数理解析研究所講究録
1013 巻 1997 年 85-91



# 作用素環における力学系エントロピー

大阪教育大学 長田 まりゑ (Marie Choda)

## 1. Introduction

The entropy invariant of Kolmogorov-Sinai is extended as Connes-St$\phi$rmer entropy $H(\cdot)$ to trace preserving automorphisms of finite von Neumann algebras ([10]). Replacing a finite trace to an invariant state $\phi$, Connes-Narnhofer-Thirring entropy $h_\phi(\cdot)$ is defined for automorphisms of $C^*$-algebras as a generalization of $H(\cdot)$ ([11]).

Many interesting automorphisms to compute the entropies are given on the algebra constructed from $\mathbb{Z}$-copies of an algebra and they are induced by the shift $\alpha : n(\in \mathbb{Z}) \to n+1$. That is, they are "shift" type automorphisms. The first typical example of shift type automorphisms is the Bernoulli shift $\beta_n$ on the infinite product space of $n$-point sets.

In the context of operator algebras (von Neumann algebras or $C^*$-algebras), the non-commutative Bernoulli shift $\alpha_n$ takes the place of $\beta_n$. It is the shift automorphism on the infinite tensor product $M = \bigotimes_{i=-\infty}^{\infty} M_i$, where $M_i$ is the $n\times n$-matrix algebra for all $i \in \mathbb{Z}$. The notion of non-commutative Bernoulli shift is extended to a large class of automorphisms coming from Jones' index theory for subfactors.

These non-commutative Bernoulli shifts satisfy some "sub-commutative" properties. Completely non-commutative shifts are automorphisms on the reduced free product of $C^*$-algebras indexed by $\mathbb{Z}$. The automorphism is called the free shift. The Cuntz algebra $\mathcal{O}_\infty$ appeared as one of such reduced free products.

The above entropies are available to unital *-endomorphisms, which are not always automorphisms. Then Connes-St$\phi$rmer entropies for shift type *- endomorphisms on the hyperfnite $\mathrm{II}_1$ factor have connection with indices of subfactors or the relative entropies of subfactors, which are given as the ranges of those *-endomorphisms ([2, 3, 8, 14, 15]).

On the Cuntz algebra $\mathcal{O}_n$, $(n \geq 2)$, the most interesting *-endomorphism appears as the extension of the *-endomorphism of non-commutative Bernoulli shift type on the half sided infinite tensor product $N = \bigotimes_{i=0}^{\infty} M_i$ of the $n \times n$-matrix algebra $M_i$s. The *-endomorphism is called Cuntz's canonical *-endomorphism.

In this note, we summarize results in [6, 7, 9] about entropies of automorphisms related to free shifts and Cuntz's canonical *-endomorphisms.

## 2. Entropies for automorphisms related to free shifts

Let $A_0$ be a unital $C^*$-algebra and let $\phi_0$ be a state of $A_0$. Let $A_i = A_0$ and $\phi_i = \phi_0$ for all $i \in \mathbb{Z}$. Every $A_i$ acts on the Hilbert space $H_i$ standardly. Let $\xi_i$ be the canonical vector in $H_i$ for the state $\phi_i$. Then the free product Hilbert space $(H,\xi) = (*H_i, *\xi_i)_{i\in\mathbb{Z}}$ is defined. Let $A$ be the reduced free product $C^*$-algebra $A = *_{i\in\mathbb{Z}} A_i$ with respect to states $\{\phi_i\}_{i\in\mathbb{Z}}$ defined by Arvitzour ([1]) and Voiculescu ([29, 31]) independently. Then $A$ is acting on $H$.

The vector state $\phi$ of $A$ defined by $\xi$ is called the free product of $\{\phi_i\}_{i\in\mathbb{Z}}$. We denote the $\phi$ by $*_{i\in\mathbb{Z}}\phi_i$. The free shift $\alpha$ is the automorphism on $A$, which is induced by the shift on $\mathbb{Z}$. It is obvious that $\phi \cdot \alpha = \phi$.

Let $(B,\beta,\mu)$ (resp. $(C,\gamma,\nu)$) be a triplet of a unital $C^*$- algebra $B$ (resp. $C$), a *-automorphism $\beta$ (resp. $\gamma$) of $B$ (resp. $C$) and a state $\mu$ (resp. $\rho$) of $B$ (resp. $C$) with $\mu \cdot \beta = \mu$ (resp. $\rho \cdot \gamma = \rho$). Now we consider the reduced free product $A * C$ with respect to $\{\phi, \rho\}$. We put

$$\mathcal{A} = (A * C) \otimes B.$$

The $\mathcal{A}$ contains the tensor product $C \otimes B$ as a $C^*$-subalgebra. Then we have a conditional expectation $F$ of $\mathcal{A}$ onto $C \otimes B$ which is given by

$$F = (E_\phi * id_C) \otimes id_B,$$

where $E_\phi(a) = \phi(a)1, (a \in A)$, $id_C$ is the identity on $C$, and $E_\phi * id_C$ is the free product of $E_\phi$ and $id_C$.

**2.1 Proposition ([6]).** *Let $\psi$ be a state on $\mathcal{A}$ and $(\alpha * \gamma) \otimes \beta$ the tensor product of the automorphism $\alpha * \gamma$ on $A * C$ (which is the free product of $\alpha$ and $\gamma$) and $\beta$. Then*

$$\psi \cdot (\alpha * \gamma) \otimes \beta = \psi$$

*if and only if there exists a state $\omega$ on $C \otimes B$ such that*

$$\omega \cdot \gamma \otimes \beta = \omega \quad \text{and} \quad \psi = \omega \cdot F.$$

In Proposition 2.1, if we put $C = \mathbb{C}1$, then we have [1 : 4.1 Proposition].

Sauvageot-Thouvenot defined the entropy $H_\phi(\cdot)$ as an alternate of Connes-Narnhofer-Thirring entropy $h_\phi(\cdot)$ ([24]). Proposition 2.1 is used to show the following relations about Sauvageot- Thouvenot entropies for two automorphisms, one of which is given as reduced free product with the free shift $\alpha$ and the other is the tensor product with $\alpha$.

**2.2 Theorem ([6]).** *For an arbitrary triplet $(B,\beta,\mu)$, we have*

$$H_{\phi*\mu}(\alpha * \beta) = H_\mu(\beta) = H_{\phi\otimes\mu}(\alpha \otimes \beta).$$

Two entropies $H_\phi(\cdot)$ and $h_\phi(\cdot)$ are equal for automorphisms on nuclear $C^*$-algebras. Hence we have :

**2.3 Corollary.** *If $A$ and $B$ are nuclear, then*

$$h_\mu(\beta) = h_{\phi\otimes\mu}(\alpha \otimes \beta).$$

The Cuntz algebra $\mathcal{O}_\infty$ is given as the reduced free product $A = *_{i\in\mathbb{Z}} A_i$. Here $A_i$ is the $C^*$-algebra of the semigroup of natural numbers $\mathbb{N}$ with respect to the vector state $\phi_i$ determined by the characteristic function of the unit. Then the free shift $\alpha$ on $\mathcal{O}_\infty$ is given as the automorphism $\alpha : S_i \to S_{i+1}$, for isometries $\{S_i; i \in \mathbb{Z}\}$ which generate $\mathcal{O}_\infty$. It is well known that $\mathcal{O}_\infty$ is nuclear.

In particular, if $B$ in Theorem 3 is the trivial algebra $\mathbb{C}1$, then we have :

**2.4 Corollary.** *If $\alpha$ on $\mathcal{O}_\infty$ is the free shift $\alpha$ on $\mathcal{O}_\infty$ and $\phi$ is the state of $\mathcal{O}_\infty$ defined by $\phi(w) = 0$ for each non-trivial word $w$ on $\{S_i; i \in \mathbb{Z}\}$, then*

$$h_\phi(\alpha) = 0.$$

Compare this Corollary with St$\phi$rmer's result ([S?]) that the free shift $\alpha$ on the algebra generated by the left regular representation of the free group on countably infinite generators $\{g_i\}_{i\in\mathbb{Z}}$. Then the $\alpha$ is defined by $\alpha : g_i \to g_{i+1}$ and has also same entropy 0 for the unique tracial state $\phi$.

As an application of Theorem 2.3 and Corollary 2.4, we have the following :

**2.5 Remark.** *The free shift $\alpha$ satisfies the additivity for tensor product :*

$$h_{\phi\otimes\mu}(\alpha \otimes \beta) = h_\phi(\alpha) + h_\mu(\beta),$$

*for an arbitrary automorphism $\beta$.*

This remark has a relation to a question in [28] about the entropies for the tensor product. They ask whether Connes-Narnhofer-Thirring entropy satisfies the additivity for tensor product. The negative answer is given in [20] by showing a counter example. Remark 2.5 means that it holds when one of automorphisms on nuclear $C^*$-algebras is the free shifts.

## 3. Inner automorphism on the crossed product induced by free shift

Let $(B, \beta, \mu)$ be a triplet as in section 2. Then we have the implimenting unitary $u(\beta)$ in the crossed product $B \rtimes_\beta \mathbb{Z}$. The $\beta$-invariant state $\mu$ of $B$ is extended to the state $\mu \cdot E_B$ of $B \rtimes_\beta \mathbb{Z}$, where $E_B$ is the conditional expectation of $B \rtimes_\beta \mathbb{Z}$ onto the original algebra $B$ with $E_B(u(\beta)^n) = 0$ for all non-zero $n \in \mathbb{Z}$. Then the inner automorphism $Ad(u(\beta))$ preserves the state $\mu \cdot E_B$. A general property of entropy says that we have the inequality

$$h_{\mu\cdot E_B}(Ad(u(\beta))) \geq h_\mu(\beta).$$

In [25], St$\phi$rmer asks whether we have equality here. Voiculescu shows in [29] this equality of Connes-Narnhofer-Thirring entropy for the classical Bernoulli shifts.

Here We show the equality for automorphisms related to the free shift $\alpha$. We use the same notations as in the section 2.

In this section 2, we denote simply by $E$ the conditional expectation of the crossed product onto the original algebra. We denote by $C^*(C \otimes B, u((\alpha * \gamma) \otimes \beta)$ the $C^*$-subalgebra of $((A * C) \otimes B) \rtimes_{(\alpha*\gamma)\otimes\beta} \mathbb{Z}$ generated by $C \otimes B$ and the unitary $u((\alpha * \gamma) \otimes \beta)$.

**Lemma 3.1** **([9]).** *There exists a conditional expectation $\epsilon$ of $((A*C)\otimes B) \rtimes_{(\alpha*\gamma)\otimes\beta} \mathbb{Z}$ onto $C^*(C \otimes B, u((\alpha * \gamma) \otimes \beta)$ which satisfies the following properties :*

*(1)* $((\phi * \rho) \otimes \mu) \cdot E \cdot \epsilon = ((\phi * \rho) \otimes \mu) \cdot E$

*(2)* $\epsilon(xu) = F(x)u$, *for* $x \in (A * C) \otimes B$.

*(3) For each $x \in ((A * C) \otimes B) \rtimes_{(\alpha*\gamma)\otimes\beta} \mathbb{Z}$ and any $\varepsilon > 0$, there are an $p \in \mathbb{N}$ and $n_i \in \mathbb{N}(i = 1, \cdots, p)$ so that*

$$\|\epsilon(x) = \frac{1}{p}\sum_{i=1}^{p}((\alpha * id_C) \otimes id_B)_i^n(x)\| < \varepsilon.$$

This conditional expectation $\epsilon$ plays a main role to compute the entropy. A necessary and sufficient condition that a state $\varphi$ on $((A * C) \otimes B) \rtimes_{(\alpha*\gamma)\otimes\beta} \mathbb{Z}$ is invariant under the inner automorphism $Ad(u(\alpha * \gamma) \otimes \beta)$ is that $\varphi$ rises from a state of $C^*(C \otimes B, u((\alpha * \gamma) \otimes \beta)$ by composition with the $\epsilon$. This fact corresponds to Lemma 2.1 and implies the following :

**Theorem 3.2** ([9]).

$$H_{(\phi*\mu)\cdot E}(Ad(u(\alpha * \beta))) = H_{\mu\cdot E}(Ad(u(\beta))) = H_{(\phi\otimes\mu)\cdot E}(Ad(u(\alpha \otimes \beta))).$$

*In particular,*

$$H_{\phi\cdot E}(Ad(u(\alpha))) = 0 = H_{\phi}(\alpha).$$

If we let $A$ be $\mathcal{O}_\infty$ in Theorem 3.2, then we have :

**Corollary 3.3.** *Let $\phi$ be the free state of the Cuntz algebra $O_\infty$ descrived above and let $\alpha$ be the free shift on $O_\infty$. Then*

$$h_{\phi\cdot E}(Ad(u(\alpha))) = 0 = h_{\phi}(\alpha).$$

*More generally, if $B$ is nuclear, then*

$$h_{(\phi\otimes\mu)\cdot E}(Ad(u(\alpha \otimes \beta))) = h_{\mu\cdot E}(Ad(u(\beta)))$$

*for any $\mu$-preserving automorphism $\beta$ of $B$.*

**3.4.** We apply these to the Bernoulli shift $\beta$. Let $B = C(X)$ for the space product space $X$ of $\mathbb{Z}$ copies of an $n$ point set and let $\mu$ be the state on $B$ indeced by the product measure of $\mu_0$ with $\mu_0(\cdot) = 1/n$. The Bernoulli shift $\beta$ is the shift automorphism on $B$. Voiculescu ([30]) proved that $h_{\mu\cdot E}(Ad(u(\beta))) = \log n$. We combine this result with above Corollary 2.7, then the free shift $\alpha$ of $\mathcal{O}_\infty$ and the Bernoulli shift $\beta$ satisfies the following relations :

$$\begin{aligned} h_{(\phi\otimes\mu)\cdot E}(Ad(u(\alpha \otimes \beta))) &= h_{\mu\cdot E}(Ad(u(\beta))) \\ &= \log n = h_{\mu}(\beta) = h_{\phi\otimes\mu}(\alpha \otimes \beta). \end{aligned}$$

## 4. Inner automorphism on the crossed product induced by non-commutative Bernoulli shift

In this section, we replace the free shift to the non-commutative Bernouli shift $\beta_n$ on the UHF algebra $M = \otimes_{i\in\mathbb{Z}} M_i$ of the $n \times n$-matrix algebra $M_i$ and compute the entropy for the inner automorphism $Ad(u(\beta_n))$ of $M \rtimes_{\beta_n} \mathbb{Z}$ as in the section 3.

We state the two entropies of $Ad(u(\beta_n))$. One is Connes-Narnhofer-Thirring entropy $h_\phi(\cdot)$. Another is the topological entropy $ht(\cdot)$ defined by Voiculescu ([30]). He defined the entropy $ht(\cdot)$ for automorphisms of nuclear $C^*$-algebras. This $ht(\cdot)$ does not depend on any state but is based on approximations. Similarly to the free shift, the shift $\beta_n$ does not change these entropies in the process of the crossed product. First, we compute the topological entropy for $Ad(u(\beta_n))$.

**4.1 Theorem ([7]).** *Let $\beta_n$ be the non-commutative Bernoulli shift. Then*

$$ht(Ad(u(\beta_n))) = \log n.$$

The topological entropy satisfies $ht(\cdot) \geq h_\phi(\cdot)$ for $\phi$-preserving automorphisms in general. Since $Ad(u(\beta_n))$ is the extension of $\beta_n$ and there exists a conditional expectation $E$ of the crossed product to the original algebra, we have

**4.2 Corollary ([7]).** *Then*

$$h_{\tau \cdot E}(Ad(u(\beta_n))) = \log n = h_\tau(\beta_n).$$

## 5. Entropies for Cuntz's canonical *-endomorphisms

Let $n (n \geq 2)$ be an integer. The Cuntz algebra $\mathcal{O}_n$ is the $C^*$-algebra generated by $n$ isometries $\{S_i : i = 1, 2, \cdots, n\}$ with $\sum_{i=1}^n S_i = 1$. Cuntz's canonical inner enodomorphism $\Phi$ is defined by

$$\Phi(x) = \sum_{i=1}^n S_i x S_i^*, \quad x \in \mathcal{O}_n.$$

The algebra $\mathcal{O}_n$ has the unique $\log n$-KMS state $\phi$ ([21]). Let $B$ be the half sided infinite tensor product $\otimes_{i=1}^\infty M_i$ of the $n \times n$ - matrix algebra and let $\sigma$ be the shift endomorphism of $B$ induced by the shift on the set of the natural numbers, $\alpha : i (\in \mathbb{N}) \rightarrow i+1$. Then the $\mathcal{O}_n$ is represented as the $C^*$-crossed product $B \rtimes_\sigma \mathbb{N}$ of $B$ by the corner *-endomorphism induced by $\sigma$ ([5, 16, 22, 23]). The $B \rtimes_\sigma \mathbb{N}$ is the $C^*$-algebra $C^*(B, w)$ generated by the UHF algebra $B$ and an isometry $w$ such that $wbw^* = \sigma(b)e$ $(b \in B)$, for some minimal projection $e \in M_1$. There exists a conditional expectation $E$ of $C^*(B, w)$ onto $B$ with $E(w^k) = 0$ for all $k \in \mathbb{N}$. Let $\tau$ be the unique tracial state of $B$. Then the $\log n$-KMS state $\phi$ on $C^*(B, w)$ is nothing but the state $\tau \cdot E$ and the *- endomorphism $\Phi$ on $C^*(B, w)$ is the extension of the shift $\sigma$. It is obvious that $\phi$ is $\Phi$-invariant.

In this section, we state results on the two entropies of $\Phi$.

The entropies $h_\phi(\cdot)$ and $ht(\cdot)$ are defined for automorphisms on $C^*$-algebras. However, these notions are available for unital *-endomorphisms on unital $C^*$-algebras. We replace the UHF algebra $M$ to $B$ and we take an analogy of the method to compute entropies for $Ad(u(\beta_n))$ in the section 4. Then we obtain the value of entropies of $\Phi$.

**5.1 Theorem ([7]).** *Let $\Phi$ be Cuntz's canonical inner endomorphism of $\mathcal{O}_n$. Then*

$$ht(\Phi) = \log n = h_\phi(\Phi).$$

### 5.3. Application to Longo's canonical *-endomorphism..

Let $\pi_\phi$ be the GNS representaion of $\mathcal{O}_n$ by $\phi$. We denote by $M$ the von Neumann algera generated by $\pi_\phi(\mathcal{O}_n)$. Then $\Phi$ is extended to the *-endomorphism on $M$, which we denote by $\Gamma$. Then $\Gamma$ is Longo's canonical endomorphism ([4]) and we have

$$h_\phi(\Gamma) = \log n.$$

## REFERENCES


[1] D. Avitzour: *Free products of $C^*$-algebras*, Trans. Amer. Math. Soc., **271** (1982), 423-435.

[2] M. Choda : *Entropy for *-endomorphisms and relative entropy for subalgebras*, J. Operator Theory, **25** (1991), 125-140.

[3] M. Choda : *Entropy for canonical shifts*, Trans. of Amer. Math. Soc, **334** (1992), 827-849.

[4] M. Choda : *Extension algebras of $II_1$ factors via *-endomorphisms*, Internat. J. Math., **5** (1994), 635-655.

[5] M. Choda : *Canonical *-endomorphisms and simple $C^*$- algebras*, J. Operator Theory, **33** (1995), 235-248.

[6] M. Choda : *Reduced free products of completely positive maps and entropy for free products of automorphisms*, Publ. RIMS, Kyoto Univ., **32** (1996), 179-190.

[7] M. Choda : *Entropy of Cuntz's canonical endmorphisms*, Preprint, (1996).

[8] M. Choda and F. Hiai: *Entropy for canonical shifts II*, Publ. RIMS, Kyoto Univ., **27** (1991), 461-489.

[9] M. Choda and T. Natume : *Reduced $C^*$-crossed products by free shifts*, Preprint, (1996).

[10] A. Connes and E. St$\phi$rmer: *Entropy of $II_1$ von Neumann algebras*, Acta Math., **134** (1975), 289-306.

[11] A. Connes, H. Narnhofer and W. Thirring: *Dynamical entropy of $C^*$ algebras and von Neumann algebras*, Comm. Math. Phys., **112** (1987), 691-719.

[12] J. Cuntz: *Simple $C^*$-algebras generated by isometries*, Comm. Math. Phys., **57** (1977), 173-185.

[13] J. Cuntz : *Regular actions of Hopf algebras on the $C^*$-algebra generated by a Hilbert space*, in "Operator algebras, mathematical physics, and low dimensional topology (Istanbul, 1991)" Res, Notes Math. Vol. **5**, Wellesley, MA, (1993), 87-100.

[14] F. Hiai: *Entropy and growth for derived towers of subfactors*,in Subfactors, World Scientific, Singapore, (1994), 206-232.

[15] F. Hiai : *Entropy for canonical shifts and strong amenability*, Internat. J. Math., **6**, (1995), 381-396.

[16] M. Izumi : *Subalgebras of infinite $C^*$-algebras with finite indices. II. Cuntz-Krieger algebras*, Preprint.

[17] R. Longo : *Index of subfactors and statistics of quantum fields. I*, Commun. Math. Phys., **126** (1989), 145-155.

[18] R. Longo : *Duality for Hopf algebras and for subfactors*, Comm. Math. Phys., **159** (1994), 133-150.

[19] R. Longo and J.E. Roberts : *A theory of dimension*, Preprint.

[20] H.Narnhofer, E. St$\phi$rmer and W. Thirring : *$C^*$-dynamical systems for which the tensor product formula for entropy fails*, Ergod. Th. & Dynam. Sys., **15**, (1995), 961-968.

[21] D. Olesen and G. K. Pedersen : *Some $C^*$-dynamical systems with a single KMS state*, Math. Scand., **42** (1978), 111-118.

[22] W. Paschke : *The crossed product of a $C^*$-algebra by an endomorphism*, Proc. Amer. Math. Soc., **80** (1980), 113-118.

[23] M. R$\phi$rdam : *Classification of certain infinite simple $C^*$-algebras*, J. Funct. Anal., **131** (1995), 415-458.

[24] J-L. Sauvageot and J-P Thouvenot: *Une nouvelle définition de l'entropie dynamique des systèmes non commutatifs*, Commun. Math. Phys., **145** (1992), 411-423.

[25] E. St$\phi$rmer : *Entropy in operator algebras*, Preprint, (1992).

[26] E. St$\phi$rmer : *Entropy of some automorphisms of the $II_1$ factor of the free group in infinite number of generators*, Invent. Math., **110** (1992), 63-73.

[27] E. St$\phi$rmer : *States and shifts on infinite free products of $C^*$-algebras*, Pre-print, University of Oslo, (1995).

[28] E. St$\phi$rmer, D. Voiculescu : *Entropy of Bogoliubov automorphisms of the canonical anticommutation relations*, Commun. Math. Phys., **133**, (1990), 521-542.

[29] D. Voiculescu : *Symmetries of some reduced free product $C^*$- algebras*, Operator Algebras and Their Connection with Topology and Ergodic Theory (Lecture Notes in Mathematics, Vol. **1132**). Springer Verlag, (1985), 556-588.

[30] D. Voiculescu: *Dynamical approximation entropies and topological entropy in operator algebras*, Commun. Math. Phys., **170** (1995), 249-281.

[31] D. Voiculescu, K. Dykema and A. Nica : *Free random variables*, (CRM Monograph Vol. **1**). AMS, (1992).