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On meromorphic α-starlike functions

by

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Abstract

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $E = \{z : |z| < 1\}$, let for a real number α

$$\mathrm{Re}\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > 0 \qquad \text{in} \quad E.$$

Then it is well known that [1, 2]

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \qquad \text{in} \quad E.$$

Corresponding to this, we take the analytic function $f(z) = 1/z + \sum_{n=0}^{\infty} a_n z^n$ in the punctured disk $U = \{z : 0 < |z| < 1\}$ satisfying

$$\mathrm{Re}\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] < 0 \qquad \text{in} \quad E.$$

Then we prove

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < 0 \qquad \text{in} \quad E.$$

1. Introduction.

Let Σ denote the class of function of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$$

which are analytic in the punctured disk $U = \{z : 0 < |z| < 1\}$.

A function $f(z)$ belonging to the class is said to be meromorphic starlike of order α $(0 \le \alpha < 1)$ in $E = \{z : |z| < 1\}$ if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < -\alpha$$

for all $z \in E$. We denote by $\Sigma^*(\alpha)$ the class of all functions in Σ which are meromorphic starlike of order α in U. We note also that

$$\Sigma^*(\alpha) \subseteq \Sigma^*(0) \equiv \Sigma^* \qquad (0 \le \alpha < 1),$$

where Σ^* denote the subclass of A consisting of functions which are meromorphic starlike in U. The meromorphic starlike is meant that the complement of $f(E)$ is starlike with respect to the origin.

Difinition 1. Let α be a real number and suppose that $f(z) \in \Sigma$ with $f(z)f'(z) \neq 0$ in U. If $f(z)$ satisfies the condition

$$\mathrm{Re}\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] < 0 \quad \text{in} \quad E,$$

then $f(z)$ is said to be a meromorphic α-starlike function.

2. Preliminary Results.

Lemma 1. *Let $p(z)$ be analytic in E, $p(0) = 1$ and suppose that there exists a point $z_0 \in E$ such that*

$$\mathrm{Re}\{p(z)\} > 0 \quad \text{for} \quad |z| < |z_0|,$$
$$\mathrm{Re}\{p(z_0)\} = 0 \quad \text{and} \quad p(z_0) = ia \quad (a \neq 0).$$

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik,$$

where

(1)
$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \geq 1 \quad \text{when} \quad a > 0$$

and

(2)
$$k \leq \frac{1}{2}\left(a + \frac{1}{a}\right) \leq -1 \quad \text{when} \quad a < 0.$$

We owe this lemma to [3, Theorem 1] .

Lemma 2. *Let α, β be positive real number $(\alpha > 1, 0 < \beta < 1)$ and $p(z)$ be analytic in E, $p(0) = 1$, $p(z) \neq \beta$ in E, and suppose that*

(i) *for the case $0 < \beta \leq 1/2$*

$$\mathrm{Re}\left(\alpha\frac{zp'(z)}{p(z)} - p(z)\right) > -\frac{\alpha\beta}{2(1-\beta)} - \beta \quad \text{in} \quad E,$$

where $\alpha > 2(1-\beta)^2/\beta$;

(ii) *for the case $1/2 < \beta < 1$*

$$\mathrm{Re}\left(\alpha\frac{zp'(z)}{p(z)} - p(z)\right) > -\frac{\alpha(1-\beta)}{2\beta} - \beta \quad \text{in} \quad E,$$

where $\alpha > 2\beta$.

Then we have

$$\mathrm{Re}\{p(z)\} > \beta \quad \text{in} \quad E.$$

Proof. If we put

$$q(z) = \frac{1-\beta}{p(z)-\beta},$$

then $q(z)$ is analytic in E, $q(0)=1$ and $q(z) \neq 0$ in E.

At first, we want to prove $\operatorname{Re}\{p(z)\} > \beta$ in E, *i.e.* $\operatorname{Re}\{q(z)\} > 0$ in E. If there exists a point $z_0 \in E$ such that

$$\operatorname{Re}\{q(z)\} > 0 \quad \text{for} \quad |z| < |z_0| < 1,$$

$$\operatorname{Re}\{q(z_0)\} = 0 \quad \text{and} \quad q(z_0) = ia \quad (a \neq 0),$$

then from Lemma 1, we have

$$\begin{aligned} \operatorname{Re}\left(\alpha\frac{z_0p'(z_0)}{p(z_0)} - p(z_0)\right) &= \operatorname{Re}\left(-\alpha\frac{1-\beta}{1-\beta+\beta ia}ik - \frac{1-\beta+\beta ia}{ia}\right) \\ &= -\frac{\alpha\beta ka(1-\beta)}{(1-\beta)^2+\beta^2a^2} - \beta \\ &\leq -\frac{\alpha\beta(1-\beta)}{2}\frac{1+a^2}{(1-\beta)^2+a^2\beta^2} - \beta \end{aligned}$$

by virtue of (1), (2). Let us put

$$\varphi(x) = \frac{1+x^2}{(1-\beta)^2+x^2\beta^2}$$

and simple calculation leads to

$$\varphi'(x) = \frac{2x(1-2\beta)}{((1-\beta)^2+x^2\beta^2)^2}.$$

For the case $0 < \beta \leq 1/2$, $\varphi(x)$ takes its minimum value at $x=0$

$$\varphi(0) = \frac{1}{(1-\beta)^2}.$$

Therefore we have

$$\operatorname{Re}\left(\alpha\frac{z_0p'(z_0)}{p(z_0)} - p(z_0)\right) \leq -\frac{\alpha\beta}{2(1-\beta)} - \beta.$$

Next, if $1/2 < \beta < 1$, $\varphi(x)$ takes its minimum at $x=\infty$

$$\lim_{x\to\infty}\varphi(x) = \lim_{x\to\infty}\frac{1+x^2}{(1-\beta)^2+x^2\beta^2} = \frac{1}{\beta^2},$$

and we have

$$\operatorname{Re}\left(\alpha\frac{z_0p'(z_0)}{p(z_0)} - p(z_0)\right) \leq -\frac{\alpha(1-\beta)}{2\beta} - \beta.$$

This contradicts the assumption of Lemma 2. Therefore we have $\operatorname{Re}\{q(z)\} > 0$ in E and then

$$\operatorname{Re}\{p(z)\} > \beta \quad \text{in} \quad E.$$

This completes our proof.

3. Main Results.

Theorem 1. Let $f(z)$ be a meromorphic α-starlike function, and suppose that

$$\operatorname{Re}\left[(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right]<0 \quad \text{in} \quad E, \tag{3}$$

where α is a real number. Then we have

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}<0 \quad \text{in} \quad E.$$

Proof. Let us put

$$p(z)=-\frac{zf'(z)}{f(z)}. \tag{4}$$

By simple calculation, we obtain

$$\frac{zp'(z)}{p(z)}-p(z)=1+\frac{zf''(z)}{f'(z)}, \tag{5}$$

or

$$\operatorname{Re}\left[(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right]=\operatorname{Re}\left[\alpha\frac{zp'(z)}{p(z)}-p(z)\right]. \tag{6}$$

At first, we want to prove $\operatorname{Re}\{zf'(z)/f(z)\}<0$ in E, which means $\operatorname{Re}\{p(z)\}>0$ in E. If there exists a point $z_0\in E$ such that

$$\operatorname{Re}\{p(z)\}>0 \quad \text{for} \quad |z|<|z_0|,$$

$$\operatorname{Re}\{p(z_0)\}=0 \quad \text{and} \quad p(z_0)=ia \quad (a\neq 0),$$

then from Lemma 1 we have

$$\frac{z_0p'(z_0)}{p(z_0)}=ik,$$

where k is real and $|k|\geq 1$. Thus

$$\operatorname{Re}\left[\alpha\frac{z_0p'(z_0)}{p(z_0)}-p(z_0)\right]=\operatorname{Re}\left[\alpha ik-ia\right]=0.$$

This contradicts the assumption of the theorem. Therefore we have

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}<0 \quad \text{in} \quad E.$$

This completes our proof.

Theorem 2. *Let α, β be positive real number* $(\alpha > 1, 0 < \beta < 1)$, $f(z)$ *be a meromorphic α-starlike function and suppose that*

(i) *for the case* $0 < \beta \leq 1/2$

$$\mathrm{Re}\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > -\frac{\alpha\beta}{2(1-\beta)} - \beta \quad \text{in} \quad E,$$

where $\alpha > 2(\beta - 1)^2/\beta$;

(ii) *for the case* $1/2 < \beta < 1$

$$\mathrm{Re}\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > -\frac{\alpha(1-\beta)}{2\beta} - \beta \quad \text{in} \quad E,$$

where $\alpha > 2\beta$.

Then we have

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < -\beta \quad \text{in} \quad E.$$

Proof. Applying (4), (5) and (6), we can easily prove the theorem. Therefore from the assumption of the theorem and Lemma 2, we have

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = \mathrm{Re}\{-p(z)\} < -\beta \quad \text{in} \quad E.$$

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