f_n(z)\}$ is a degenerating sequence for $\mu(z)$ with $\|f_n\|_1 = 1$, by Theorem B and (2.11), we can choose a ρ' with $\rho_0 < \rho' < 1$ such that

$$\begin{aligned}
|L_\mu(f_n)| \le \frac{4\|\mu\|_\infty}{\pi}\iint_{|z|\le\rho'} |f_n(re^{i\theta})|r\,drd\theta + \sup_{\rho'<|z|<1}(1-|z|^2)|g(z)| \\
+ \sup_{\rho'<|z|<1}(1-|z|^2)|g'(z)| < 1, \quad \text{for } n\to\infty,
\end{aligned}$$

which contradicts that $\{f_n(z)\}$ is a degenerating sequence. By Theorem A, Theorem 1 is proved.

The following example 1 shows that there is non-extremal Beltrami coefficient $\mu(z)$ with the bound $\sup_{\rho_0<|z|<1}(1-|z|^2)|g'(z)| = \frac{2}{\pi}$.

Example 1. Set Beltrami coefficient

$$\mu(z) = \begin{cases} 1, & \text{for } \Im z \geq 0, |z| < 1 \\ 0, & \text{for } \Im z < 0, |z| < 1. \end{cases}$$

Then by [8, Theorem 1], we see that $\mu(z)$ is not extremal. In this case, by calculation, we have

$$g'(z) = 2 + \frac{2i}{\pi}[z + \frac{1}{3}z^3 + \cdots + \frac{1}{2n-1}z^{2n-1} + \cdots]$$

and $\lim_{|z|\to 1}(1-|z|^2)|g'(z)| = \frac{2}{\pi}$.

Next we will investigate the relationship between extremal Beltrami coefficient μ and the coefficients of $g(z)$ defined in (2.6).

From [11] and Theorem 1, we know that if $\mu(z)$ is extremal and the determained analytic function $g(z) \in B_0$, then $\lim_{n\to\infty} |b_n| = 0$. However, we also know that even if $f(z) \in B$ and $\lim_{n\to\infty} |b_n| = 0$, one can not derive that $f(z) \in B_0$. From this we will prove the following

Corollary 1. *Suppose $\mu(z)$ is extremal, and let $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be defined in (2.6), if there exist a positive number N_0 and l, $0 < l < \frac{1}{2}$, such that*

$$|b_n| < \frac{l}{n}, \quad \textit{holds for } n > N_0,$$

then there exists a $\varphi_0(z) \in A(D)$ with

$$\mu(z) = \|\mu\|_\infty \bar{\varphi}_0/|\varphi_0|, \quad \textit{for almost all } z \in D.$$

The proof of Collary 1. If $\mu(z)$ is extremal, and let $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be defined in (2.6), we have

$$\begin{aligned} |g'(z)| &\leq |\sum_{n=0}^{N_0} n b_n z^n| + \sum_{n=N_0+1}^{\infty} l|z^n| \\ &= |\sum_{n=0}^{N_0} n b_n z^n| + l\frac{|z|^{N_0+1}}{1-|z|}, \end{aligned}$$

thus there exists a $\rho_0 > 0$, such that $\sup_{\rho_0<|z|<1}(1-|z|^2)|g'(z)| < 1$, by Theorem 1, we obtain the assertion.

Let Π denote the subset of $T(1)$ consisting of elements of $[f]$ which correspond to Teichmüller mappings of finite type whose complex dilatations $\mu = \mu_f$ satisfy the following condition: There exists a ρ_0, $0 < \rho_0 < 1$, such that $\sup_{\rho_0<|\zeta|<1}(1 - |\zeta|^2)|g'(\zeta)| < 1$, where $g(\zeta)$ is defined in (2.6). We will prove the following

Theorem 2. *For $[f] \in \Pi$, then $\bar{d}(0, \pi([f])) < d(0, [f])$.*

In order to prove Theorem 2, we need the following Theorem D due to Gardiner [2].

Theorem D. *For every* $[f] \in T(1)$, *then* $\bar{k}_f = \bar{k}_0(f)$ *if and only if*

$$\sup_{\{\varphi_n\}} \limsup_{n\to\infty} |Re \iint_D \varphi_n \mu_f \, dxdy| = \bar{k}_f,$$

where the supremum is taken over all degenerating sequences $\{\varphi_n\}$ *for* μ_f *with* $\|\varphi_n\|_1 = 1$ *in* $A(D)$.

The proof of Theorem 2. We use the same way as in [3] to prove Theorem 2. If $[f] \in \Pi$, then we conclude that $\bar{k}_0(f) = k_0(f)$. On the contrary, by Theorem D, we can find a degenerating sequence $\{\varphi_n\}$ with $\|\varphi_n\|_1 = 1$ such that

$$\lim_{n\to\infty} \text{Re} \iint_D \varphi_n \mu_f \, dxdy = \|\mu_f\|_\infty = k_0(f) = \bar{k}_0(f),$$

which is impossible by Theorem 1.

Thus we have $\bar{k}_0(f) < k_0(f)$, which is equivalent to $\bar{d}(0, \pi([f])) < d(0, [f])$.

On the other hand, comparing with Theorem 2, we will prove the following

Theorem 3. *Suppose* $[f] \in T(1)$, *and* $b_n = \frac{n+2}{\pi} \iint_D \mu_f z^n \, dxdy$, *if* $\overline{\lim}_{n\to\infty} b_n = 2\|\mu_f\|_\infty$, *then* $\bar{d}(0, \pi([f])) = d(0, [f])$. *The constant 2 is the best.*

The proof of Theorem 3. First, from Fehlmann and Sakan's paper in [10], we know that the subset of $T(1)$ satisfying the conditions in Theorem 3 is not empty, and by the example of Fehlmann and Sakan made in [10], there exists an extremal Beltrami coefficient μ such that the coefficients of $g(z)$ satisfy $\overline{\lim}_{n\to\infty} b_n = 2\|\mu\|_\infty$, thus the constant 2 is the best. Now, if $\overline{\lim}_{n\to\infty} b_n = 2\|\mu_f\|_\infty$, then we have $\lim_{j\to\infty} b_{n_j} = 2\|\mu_f\|_\infty$, and the sequence $\{\varphi_{n_j}(z) = \frac{n_j+2}{2} z^{n_j}\}$ is a degenerating sequence for the Beltrami coefficient μ_f, with $\|\varphi_{n_j}\|_1 = 1$, by Theorem D, we conclude that $\bar{k}_0(f) = k_0(f)$, thus $d(0, \pi([f])) = d(0, [f])$.

To consider the contraction of Teichmüller metrics, we need the following Principle of Teichmüller contraction due to Gardiner [2].

Principle of Teichmüller contraction. *Assume* $\|\mu\| = 1$, $0 < k_1 < k_2 < 1$, *and* $d(0, [f^{k_1}]) \le \lambda_1 d_p(0, k_1)$ *or* $\bar{d}(0, \pi([f^{k_1}])) \le \lambda_1 d_p(0, k_1)$ *with some* $\lambda_1 < 1$, *where and in the sequel,* f^k *is the quasiconformal mapping of* D *on to itself such that* $\mu_f = k\mu$ *for every positive* $k < 1$. *Then there exists a* $\lambda_2 < 1$ *depending only on* k_1, k_2, *and* λ_1 *such that*

$$d(0, [f^k]) \le \lambda_2 d_p(0, k) \qquad \text{or} \qquad \bar{d}(0, \pi([f^k])) \le \lambda_2 d_p(0, k)$$

respectively, for all k *with* $0 \le k \le k_2$.

Using Theorem 2 and the Principle of Teichmüller contraction, we can obtain the following

Corollary 2. *Under the same circumstance as in Theorem 2, let* $k = \|\mu_f\|_\infty$ *and* $\lambda = \bar{d}(0, \pi([f]))/d(0,[f])$. *Fix* $k' < 1$ *and let* f^t *be the quasiconformal mapping of* D *onto itself such that* $\mu_{f^t} = (t/k)\mu_f$ *for every* $t \in [0, k')$. *Then there exists* $\lambda' < 1$ *depending only on* k, k', *and* λ *such that*

$$\bar{d}(0, \pi([f^t])) \leq \lambda' d_p(0, t),$$

for every t *with* $0 \leq t \leq k'$, *where* d_p *denotes the Poincaré metric on* D.

The proof of Corollary 2. By Theorem 2, we have $d(0,[f]) = d_p(0,k)$ and $\lambda = \bar{d}(0, \pi[f])/d(0,[f]) < 1$, using the principle of Teichmüller contraction, the Corollary 2 is obtained.

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Department of Mathematics, Huaqiao University, Quanzhou, Fujian 362011, China
Department of Mathematics, Kinki University, Higashi-Osaka, Osaka 577, Japan