数理解析研究所講究録
1012 巻 1997 年 25-46

Distortion and Characterization Theorems for Starlike and Convex Functions Related to Generalized Fractional Calculus

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1. Introduction

Let $A(n)$ denote the class of functions of the form

$$f(z)=z+\sum_{k=n+1}^{\infty}a_kz^k \qquad (n\in\mathbb{N}=\{1,2,3,\cdots\}), \tag{1}$$

which are analytic in the unit disk $U=\{z:|z|<1\}$, and let $S(n)$ denote the subclass of $A(n)$ of *univalent functions* in U. Further, a function $f(z)$ belonging to $S(n)$ is said to be *starlike of order* δ $(0\leqq\delta<1)$ if and only if it satisfies the inequality

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\delta \qquad (z\in U) \tag{2}$$

and such a subclass is denoted by $S_\delta(n)$. Also, $f(z)\in S(n)$ is said to be *convex of order* δ $(0\leqq\delta<1)$ if and only if

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\delta \qquad (z\in U) \tag{3}$$

and the subclass by $K_\delta(n)$. We note that $f(z)\in K_\delta(n)$ if and only if $zf'(z)\in S_\delta(n)$, and also for any $0\leqq\delta<1$,

$$S_\delta(n)\subseteq S_0(n), \qquad K_\delta(n)\subseteq K_0(n) \qquad \text{and} \qquad K_\delta(n)\subset S_\delta(n). \tag{4}$$

The classes $S_\delta(n)$ and $K_\delta(n)$ have been recently studied by Srivastava, Owa and Chatterjea [22]. For $n=1$, these denotations are usually used as $S_\delta(1)=S^*(\delta)$, $K_\delta(1)=K(\delta)$,

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which are introduced earlier by Robertson [12]. Especially, taking $\delta = 0$, we obtain the well-known classes S^* and K of starlike and convex functions in U, respectively.

Further, we consider the so-called subclasses of functions with *negative coefficients*, namely denoting by $T(n) \subset S(n)$ the functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad \text{with} \quad a_k \geqq 0 \ \ (k = n+1, n+2, \cdots), \tag{5}$$

and taking respective intersections for $0 \leqq \delta < 1, n \in \mathbb{N}$:

$$T_\delta(n) = S_\delta(n) \cap T(n), \quad L_\delta(n) = K_\delta(n) \cap T(n). \tag{6}$$

The latter classes were considered by Chatterjea [1] and in particular, case $n = 1$ gives the Silverman classes $T^*(\delta)$ and $L(\delta)$, [19].

For functions of these classes we propose some *distortion inequalities* and other *characterization theorems*, in terms of the generalized fractional calculus operators defined in [5], [7], [8]. As applications of these general results we derive the same kind ones for the Saigo's operator ([14], [15], [16], [23]), Hohlov's operator ([3], [4]) as well as for the fractional integrals and derivatives involving the Appell's F_3-function, recently studied by Saigo et al. [17], [18].

2. Generalized Fractional Calculus Operators

First we need the definition of the generalized hypergeometric function known as *Meijer's G-function*:

$$G_{p,q}^{m,n}(\sigma) = G_{p,q}^{m,n}\left[\sigma \,\middle|\, \begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix}\right] = G_{p,q}^{m,n}\left[\sigma \,\middle|\, \begin{matrix} (a_j)_1^p \\ (b_j)_1^q \end{matrix}\right]$$
$$= \frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{\prod_{i=1}^{m} \Gamma(b_i - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s)}{\prod_{i=m+1}^{q} \Gamma(1 - b_i + s) \prod_{j=n+1}^{p} \Gamma(a_j - s)} \sigma^s \, ds \qquad (\sigma \neq 0), \tag{7}$$

where $a_1, \cdots, a_p, b_1, \cdots, b_q \in \mathbb{C}$ with $\mathbb{C}$ being the field of complex numbers and $\mathfrak{C}$ is a certain contour on the complex plane (see for precise [2], [11], [7]).

Using a Meijer's G-function of peculiar order $(m, 0, m, m)$, in [5], [7], a *generalized fractional calculus* has been developed that includes as special cases almost all the known operators of fractional integration and differentiation studied by many authors.

Let $n \in \mathbb{N}$, $\beta \in \mathbb{R}^+ = (0, \infty)$, $\gamma_i \in \mathbb{R} = (-\infty, \infty)$ $(i = 1, \cdots, n)$ and $\delta_i \in \mathbb{R}_0^+ = [0, \infty)$ $(i = 1, \cdots, n)$. $\delta = (\delta_1, \cdots, \delta_n)$ is considered as a *fractional multiorder of integration.* The following basic notion of the *generalized operator of fractional integration* (*generalized fractional integral*) is introduced:

$$I_{\beta,m}^{(\gamma_i),(\delta_i)} f(z) = \begin{cases} \int\limits_0^1 G_{m,m}^{m,0}\left[\sigma \left| \begin{matrix} (\gamma_i+\delta_i)_1^n \\ (\gamma_i)_1^m \end{matrix} \right.\right] f\left(z\sigma^{1/\beta}\right) d\sigma, & \text{if } \sum\limits_{i=1}^m \delta_i > 0 \\ f(z), & \text{if } \sum\limits_{i=1}^m \delta_i = 0. \end{cases} \tag{8}$$

The corresponding generalized fractional derivative is denoted by $D_{\beta,m}^{(\gamma_i),(\delta_i)}$ and defined by means of an explicit differ-integral expression. By a suitable choice of parameters, one can derive as very special cases of (8), the classical fractional integral and derivative R^δ of *Riemann-Liouville* and the *Erdélyi-Kober integral* $I_\beta^{\gamma,\delta}$, widely used in the applied mathematical analysis (see [20], [7]):

$$\left.\begin{aligned} R^\delta f(z) &= z^\delta \int\limits_0^1 \frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)} f(z\sigma)\, d\sigma \qquad (\delta>0); \\ I_\beta^{\gamma,\delta} f(z) &= \int\limits_0^1 \frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)} \sigma^\gamma f(z\sigma^{1/\beta}) d\sigma \quad (\delta>0, \gamma\in\mathbb{R}), \end{aligned}\right\} \tag{9}$$

namely:

$$R^\delta f(z) = z^\delta I_{1,1}^{0,\delta} f(z); \quad I_\beta^{\gamma,\delta} f(z) = I_{\beta,1}^{\gamma,\delta} f(z)$$

as well as the hypergeometric fractional integrals and many other generalized integrations and differentiations.

A detailed theory, called *generalized fractional calculus* and an analogue of the classical fractional calculus and its different applications are proposed in [7].

The most useful property of the generalized fractional integrals is their alternative representation as products of commuting E-K fractional integrals:

$$\begin{aligned} I_{\beta,m}^{(\gamma_i),(\delta_i)} f(z) &= I_\beta^{\gamma_1,\delta_1} \cdots I_\beta^{\gamma_m,\delta_m} f(z) \\ &= \int\limits_0^1 \cdots \int\limits_0^1 \left\{ \prod_{i=1}^m \frac{(1-\sigma_i)^{\delta_i-1}\sigma_i^{\gamma_i}}{\Gamma(\delta_i)} \right\} f\left[z\left(\sigma_1\cdots\sigma_m\right)^{1/\beta}\right] d\sigma_1\cdots d\sigma_m. \end{aligned} \tag{10}$$

In [5], [8] we have considered the above operator and its properties in classes of analytic functions in starlike domains and in particular, in the disk $\{|z|<R\}$ $(R>0)$, but for the purposes here we restrict ourselves only to the *unit disk* $U=\{|z|<1\}$ and to the classes $A(n)$ of functions of form (1).

Using only the simple properties of Meijer's G-function ([2]), one easily obtains

Lemma 0. *For* $\delta_i \geqq 0$ $(i=1,\cdots,m)$,

$$I_{\beta,m}^{(\gamma_i),(\delta_i)}\{z^p\} = c_p z^p \quad \textit{for} \quad p > \max_{1\leqq i\leqq m}\left[-\beta(\gamma_i+1)\right], \tag{11}$$

where

$$c_p = \prod_{i=1}^{m} \frac{\Gamma(\gamma_i + 1 + p/\beta)}{\Gamma(\gamma_i + \delta_i + 1 + p/\beta)} > 0.$$

P r o o f . To evaluate the $I_{\beta,m}^{(\gamma_i),(\delta_i)}$-image of an arbitrary power function $f(z) = z^p$, we use an extension of formula [2, Vol.1, §5.5.2, (5)], namely: [7, Appendix, p.324, Lemma B.2]:

$$\int_0^1 G_{m,m}^{m,0} \left[\sigma \left| \begin{matrix} (a_i)_1^m \\ (b_i)_1^m \end{matrix} \right. \right] d\sigma = \prod_{i=1}^{m} \frac{\Gamma(b_i + 1)}{\Gamma(a_i + 1)} \quad \text{for } a_i > b_i > -1 \ (i = 1, \cdots, m).$$

Then, according to the G-function's property [2, Vol.1, §5.3.1, (8)], we obtain

$$I_{\beta,m}^{(\gamma_i),(\delta_i)}\{z^p\} = z^p \int_0^1 G_{m,m}^{m,0} \left[\sigma \left| \begin{matrix} (\gamma_i + \delta_i)_1^m \\ (\gamma_i)_1^m \end{matrix} \right. \right] \sigma^{p/\beta} \, d\sigma$$

$$= z^p \int_0^1 G_{m,m}^{m,0} \left[\sigma \left| \begin{matrix} (\gamma_i + \delta_i + p/\beta)_1^m \\ (\gamma_i + p/\beta)_1^m \end{matrix} \right. \right] d\sigma = z^p \prod_{i=1}^{m} \frac{\Gamma(\gamma_i + 1 + p/\beta)}{\Gamma(\gamma_i + \delta_i + 1 + p/\beta)} = c_p z^p,$$

where the conditions $\gamma_i + \delta_i + \beta > \gamma_i + p/\beta > -1$ $(i = 1, \cdots, m)$ are ensured by $\delta_i \geqq 0$ and $\gamma_i > -1 - p/\beta$ $(i = 1, \cdots, m)$, i.e. $p > \max\limits_{1 \leqq i \leqq m} [-\beta(\gamma_i + 1)]$.

For the sake of brevity, this paper is considered only for the simpler case (with respect to denotations), when $\beta = 1$. Practically, the integral, differential or integro-differential operators used by different authors in univalent function theory, follow as special cases of operator (8) with $\beta = 1$, but recently some more general fractional calculus operators have been also used that correspond to our operator (8) with arbitrary $\beta > 0$, or even to (10) with different parameters $\beta_i > 0$ $(i = 1, \cdots, m)$ (in this case operator (8) has a more general form with Fox's H-function as a kernel).

While considering functions in the classes $A(n)$, it is suitable to *normalize* operator (8) by means of multiplication by the constant c_1^{-1} $(p = 1)$. Then, further we consider the generalized fractional integrals (using the same name for the normalized version, but stressing this fact by additional "tilde" in denotation: $\widetilde{I}_{1,m}^{(\gamma_i),(\delta_i)} := c_1^{-1} I_{1,m}^{(\gamma_i),(\delta_i)}$), i.e.

$$\widetilde{I}_{1,m}^{(\gamma_i),(\delta_i)} f(z) := \prod_{i=1}^{m} \frac{\Gamma(\gamma_i + \delta_i + 2)}{\Gamma(\gamma_i + 2)} \, I_{1,m}^{(\gamma_i),(\delta_i)} f(z). \tag{12}$$

Then, from Lemma 0 and the more general results in [5], [8] (Theorem 1) and [7], we easily obtain the following:

Theorem 1. *Under the parameters' conditions*

$$\gamma_i > -2, \quad \delta_i \geqq 0 \quad (i = 1, \cdots, m) \tag{13}$$

the generalized fractional integral $\widetilde{I}_{1,m}^{(\gamma_i),(\delta_i)}$ maps the class $A(n)$ into itself, and the image of a power series (1) has the form

$$\widetilde{I}f(z) = \widetilde{I}_{1,m}^{(\gamma_i),(\delta_i)}\left\{z + \sum_{k=n+1}^{\infty} a_k z^k\right\} = z + \sum_{k=n+1}^{\infty} \Psi(k) a_k z^k \in A(n), \tag{14}$$

where the multipliers are

$$\Psi(k) = \prod_{i=1}^{m} \frac{(\gamma_i + 2)_{k-1}}{(\gamma_i + \delta_i + 2)_{k-1}} > 0 \qquad (k = n+1, n+2, \cdots) \tag{15}$$

with $(a)_k = \Gamma(a+k)/\Gamma(a)$ denoting the known Pochhammer symbol.

Proof. In order to have (11) valid for $\beta = 1$ with $p = 1$ and $p = n+1, n+2, \cdots$, we require $\gamma_i > -2$ $(i = 1, \cdots, m)$. Then,

$$\widetilde{I}_{1,m}^{(\gamma_i),(\delta_i)}\{z\} = z \quad \text{and}$$

$$\widetilde{I}_{1,m}^{(\gamma_i),(\delta_i)}\{z^k\} = \frac{c_k}{c_1} z^k = z^k \prod_{i=1}^{m} \frac{\Gamma(\gamma_i + 1 + k)\Gamma(\gamma_i + \delta_i + 2)}{\Gamma(\gamma_i + \delta_i + 1 + k)\Gamma(\gamma_i + 2)}$$

$$= z^k \prod_{i=1}^{m} \frac{(\gamma_i + 2)_{k-1}}{(\gamma_i + \delta_i + 2)_{k-1}} = \Psi(k) z^k$$

and the term-by-term integration of power series (1) gives series (14). By virtue of the Cauchy-Hadamard formula, the radius of convergence of the latter series is calculated by

$$R = \left\{\overline{\lim_{k\to\infty}}\, |a_k|^{1/k} |\Psi(k)|^{1/k}\right\}^{-1}.$$

Since the series (1) is analytic function in the unit disc, we find $\overline{\lim\limits_{k\to\infty}}\, |a_k|^{1/k} \geqq 1$. On the other hand,

$$\lim_{k\to\infty} |\Psi(k)|^{1/k} = \lim_{k\to\infty} \prod_{i=1}^{m} \left[\frac{\Gamma(\gamma_i + 1 + k)}{\Gamma(\gamma_i + \delta_i + 1 + k)}\right]^{1/k} \left[\frac{\Gamma(\gamma_i + \delta_i + 2)}{\Gamma(\gamma_i + 2)}\right]^{1/k}$$

$$= \lim_{k\to\infty} \prod_{i=1}^{m} \left(k^{1/k}\right)^{-\delta_i} = 1$$

by using the known asymptotics

$$\frac{\Gamma(b+k)}{\Gamma(a+k)} \sim k^{b-a} \qquad (k \to \infty),$$

then, it follows $R \geqq 1$ and the image $\widetilde{I}_{1,m}^{(\gamma_i),(\delta_i)} f(z)$ given by series (14) is analytic in the unit disc.

The *Hadamard product* (*convolution*) of two analytic functions in U

$$f(z)=\sum_{k=0}^{\infty}a_kz^k,\qquad g(z)=\sum_{k=0}^{\infty}b_kz^k$$

is defined by

$$f*g(z):=\sum_{k=0}^{\infty}a_kb_kz^k. \tag{16}$$

Theorem 2. *In the class $A(n)$ the generalized fractional integral (12) can be represented as the Hadamard product*

$$\widetilde{I}_{1,m}^{(\gamma_i),(\delta_i)}f(z)=h(z)*f(z), \tag{14*}$$

where the function $h(z)\in A(n)$ is expressed by the generalized hypergeometric function:

$$h(z)=z+\sum_{k=n+1}^{\infty}\Psi(k)z^k=z+\sum_{k=n+1}^{\infty}\left[\prod_{i=1}^{m}\frac{(\gamma_i+2)_{k-1}}{(\gamma_i+\delta_i+2)_{k-1}}\right]z^k$$

$$=z+\left[\prod_{i=1}^{m}\frac{(\gamma_i+2)_n}{(\gamma_i+\delta_i+2)_n}\right]z^{n+1}\ {}_{m+1}F_m\left(\begin{matrix}1,(\gamma_i+2+n)_1^m\\ \\ (\gamma_i+\delta_i+2+n)_1^m\end{matrix};z\right). \tag{17}$$

Special cases of operator (12), or of its modified form

$$Rf(z)=cz^{\delta_0}\widetilde{I}_{1,m}^{(\gamma_i),(\delta_i)}f(z)\qquad\text{with } c=\text{const and } \delta_0\geqq 0, \tag{12*}$$

have been used very often in the univalent function theory, like the known operators of: Biernacki, Komatu, Libera, Rusheweyh, Owa and Srivastava, Carlson and Shaffer, Saigo, Hohlov, etc. (see Examples 1 - 9 in [8]). Thus, the results below give as corollaries corresponding properties of all these operators.

3. Distortion Inequalities in the Classes $T_\delta(n)$ and $L_\delta(n)$

We need the following lemmas given by Chatterjea [1].

Lemma 1. *Let the function $f(z)$ be defined by (1). Then $f(z)$ is in the class $T_\delta(n)$ if and only if*

$$\sum_{k=n+1}^{\infty}\frac{k-\delta}{1-\delta}\,a_k\leqq 1. \tag{18}$$

Lemma 2. *Let the function $f(z)$ be defined by (1). Then $f(z)$ is in the class $L_\delta(n)$ if and only if*

$$\sum_{k=n+1}^{\infty} \frac{k(k-\delta)}{1-\delta}\, a_k \leqq 1. \tag{19}$$

Applying Lemma 1 and Theorem 1, we obtain

Theorem 3. *Let condition (13) be satisfied and the function $f(z)$ defined by (1) belong to the class $T_\delta(n)$. Then the following distortion inequalities hold for $z \in U$:*

$$\left|\widetilde{I}_{1,m}^{(\gamma_i),(\delta_i)} f(z)\right| \geqq |z| - \frac{1-\delta}{n+1-\delta}\, \Psi(n+1)\, |z|^{n+1} \tag{20}$$

and

$$\left|\widetilde{I}_{1,m}^{(\gamma_i),(\delta_i)} f(z)\right| \geqq |z| + \frac{1-\delta}{n+1-\delta}\, \Psi(n+1)\, |z|^{n+1}, \tag{21}$$

where the multiplier $\Psi(n+1)$ is defined as in (15), namely:

$$\Psi(n+1) = \prod_{i=1}^{m} \frac{(\gamma_i+2)_n}{(\gamma_i+\delta_i+2)_n} > 0. \tag{22}$$

Equalities in (20) and (21) are attained by the function

$$f(z) = z - \frac{1-\delta}{n+1-\delta}\, z^{n+1}. \tag{23}$$

Theorem 4. *Let condition (13) be satisfied and the function $f(z)$ defined by (1) belong to the class $L_\delta(n)$. Then the following inequalities hold for $z \in U$:*

$$\left|\widetilde{I}_{1,m}^{(\gamma_i),(\delta_i)} f(z)\right| \geqq |z| - \frac{1-\delta}{n+1-\delta}\, \frac{\Psi(n+1)}{n+1}\, |z|^{n+1} \tag{24}$$

and

$$\left|\widetilde{I}_{1,m}^{(\gamma_i),(\delta_i)} f(z)\right| \leqq |z| + \frac{1-\delta}{n+1-\delta}\, \frac{\Psi(n+1)}{n+1}\, |z|^{n+1}, \tag{25}$$

where the multiplier $\Psi(n+1)$ is defined in (22). Equalities in (24) and (25) are attained by the function

$$f(z) = z - \frac{1-\delta}{(n+1)(n+1-\delta)}\, z^{n+1}. \tag{26}$$

Proof of Theorems 3 and 4. It is easily seen that under the assumption (13), the function $\Psi(k)$ is nonincreasing for all integers $k \geqq n+1$, since

$$\frac{\Psi(k+1)}{\Psi(k+2)} = \prod_{i=1}^{m} \frac{(\gamma_i+2)_k}{(\gamma_i+2)_{k+1}} \cdot \frac{(\gamma_i+\delta_i+2)_{k+1}}{(\gamma_i+\delta_i+2)_k}$$

$$= \prod_{i=1}^{m} \frac{\gamma_i+\delta_i+2+k}{\gamma_i+2+k} \geqq 1,$$

because of

$$\frac{(a)_k}{(a)_{k+1}} = \frac{1}{a+k} \quad \text{and} \quad \frac{\gamma_i+\delta_i+2+k}{\gamma_i+2+k} \geqq 1.$$

Hence,

$$0 < \Psi(k) \leqq \Psi(n+1) \quad (k \geqq n+1)$$

and for $f(z)$ of form (5),

$$\left|\tilde{I}_{1,m}^{(\gamma_i),(\delta_i)} f(z)\right| \geqq |z| - \left|\sum_{k=n+1}^{\infty} \Psi(k)\, a_k z^k\right| \geqq |z| - \Psi(n+1)|z|^{n+1} \sum_{k=n+1}^{\infty} a_k.$$

Using (18), we obtain inequality (20). The inequality (21) can be proved similarly, and Theorem 4 follows in analogous way by applying Lemma 2.

Remark. If we set $n = 1$ and $\delta = 0$, we obtain

$$f \in S^* \cap T(1) \Longrightarrow \left|\tilde{I}f(z)\right| \geqq |z| - \frac{\Psi(2)}{2}\,|z|^2, \qquad \left|\tilde{I}f(z)\right| \leqq |z| + \frac{\Psi(2)}{2}\,|z|^2$$

$$f \in K \cap T(1) \Longrightarrow \left|\tilde{I}f(z)\right| \geqq |z| - \frac{\Psi(2)}{4}\,|z|^2, \qquad \left|\tilde{I}f(z)\right| \leqq |z| + \frac{\Psi(2)}{4}\,|z|^2$$

with $\Psi(2) = \prod_{i=1}^{m}(\gamma_i+2)/(\gamma_i+\delta_i+2)$. The case $m = 1$ (simply omitting the sign $\prod_{i=1}^{m}$ in (22)) gives estimates for the classical Erdélyi-Kober operator (9).

4. Characterization Theorems in the Classes $S^*(n)$ and $K(n)$

Now we consider some sufficient conditions for starlike and convex functions of form (1). Namely, we denote by $S^*(n)$ the subclass of $A(n)$ of functions satisfying (2) with $\delta = 0$, i.e. $S^*(n) := S_0(n)$. Analogously, $K(n) := K_0(n)$ is the subclass of $A(n)$ of functions $f(z)$ satisfying (3) with $\delta = 0$.

From Silverman's results [19], one can formulate the following auxiliary lemmas.

Lemma 3. *If the function $f(z)$ defined by* (1) *satisfies the condition*

$$\sum_{k=n+1}^{\infty} k|a_k| \leqq 1, \tag{27}$$

then $f(z) \in S^*(n)$. The equality in (27) is attained by the function

$$g_1(z) = z + \frac{z^k}{k} \quad (z \in U) \tag{28}$$

for some $k \geqq n+1$.

Lemma 4. *If the function $f(z)$ defined by (1) satisfies the condition*

$$\sum_{k=n+1}^{\infty} k^2 |a_k| \leqq 1, \tag{29}$$

then $f(z) \in K(n)$. The equality in (29) is attained by the function

$$g_2(z) = z + \frac{z^k}{k^2} \quad (z \in U) \tag{30}$$

for some $k \geqq n+1$.

For the generalized fractional integral (12) we obtain then the following sufficient conditions.

Theorem 5. *Under the condition (13), if the function $f(z)$ defined by (1) satisfies*

$$\sum_{k=n+1}^{\infty} k|a_k| \leqq \frac{1}{\Psi(n+1)} = \prod_{i=1}^{m} \frac{(\gamma_i + \delta_i + 2)_n}{(\gamma_i + 2)_n}, \tag{31}$$

then $\widetilde{I}_{1,m}^{(\gamma_i),(\delta_i)} f(z)$ belongs to the class $S^*(n)$.

Proof. We use again the inequality $0 < \Psi(k) \leqq \Psi(n+1)$, valid for each $k \geqq n+1$ and each $n \in \mathbb{N}$. Then, for the function $\widetilde{I}f(z) = z + \sum\limits_{k=n+1}^{\infty} b_k z^k$ with coefficients $b_k = \Psi(k) a_k$, we obtain $\sum\limits_{k=n+1}^{\infty} k|b_k| \leqq \Psi(n+1) \sum\limits_{k=n+1}^{\infty} k|a_k| \leqq 1$.

Analogously, using Lemma 4, we obtain

Theorem 6. *Under the condition (13), if the function $f(z)$ defined by (1) satisfies*

$$\sum_{k=n+1}^{\infty} k^2|a_k| \leqq \frac{1}{\Psi(n+1)} = \prod_{i=1}^{m} \frac{(\gamma_i + \delta_i + 2)_n}{(\gamma_i + 2)_n}, \tag{32}$$

then $\widetilde{I}_{1,m}^{(\gamma_i),(\delta_i)} f(z)$ belongs to the class $K(n)$.

Remark. Examples of functions satisfying conditions (31) and (32) are the following functions

$$g_3(z) = z + \frac{1}{\Psi(k_0)} \frac{z^{k_0}}{k_0} \quad \text{and} \quad g_4(z) = z + \frac{1}{\Psi(k_0)} \frac{z^{k_0}}{k_0^2},$$

respectively, with some $k_0 \geqq n+1$.

We use also the following result due to Rusheweyh and Sheil-Small [13].

Lemma 5. *Let $h(z)$ and $f(z)$ be analytic in U and satisfy the condition:*

$$h(0) = f(0) = 0, \quad h(z) * \left\{ \frac{1+\rho\sigma z}{1-\sigma z} f(z) \right\} \neq 0 \quad (z \in U \backslash \{0\}). \tag{33}$$

for any $\rho, \sigma \in \mathbb{C}$ ($|\rho| = |\sigma| = 1$) with $*$ denoting the Hadamard product (16)*. Then for a function $F(z)$ analytic in U and satisfying*

$$\operatorname{Re}\{F(z)\} > 0 \quad (z \in U),$$

the inequality

$$\operatorname{Re}\left\{ \frac{(h * Ff)(z)}{(h * f)(z)} \right\} > 0 \quad (z \in U) \tag{34}$$

follows.

Now we state some characterization theorems in terms of the Hadamard product.

Theorem 7. *Let us assume condition* (13)*, and let the function $f(z)$ defined by* (1) *belong to $S^*(n)$ and satisfy*

$$h(z) * \left\{ \frac{1+\rho\sigma z}{1-\sigma z} f(z) \right\} \neq 0 \quad (z \in U \backslash \{0\}) \tag{35}$$

for any $\rho, \sigma \in \mathbb{C}$ ($|\rho| = |\sigma| = 1$) and for the function $h(z)$ defined by (17)*. Then, $\widetilde{I}_{1,m}^{(\gamma_i),(\delta_i)} f(z)$ also belongs to $S^*(n)$, i.e. under such conditions the generalized fractional integral preserves the class $S^*(n)$.*

P r o o f . By Theorem 2,

$$\widetilde{I}_{1,m}^{(\gamma_i),(\delta_i)} f(z) = z + \sum_{k=n+1}^{\infty} \Psi(k) z^k = h(z) * f(z).$$

Since it is easy to check that

$$\frac{z(h*f)'(z)}{(h*f)(z)} = \frac{(h*(zf'))}{(h*f)(z)} \quad \text{for each } h, f \in A(n),$$

it follows, if we set $F(z) = zf'(z)/f(z)$,

$$\frac{z\left(\widetilde{I}f(z)\right)'}{\widetilde{I}f(z)} = \frac{h*zf'}{h*f} = \frac{h*Ff}{h*f}.$$

Using that $f \in S^*(n)$ implies $\mathrm{Re}\{F(z)\} > 0$, we obtain from Lemma 5

$$\mathrm{Re}\left\{\frac{z\left(\widetilde{I}f(z)\right)'}{\widetilde{I}f(z)}\right\} = \mathrm{Re}\left\{\frac{(h * Ff)(z)}{(h * f)(z)}\right\} > 0 \quad \Longrightarrow \quad \widetilde{I}f(z) \in S^*(n).$$

For a subclass of the convex functions, an analogous theorem can be read as follows.

Theorem 8. *Let us assume condition* (13), *and let the function* $f(z)$ *defined by* (1) *belong to* $K(n)$ *and satisfy*

$$h(z) * \left\{\frac{1+\rho\sigma z}{1-\sigma z} z f'(z)\right\} \neq 0 \quad (z \in U\backslash\{0\}) \tag{36}$$

for any $\rho, \sigma \in \mathbb{C}$ $(|\rho| = |\sigma| = 1)$ *and for the function* $h(z)$ *defined by* (17). *Then,* $\widetilde{I}_{1,m}^{(\gamma_i),(\delta_i)} f(z)$ *also belongs to* $K(n)$, *i.e. under such conditions the generalized fractional integrals preserve the class* $K(n)$.

Proof. Note that in (36) we have $zf'(z)$ instead of $f(z)$ in (35). We use the fact that $f \in K(n) \Longleftrightarrow zf' \in S^*(n)$ and Theorem 7.

Lemma 6. (Rusheweyh and Sheil-Small [13]) *Let* $h(z)$ *be convex and* $f(z)$ *be starlike in* U. *Then, for each function* $F(z)$ *analytic in* U *and satisfying* $\mathrm{Re}\{F(z)\} > 0$ $(z \in U)$, *the inequality*

$$\mathrm{Re}\left\{\frac{(h * Ff)(z)}{(h * f)(z)}\right\} > 0 \quad (z \in U) \tag{37}$$

holds valid.

Whence, in a way similar like in Theorems 7, 8 we have the following characterization theorems.

Theorem 9. *Let us assume condition* (13), *and let the function* $f(z)$ *defined by* (1) *belong to* $S^*(n)$ *and* $h(z)$ *defined by* (17) *belong to* $K(n)$. *Then,* $\widetilde{I}_{1,m}^{(\gamma_i),(\delta_i)} f(z)$ *belongs to* $S^*(n)$, *i.e.*

$$f(z) \in S^*(n), \quad h(z) \in K(n) \quad \Longrightarrow \quad \widetilde{I}_{1,m}^{(\gamma_i),(\delta_i)} f(z) \in S^*(n). \tag{38}$$

Theorem 10. *Let us assume condition* (13), *and let the functions* $f(z)$ *defined by* (1) *and* $h(z)$ *defined by* (17) *belong to* $K(n)$. *Then,* $\widetilde{I}_{1,m}^{(\gamma_i),(\delta_i)} f(z)$ *belongs to* $K(n)$, *i.e.*

$$f(z) \in K(n), \quad h(z) \in K(n) \quad \Longrightarrow \quad \widetilde{I}_{1,m}^{(\gamma_i),(\delta_i)} f(z) \in K(n). \tag{39}$$

Summarized, (38) and (39) mean that *if the "kernel function" (17) of generalized fractional integrals* (12), (14*) *belongs to* $K(n)$, *then this operator* $\widetilde{I}_{1,m}^{(\gamma_i),(\delta_i)}$ *preserves both classes* $S^*(n), K(n)$.

5. Saigo's and Hohlov's Operators (m=2)

In [14], [16], Saigo introduced operators of generalized fractional integration and differentiation, involving the *Gauss hypergeometric function.* For real numbers $\alpha > 0, \beta, \eta$, the fractional integral operator $I^{\alpha,\beta,\eta}$ is defined by

$$I^{\alpha,\beta,\eta} f(z) = z^{-\alpha-\beta} \int_0^z \frac{(z-\xi)^{\alpha-1}}{\Gamma(\alpha)} \; {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\xi}{z}\right) f(\xi) d\xi, \tag{40}$$

where $f(z)$ is an analytic function in a simply-connected domain of the z-plane, containing the origin $z=0$ such that $f(z) = O(|z|^\varepsilon)$ $(z \to 0)$ with $\varepsilon > \max\{0, \beta-\eta\} - 1$, and it is assumed that the multiplicity of $(z-\xi)^{\alpha-1}$ is removed by requiring $\log(z-\xi)$ to be real for $z - \xi > 0$.

The operator (40) has been first considered for real-valued functions and used in solving boundary value problems [15], [23] for the Euler-Darboux equation, but recently Srivastava, Saigo and Owa (see [24], [10]) have applied them to classes of univalent functions.

The operator (40) can be represented also as products of two classical Erdélyi-Kober integrals ([14], [16]) and thus, as pointed by Kiryakova [7], it is an important example of the generalized fractional integral (8) with multiplicity $m = 2$, when the kernel $G_{2,2}^{2,0}$-function turns into a Gauss hypergeometric function. Namely, the following representation of (40) in terms of (8) holds:

$$I^{\alpha,\beta,\eta} f(z) = z^{-\beta} \int_0^1 \frac{(1-\sigma)^{\alpha-1}}{\Gamma(\alpha)} \; {}_2F_1(\alpha+\beta, -\eta; \alpha; 1-\sigma) f(z\sigma) d\sigma$$

$$= z^{-\beta} \int_0^1 G_{2,2}^{2,0}\left[\sigma \left| \begin{matrix} -\beta, \alpha+\eta \\ \eta-\beta, 0 \end{matrix} \right.\right] f(z\sigma) d\sigma = z^{-\beta} I_{1,2}^{(\eta-\beta,o),(-\eta,\alpha+\eta)} f(z), \tag{41}$$

with

$$m \to 2, \beta \to 1; \; \gamma_1 \to \eta - \beta, \gamma_2 \to 0; \; \delta_1 \to -\eta, \delta_2 \to \alpha + \eta$$

in (8).

In view of Lemma 0 and Theorem 1, it is suitably to "normalize" operator (40), (41) multiplying by $c_1^{-1} z^\beta$, as already done in [7], [5]. Thus, further we consider "*normalized*" *Saigo's operator*

$$\widetilde{I}^{\alpha,\beta,\eta} f(z) := \frac{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)}{\Gamma(2-\beta+\eta)} \; z^\beta \; I^{\alpha,\beta,\eta} f(z), \tag{42}$$

which preserves the classes $A(n)$ $(n \in \mathbb{N})$:

$$\widetilde{I}^{\alpha,\beta,\eta} \left\{ z + \sum_{k=n+1}^{\infty} a_k z^k \right\} = z + \sum_{k=n+1}^{\infty} \Psi(k) a_k z^k, \tag{43}$$

where
$$\Psi(k) = \frac{(-\beta+\eta+2)_{k-1}k!}{(-\beta+2)_{k-1}(\alpha+\eta+2)_{k-1}}$$
for which compare with [24, (3.10)]. Or, in terms of the Hadamard product in $A(n)$,
$$\widetilde{I}^{\alpha,\beta,\eta} f(z) = h(z) * f(z)$$
with
$$h(z) = z + \sum_{k=n+1}^{\infty} \frac{(-\beta+\eta+2)_{k-1}k!}{(-\beta+2)_{k-1}(\alpha+\eta+2)_{k-1}} z^k$$
$$= z + \frac{(-\beta+\eta+2)_n(n+1)!}{(-\beta+2)_n(\alpha+\eta+2)_n} z^{n+1} {}_3F_2\left(\begin{matrix} 1, -\beta+\eta+2+n, 2+n \\ -\beta+2+n, \alpha+\eta+2+n \end{matrix}; z\right). \quad (44)$$

Especially in the class $A = A(1)$, the convolutional representation turns into $\widetilde{I}^{\alpha,\beta,\eta} f(z) = h(z) * f(z)$ with (for $n = 1$):
$$h(z) = z + \frac{2(-\beta+\eta+2)}{(-\beta+2)(\alpha+\eta+2)} z^2 {}_3F_2\left(\begin{matrix} 1, -\beta+\eta+3, 3 \\ -\beta+3, \alpha+\eta+2 \end{matrix}; z\right)$$
$$= z \, {}_3F_2\left(\begin{matrix} 1, -\beta+\eta+2, 2 \\ -\beta+2, \alpha+\eta+2 \end{matrix}; z\right). \quad (44^*)$$

Then, from Theorems 2 – 10, one can easily write down the corresponding results for operator (42). As for the original operator $I^{\alpha,\beta,\eta}$ in (40), they follow by reverse multiplication by $c_1 z^{-\beta}$ and they have been given by [24, Theorems 1 – 2] and [10, Theorems 1 – 6]. See also interesting corollaries there concerning classical fractional derivatives $D_z^{-\lambda}$.

Remark. Note only that there is a small difference in the conditions required on parameters α, β, γ and on $n \in \mathbb{N}$, comparing results in [24] (Theorems 1 – 2) and corresponding Theorems 3 – 4 here! These two theorems hold for any integer $n \in \mathbb{N}$, while in [24] condition (3.2) for $n \geqq [\beta(\alpha+\eta)/\alpha] - 2$ is imposed. But in compensation, our conditions (13) for the parameters of the operators (40), (42), in this case reducible to:
$$\begin{cases} \beta - \eta < 2 \text{ (the same)}, \quad \alpha + \eta \geqq 0 \text{ (stronger than } \alpha+\eta > -2), \\ \eta \leqq 0 \text{ (new condition, but from it and the first above} \implies \beta < 2 \text{ as in [24])}, \end{cases} \quad (45)$$
are stronger than (3.1).

In [3], [4] Hohlov introduced a generalized fractional integration operator defined by means of Hadamard product (16) with an arbitrary Gauss hypergeometric function:
$$\mathbf{F}(a,b,c)f(z) := z \, {}_2F_1(a,b;c;z) * f(z). \quad (46)$$

This three-parameter family of operators contains as special cases most of the known linear integral or differential operators, already used in univalent functions theory (see Hohlov [3],[4] and for more details Kiryakova [7], [5], [8]). Namely (we give also in brackets their representations in terms of our operators (8)):

$$\left.\begin{array}{l}
\mathbf{F}(1,1,2)=\mathbf{B} \qquad (\text{Biernacki operator}: \mathbf{B}=I_{1,1}^{-1,1});\\
\mathbf{F}(1,\alpha+1,1)=\mathbf{B}_n^{-1}\\
\qquad (\text{Rusheweyh derivative of order } \alpha: \ \mathbf{B}_n^{-1}=\mathbf{D}^{\alpha}=\dfrac{1}{\Gamma(\alpha)}D_1^{-1,\alpha});\\
\mathbf{F}(1,c+1,c+2)=\mathbf{B}_c \qquad (\text{generalized Libera operator}: \mathbf{B}_c=(c+1)I_{1,1}^{c-1,1});\\
\mathbf{F}(1,2,3)=\mathbf{L}, \quad \mathbf{F}(1,3,2)=\mathbf{L}^{-1}\\
\qquad (\text{Libera and inverse Libera operators}: \mathbf{L}=2I_{1,1}^{01,1});\\
\mathbf{F}(1,a,c)=\mathbf{L}(a,c)\\
\qquad (\text{Carlson-Shaffer operator: } \mathbf{L}(a,c)=I_{1,1}^{a-2,c-a}); \qquad \text{etc.}
\end{array}\right\} \quad (47)$$

As shown in [7], [8], this rather general operator follows again as a *particular case of generalized fractional integrals* (8) and (12):

$$\mathbf{F}(a,b,c)f(z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\,I_{1,2}^{(a-2,b-2),(1-a,c-b)}f(z)=\widetilde{I}_{1,2}^{(a-2,b-2),(1-a,c-b)}f(z). \quad (48)$$

Thus, Theorems 1 – 10 give corresponding results for this operator, and also for all the special cases in the list (47). We only refer to the form of these results in the general case (48), by taking

$$m\to 2, \beta\to 1;\ \gamma_1\to a-2, \gamma_2\to b-2;\ \delta_1\to 1-a, \delta_2\to c-b. \quad (49)$$

Then, conditions in (13) appearing in Theorems 1 – 10 have the form:

$$0<a\leqq 1, \quad 0<b\leqq c, \quad (50)$$

the "multiplier coefficients" (15) and (22) are

$$\Psi(k)=\frac{(a)_{k-1}(b)_{k-1}}{(1)_{k-1}(c)_{k-1}} \quad \text{and} \quad \Psi(n+1)=\frac{(a)_n(b)_n}{n!(c)_n} \quad (51)$$

and the "multiplier function" (17) in $A(n)$ is:

$$h(z)=z+\frac{(a)_n(b)_n}{n!(c)_n}\,z^{n+1}\,{}_3F_2(1,a+n,b+n;c+n,1+n;z). \quad (52)$$

Remark. Note that for $n=1$, in the class $A=A(1)$, from (52) we obtain

$$h(z)=z+\frac{ab}{c}\,z^2\,{}_3F_2(1,a+1,b+1;c+1,2;z)=z\,{}_2F_1(a,b;c;z), \quad (52^*)$$

which conforms with original Hohlov's representation (46).

Remark. Comparing the "multiplier functions" $h(z)$ for Saigo's and Hohlov's operators (42) and (48) in terms of ${}_3F_2$-functions (44), (52) or (44*), (52*), one can see why these two operators, both related to the Gauss function and to $m = 2$ in (12), are not included into the other, as a special case. They have an intersection only for some special values of the parameters α, β, η and a, b, c, when their functions $h(z)$ coincide, for example: For $\beta = 1$, and any α, η set $a = \eta + 1, b = 2, c = \alpha + \eta + 2$, then

$$\mathbf{F}(\eta + 1, 2; \alpha + \eta + 2) = \widetilde{I}^{\alpha,1,\eta}. \tag{53}$$

Now we give briefly the analogues of Theorems 1 – 10 for Hohlov's operator (46) or (48).

Theorem 1*. *Under the parameters' condition* (50) *Hohlov's operator* (46) *or* (48) *maps the class* $A(n)$ *into itself, and the image of a power series* (1) *has the form*

$$\mathbf{F}(a,b,c)f(z) = \mathbf{F}(a,b,c)\left\{z + \sum_{k=n+1}^{\infty} a_k z^k\right\} = z + \sum_{k=n+1}^{\infty} \Psi(k)\ a_k z^k \in A(n),$$

with multiplier coefficient $\Psi(k)$ *in* (51).

Theorem 2*. *In the class* $A(n)$ *Hohlov's operator* (46) *can be represented as Hadamard product* $\mathbf{F}(a,b,c)f(z) = h(z) * f(z)$ *with the function* $h(z) \in A(n)$ *given by* (52).

Theorem 3*. *Let condition* (50) *be satisfied and the function* $f(z)$ *defined by* (1) *belong to the class* $T_\delta(n)$. *Then the following distortion inequalities hold for* $z \in U$:

$$|\mathbf{F}(a,b,c)f(z)| \geqq |z| - \frac{1-\delta}{n+1-\delta}\ \Psi(n+1)|z|^{n+1} \qquad \text{and}$$

$$|\mathbf{F}(a,b,c)f(z)| \leqq |z| + \frac{1-\delta}{n+1-\delta}\ \Psi(n+1)|z|^{n+1},$$

where $\Psi(n+1)$ *is defined as in* (51). *Equalities are attained by the function*

$$f(z) = z - \frac{1-\delta}{n+1-\delta}\ z^{n+1}.$$

Theorem 4*. *Let condition* (50) *be satisfied and the function* $f(z)$ *defined by* (1) *belong to the class* $L_\delta(n)$*. Then the following inequalities hold for* $z \in U$:

$$|\mathbf{F}(a,b,c)f(z)| \geqq |z| - \frac{1-\delta}{n+1-\delta}\,\frac{\Psi(n+1)}{n+1}|z|^{n+1} \quad \text{and}$$

$$|\mathbf{F}(a,b,c)f(z)| \leqq |z| + \frac{1-\delta}{n+1-\delta}\,\frac{\Psi(n+1)}{n+1}|z|^{n+1},$$

where $\Psi(n+1)$ *is defined as in* (51)*. Equalities are attained by the function*

$$f(z) = z - \frac{1-\delta}{(n+1)(n+1-\delta)}\, z^{n+1}.$$

Theorem 5*. *Let us assume condition* (50)*. If the function* $f(z)$ *defined by* (1) *satisfies*

$$\sum_{k=n+1}^{\infty} k|a_k| \leqq \frac{1}{\Psi(n+1)}$$

with $\Psi(n+1)$ *given by* (51)*, then* $\mathbf{F}(a,b,c)f(z)$ *belongs to the class* $S^*(n)$.

Theorem 6*. *Let us assume condition* (50)*. If the function* $f(z)$ *defined by* (1) *satisfies*

$$\sum_{k=n+1}^{\infty} k^2|a_k| \leqq \frac{1}{\Psi(n+1)},$$

then $\mathbf{F}(a,b,c)f(z)$ *belongs to the class* $K(n)$.

Theorem 7*. *Let us assume condition* (50) *and let the function* $f(z)$ *defined by* (1) *belong to* $S^*(n)$ *and satisfy*

$$h(z) * \left\{\frac{1+\rho\sigma z}{1-\sigma z} f(z)\right\} \neq 0 \quad (z \in U\backslash\{0\})$$

for any $\rho, \sigma \in \mathbb{C}$ $(|\rho| = |\sigma| = 1)$ *and for the function* $h(z)$ *defined by* (52)*. Then,* $\mathbf{F}(a,b,c)f(z)$ *also belongs to* $S^*(n)$*, i.e. under the above condition Hohlov's operator preserves the class* $S^*(n)$.

Theorem 8*. *Let us assume condition* (50) *and let the function* $f(z)$ *defined by* (1) *belong to* $K(n)$ *and satisfy*

$$h(z) * \left\{\frac{1+\rho\sigma z}{1-\sigma z} z f'(z)\right\} \neq 0 \quad (z \in U\backslash\{0\})$$

for any $\rho, \sigma \in \mathbb{C}$ ($|\rho| = |\sigma| = 1$) and for the function $h(z)$ defined by (52). *Then,* $\mathbf{F}(a, b, c)f(z)$ *also belongs to* $K(n)$, *i.e. under the above condition Hohlov's operator preserves the class* $K(n)$.

Theorem 9*. *Let us assume condition* (50) *and let the function $f(z)$ defined by* (1) *belong to $S^*(n)$ and $h(z)$ defined by* (52) *belong to $K(n)$. Then,* $\mathbf{F}(a, b, c)f(z)$ *belongs to* $S^*(n)$, *i.e.*

$$f(z) \in S^*(n), \quad h(z) \in K(n) \quad \Longrightarrow \quad \mathbf{F}(a, b, c)f(z) \in S^*(n).$$

Theorem 10*. *Let us assume condition* (50) *and let the functions $f(z)$ defined by* (1) *and $h(z)$ defined by* (52) *belong to $K(n)$. Then,* $\mathbf{F}(a, b, c)f(z)$ *belongs to* $K(n)$, *i.e.*

$$f(z) \in K(n), \quad h(z) \in K(n) \quad \Longrightarrow \quad \mathbf{F}(a, b, c)f(z) \in K(n).$$

It is interesting also to specialize these results for the case $n = 1$, class $A = A(1)$, where Hohlov has originally defined and studied the operator (46).

For example, Theorems 5* and 6* then read as follows: Under condition (50) for a function $f(z)$ defined by (1):

$$\sum_{k=2}^{\infty} k|a_k| \leqq \frac{c}{ab} \quad \Longrightarrow \quad \mathbf{F}(a, b, c)f(z) \in S^*,$$

$$\sum_{k=2}^{\infty} k^2|a_k| \leqq \frac{c}{ab} \quad \Longrightarrow \quad \mathbf{F}(a, b, c)f(z) \in K.$$

Similarly, Theorems 7* – 10* take place with the function $h(z) = z\ {}_2F_1(a, b; c; z)$ and concern the classes S^* and K, again.

6. Saigo's Operators Involving F_3-Appell's Function (m=3)

In [17], [18] Saigo and his co-worker investigated in details the operator of generalized fractional integration which involve so-called *Appell's F_3-function* and can be decomposed as products of three Erdélyi-Kober operators (9). Similar operators have been introduced and studied first by Marichev [9] (but in other aspects) and have been shown by Kiryakova [7] to be an example of generalized fractional integrals (8), (12) with multiplicity $m = 3$. Saigo considers such an operator in the following form and denotations:

$$I(\alpha, \alpha', \beta, \beta'; \gamma)f(z) = z^{-\alpha} \int_0^z \frac{(z-\xi)^{\gamma-1}}{\Gamma(\gamma)} \xi^{-\alpha'}\ F_3\left(\alpha, \alpha', \beta, \beta'; \gamma; 1 - \frac{\xi}{z}, 1 - \frac{z}{\xi}\right) f(\xi) d\xi \quad (54)$$

for $\gamma > 0$, but it could be put also in the form

$$I(\alpha,\alpha',\beta,\beta';\gamma)f(z)$$

$$= z^{-\alpha-\alpha'+\gamma}\int\limits_0^1 G_{3,3}^{3,0}\left[\sigma \,\middle|\, \begin{matrix}\alpha-\alpha'+\beta,\gamma-2\alpha',\gamma-\alpha'-\beta' \\ \alpha-\alpha',\beta-\alpha',\gamma-2\alpha'-\beta'\end{matrix}\right] f(z\sigma)d\sigma,$$

that is, with

$$m\to 3,\ \beta\to 1;\ \gamma_1\to\alpha-\alpha',\ \gamma_2\to\beta-\alpha',\ \gamma_3\to\gamma-2\alpha'-\beta',$$

$$\delta_1\to\beta,\ \delta_2\to\gamma-\alpha'-\beta,\ \delta_3\to\alpha',$$

we get the representation

$$I(\alpha,\alpha',\beta,\beta';\gamma)f(z) = z^{-\alpha-\alpha'+\gamma}\ I_{1,3}^{(\alpha-\alpha',\beta-\alpha',\gamma-2\alpha'-\beta'),(\beta,\gamma-\alpha'-\beta,\alpha')}f(z). \tag{55}$$

Then, for the "*normalized*" F_3-*operator*

$$\tilde{I}f(z) = \tilde{I}(\alpha,\alpha',\beta,\beta';\gamma)f(z) := z^{\alpha+\alpha'-\gamma}\ I(\alpha,\alpha',\beta,\beta';\gamma)f(z) \tag{56}$$

we can apply *all the results for classes of univalent functions, already obtained in Theorems* $1-10$.

In this case the conditions in (13) turn into the following conditions which we require for the parameters of operators (54) - (56):

$$\alpha'\geqq 0,\ \alpha>\alpha'-2,\ \beta\geqq 0,\ \beta>\alpha'-2,\ \gamma\geqq\alpha'+\beta,\ \gamma>2\alpha'+\beta'-2, \tag{57}$$

the "multiplier coefficients" (15) and (22) are

$$\left.\begin{aligned}\Psi(k) &= \frac{(\alpha-\alpha'+2)_{k-1}(\beta-\alpha'+2)_{k-1}(\gamma-2\alpha'-\beta'+2)_{k-1}}{(\alpha-\alpha'+\beta+2)_{k-1}(\gamma-2\alpha'+2)_{k-1}(\gamma-\alpha'-\beta'+2)_{k-1}} \\ \Psi(n+1) &= \frac{(\alpha-\alpha'+2)_n(\beta-\alpha'+2)_n(\gamma-2\alpha'-\beta'+2)_n}{(\alpha-\alpha'+\beta+2)_n(\gamma-2\alpha'+2)_n(\gamma-\alpha'-\beta'+2)_n}\end{aligned}\right\} \tag{58}$$

and the "multiplier function" (17) is:

$$h(z) = z + \frac{(\alpha-\alpha'+2)_n(\beta-\alpha'+2)_n(\gamma-2\alpha'-\beta'+2)_n}{(\alpha-\alpha'+\beta+2)_n(\gamma-2\alpha'+2)_n(\gamma-\alpha'-\beta'+2)_n}z^{n+1}$$

$$\times\ {}_4F_3\left(\begin{matrix}1,\alpha-\alpha'+2+n,\beta-\alpha'+2+n,\gamma-2\alpha'-\beta'+2+n \\ \alpha-\alpha'+\beta+2+n,\gamma-2\alpha'+2+n,\gamma-\alpha'-\beta'+2+n\end{matrix};z\right)\in A(n) \tag{59}$$

The following results follow as corollaries of our general results.

Theorem 1.** *Under the parameters' condition* (57) *the* F_3*-operator* (56) *map the class* $A(n)$ *into itself, and the image of a power series* (1) *has the form*

$$\widetilde{I}(\alpha,\alpha',\beta,\beta';\gamma)f(z)=\widetilde{I}(\alpha,\alpha',\beta,\beta';\gamma)\left\{z+\sum_{k=n+1}^{\infty}a_kz^k\right\}=z+\sum_{k=n+1}^{\infty}\Psi(k)\;a_kz^k\in A(n),$$

with multipliers (58).

Theorem 2.** *In the class* $A(n)$ *the* F_3*-operator* (56) *can be represented as Hadamard product* $\widetilde{I}(\alpha,\alpha',\beta,\beta';\gamma)f(z)=h(z)*f(z)$ *with the function* $h(z)\in A(n)$ *given by* (59).

Theorem 3.** *Let condition* (57) *be satisfied and the function* $f(z)$ *defined by* (1) *belong to the class* $T_\delta(n)$. *Then the following distortion inequalities hold for* $z\in U$:

$$\left|\widetilde{I}(\alpha,\alpha',\beta,\beta';\gamma)f(z)\right|\geqq|z|-\frac{1-\delta}{n+1-\delta}\;\Psi(n+1)\;|z|^{n+1}$$

and

$$\left|\widetilde{I}(\alpha,\alpha',\beta,\beta';\gamma)f(z)\right|\leqq|z|+\frac{1-\delta}{n+1-\delta}\;\Psi(n+1)\;|z|^{n+1},$$

where $\Psi(n+1)$ *is defined by* (58). *Equalities are attained by the function*

$$f(z)=z-\frac{1-\delta}{n+1-\delta}\;z^{n+1}.$$

Theorem 4.** *Let condition* (57) *be satisfied and the function* $f(z)$ *defined by* (1) *belong to the class* $L_\delta(n)$. *Then the following inequalities hold for* $z\in U$:

$$\left|\widetilde{I}(\alpha,\alpha',\beta,\beta';\gamma)f(z)\right|\geqq|z|-\frac{1-\delta}{n+1-\delta}\frac{\Psi(n+1)}{n+1}\;|z|^{n+1}$$

and

$$\left|\widetilde{I}(\alpha,\alpha',\beta,\beta';\gamma)f(z)\right|\leqq|z|+\frac{1-\delta}{n+1-\delta}\frac{\Psi(n+1)}{n+1}\;|z|^{n+1},$$

where $\Psi(n+1)$ *is defined by* (58). *Equalities are attained by the function*

$$f(z)=z-\frac{1-\delta}{(n+1)(n+1-\delta)}\;z^{n+1}.$$

Theorem 5.** *Under condition* (57), *if the function* $f(z)$ *defined by* (1) *satisfies*

$$\sum_{k=n+1}^{\infty}k|a_k|\leqq\frac{1}{\Psi(n+1)},$$

with $\Psi(n+1)$ given by (58), then $\widetilde{I}(\alpha,\alpha',\beta,\beta';\gamma)f(z)$ belongs to the class $S^*(n)$.

Theorem 6.** *Let us assume condition (57). If the function $f(z)$ defined by (1) satisfies*

$$\sum_{k=n+1}^{\infty} k^2|a_k| \leqq \frac{1}{\Psi(n+1)},$$

then $\widetilde{I}(\alpha,\alpha',\beta,\beta';\gamma)f(z)$ belongs to the class $K(n)$.

Theorem 7.** *Let condition (57) be satisfied, and let the function $f(z)$ defined by (1) belong to $S^*(n)$ and satisfy*

$$h(z) * \left\{\frac{1+\rho\sigma z}{1-\sigma z} f(z)\right\} \neq 0 \quad (z \in U\backslash\{0\})$$

for any $\rho,\sigma \in \mathbb{C}$ ($|\rho| = |\sigma| = 1$) and for the function $h(z)$ defined by (59). Then, $\widetilde{I}(\alpha,\alpha',\beta,\beta';\gamma)f(z)$ also belongs to $S^*(n)$, i.e. under the above condition the F_3-operator $\widetilde{I}(\alpha,\alpha',\beta,\beta';\gamma)$ preserves the class $S^*(n)$.

Theorem 8.** *Let condition (57) be satisfied, and let the function $f(z)$ defined by (1) belong to $K(n)$ and be such that*

$$h(z) * \left\{\frac{1+\rho\sigma z}{1-\sigma z} zf'(z)\right\} \neq 0 \quad (z \in U\backslash\{0\})$$

for any $\rho,\sigma \in \mathbb{C}$ ($|\rho| = |\sigma| = 1$) and for the function $h(z)$ defined by (59). Then, $\widetilde{I}(\alpha,\alpha',\beta,\beta';\gamma)f(z)$ also belongs to $K(n)$, i.e. under the above condition the F_3-operators $\widetilde{I}(\alpha,\alpha',\beta,\beta';\gamma)$ preserves the class $K(n)$.

Theorem 9.** *Let condition (57) be satisfied, and let the function $f(z)$ defined by (1) belong to $S^*(n)$ and $h(z)$ defined by (59) belong to $K(n)$. Then, $\widetilde{I}(\alpha,\alpha',\beta,\beta';\gamma)f(z)$ belongs to $S^*(n)$, i.e.*

$$f(z) \in S^*(n), \quad h(z) \in K(n) \quad \Longrightarrow \quad \widetilde{I}(\alpha,\alpha',\beta,\beta';\gamma)f(z) \in S^*(n).$$

Theorem 10.** *Let conditions (57) be satisfied, and let the functions $f(z)$ defined by (1) and $h(z)$ defined by (59) belong to $K(n)$. Then, $\widetilde{I}(\alpha,\alpha',\beta,\beta';\gamma)f(z)$ belongs to $K(n)$, i.e.*

$$f(z) \in K(n), \quad h(z) \in K(n) \quad \Longrightarrow \quad \widetilde{I}(\alpha,\alpha',\beta,\beta';\gamma)f(z) \in K(n).$$

It is interesting also to specialize these results for the case $n = 1$, in the class $A = A(1)$. For example, Theorems 5**, 6** then read as follows: Under conditions (57), for a function $f(z)$ defined by (1):

$$\sum_{k=2}^{\infty} k|a_k| \leqq \frac{1}{\Psi(2)} \quad \Longrightarrow \quad \tilde{I}(\alpha, \alpha', \beta, \beta'; \gamma) f(z) \in S^*,$$

$$\sum_{k=2}^{\infty} k^2|a_k| \leqq \frac{1}{\Psi(2)} \quad \Longrightarrow \quad \tilde{I}(\alpha, \alpha', \beta, \beta'; \gamma) f(z) \in K,$$

where

$$\Psi(2) = \frac{(\alpha - \alpha' + 2)(\beta - \alpha' + 2)(\gamma - 2\alpha' - \beta' + 2)}{(\alpha - \alpha' + \beta + 2)(\gamma - 2\alpha' + 2)(\gamma - \alpha' - \beta' + 2)}.$$

In the same case, Theorems 7** – 10** take place with the function

$$h(z) = z + \Psi(2)\ z^2\ {}_4F_3 \left(\begin{matrix} 1, \alpha - \alpha' + 3, \beta - \alpha' + 3, \gamma - 2\alpha' - \beta' + 3 \\ \alpha - \alpha' + \beta + 3, \gamma - 2\alpha' + 3, \gamma - \alpha' - \beta' + 3 \end{matrix} ; z \right) \in A.$$

and concern the classes S^* and K.

Acknowledgements

The present work is partly supported by Science Research Promotion Fund from the Japan Private School Promotion Foundation, and Research Project MM 433/94, Bulgarian Ministry of Education, Science and Technology.

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