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量子系の力学的エントロピーについて

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Abstract

Classical dynamical entropy is an important tool to analyse the efficiency of information transmission in communication processes.

Quantum dynamical entropy was first studied by Connes - Stormer and Emch. Since then, there have been many attempts to formulate or compute the dynamical entropy for some models. Here we review four formulations due to Connes - Narnhofer - Thirring, Ohya, Accardi - Ohya - Watanabe, Alicki - Fannes and consider the relations among these formulations. We show some concrete computations in a model.

Introduction

Classical dynamical (or Kolmogorov - Sinai) entropy $S(T)$ for a measure preserving transformation T was defined on a message space through finite partitions of the measurable space. The classical coding theorems of Shannon are important tools to analyse communication processes, which have been formulated by the mean dynamical entropy and the mean dynamical mutual entropy. The mean dynamical entropy represents the amount of information per one letter of a signal sequence sent from an input source and the mean dynamical mutual entropy does the amount of information per one letter of the signal received in an output system.

Quantum dynamical entropy (QDE for short) has been studied by Connes, Stormer [11], Emch [12], Connes, Narnhofer, Thirring [10], Alicki, Fannes [6] and others [8,23]. Their dynamical entropies were defined in the observable spaces.

Recently, the quantum dynamical entropy and the quantum dynamical mutual entropy were studied by one of the present authors [24,15]. They are formulated in the state spaces through the complexity of Information Dynamis [22,24]. Furthermore, another formulation of the dynamical entropy through the quantum Markov chain (QMC for short) was done in [4].

In §1, we briefly review the formulation by Connes-Narnhofer-Thirring (CNT for short). In §2, we explain the formulation by the complexity [24,25]. In §3, we review the formulation by QMC. In §4, we briefly explain the formulation by Alicki-Fannes (AF for short). In §5, we discuss the relations among four formulations. In §6, we compute the mean entropies in quantum communication processes.

§1. CNT Formulation

Let $(\mathcal{A}, \theta, \varphi)$ be an initial C*-system. That is, $\mathcal{A}$ is a unital C^*-algebra, θ is an automorphism of $\mathcal{A}$, and φ is an invariant state over $\mathcal{A}$ with respect to θ; $\varphi \circ \theta = \varphi$. Let $\mathcal{B}$ be a finite dimensional C* - subalgebra of $\mathcal{A}$.

The CNT entropy functional [10] for a subalgebra $\mathcal{B}$ is

$$H_\varphi(\mathcal{B}) = \sup \left\{ \sum_k \lambda_k S(\omega_k \mid \mathcal{B}, \varphi \mid \mathcal{B}) ; \varphi = \sum_k \lambda_k \omega_k \text{ finite decomposition of } \varphi \right\}.$$

where $\varphi \mid \mathcal{B}$ is the restriction of a state φ to $\mathcal{B}$ and $S(\cdot, \cdot)$ is the relative entropy for C^*-algebra [7,27,28].

The CNT dynamical entropy with respect to θ and $\mathcal{B}$ is given by

$$\tilde{H}_\varphi(\theta, \mathcal{B}) = \limsup_{N \to \infty} \frac{1}{N} H_\varphi(\mathcal{B} \vee \theta\mathcal{B} \vee \cdots \vee \theta^{N-1}\mathcal{B}).$$

The dynamical entropy for θ is defined by

$$\tilde{H}_\varphi(\theta) = \sup_{\mathcal{B}} \tilde{H}_\varphi(\theta, \mathcal{B}),$$

§2. Formulation by Complexity

In this section, we first review concepts of channel and complexity, which are the key concepts of ID (Information Dynamics) introduced by Ohya [22,24].

Let $(\mathcal{A}, \Sigma(\mathcal{A}), \alpha(G))$, $(\overline{\mathcal{A}}, \overline{\Sigma}(\overline{\mathcal{A}}), \overline{\alpha}(\overline{G}))$ be an input (initial) and an output (final) C*-systems, respectively, where $\mathcal{A}$ (resp. $\overline{\mathcal{A}}$) is a unital C^*-algebra, $\Sigma(\mathcal{A})$ (resp. $\overline{\Sigma}(\overline{\mathcal{A}})$) is the set of all states on $\mathcal{A}$ (resp. $\overline{\mathcal{A}}$) and $\alpha(G)$ (resp. $\overline{\alpha}(\overline{G})$) is the group of automorphisms of $\mathcal{A}$ (resp. $\overline{\mathcal{A}}$) indexed by a group G (resp. $\overline{G}$).

A channel is a map Λ^* from $\Sigma(\mathcal{A})$ to $\overline{\Sigma}(\bar{\mathcal{A}})$. If the dual map Λ from $\bar{\mathcal{A}}$ to $\mathcal{A}$ of Λ^* satisfies the complete positivity, the channel Λ^* is called a complete positive channel (CP channel for short).

For a weak * compact convex subset $\mathcal{S}$ of Σ, there exists a maximum measure μ with the barycenter φ such that

$$\varphi = \int_{\mathcal{S}} \omega \, d\mu$$

The compound state introduced in [18,19] exhibiting the correlation between an initial φ and its final $\Lambda^*\varphi$ is given by

$$\mathcal{E}_\mu^* \varphi = \int_{\mathcal{S}} \omega \otimes \Lambda^* \omega \, d\mu$$

In the sequel, we use a CP channel Λ^* and the compound state to formulate the dynamical entropy.

There are two types of complexity in ID. One is a complexity $C_T^{\mathcal{S}}(\varphi)$ of a system itself and another is a transmitted complexity $T^{\mathcal{S}}(\varphi\,;\Lambda^*)$ from an initial system to a final system. These complexities should satisfy the following conditions:

(i) For any $\varphi \in \mathcal{S} \subset \Sigma$,

$$C^{\mathcal{S}}(\varphi) \geq 0, \quad T^{\mathcal{S}}(\varphi\,;\Lambda^*) \geq 0,$$

(ii) If there exists bijection $j : ex\Sigma \to ex\Sigma$; the set of all extremal points in Σ, then

$$C^{j(\mathcal{S})}(j(\varphi)) = C^{\mathcal{S}}(\varphi)$$
$$T^{j(\mathcal{S})}(j(\varphi)\,;\Lambda^*) = T^{\mathcal{S}}(\varphi\,;\Lambda^*)$$

(iii) For a state $\Psi = \varphi \otimes \psi \in \mathcal{S}_t$, put $\varphi \in \mathcal{S}$, $\psi \in \overline{\mathcal{S}}$,

$$C^{\mathcal{S}_t}(\Phi) = C^{\mathcal{S}}(\varphi) + C^{\overline{\mathcal{S}}}(\psi)$$

(iv) $0 \leq T^{\mathcal{S}}(\varphi\,;\Lambda^*) \leq C^{\mathcal{S}}(\varphi)$

(v) $T^{\mathcal{S}}(\varphi\,;id) = C^{\mathcal{S}}(\varphi)$

Here we explain the formulation of three types of entropic complexity introduced in [24].

Let $(\mathcal{A},\ \Sigma(\mathcal{A}),\ \alpha(G))$, $(\overline{\mathcal{A}},\ \overline{\Sigma}(\overline{\mathcal{A}}),\ \overline{\alpha}(\overline{G}))$ and $\mathcal{S}$ as above. Let $M_\varphi(\mathcal{S})$ be the set of all maximal measures μ on $\mathcal{S}$ with the fixed barycenter φ and $F_\varphi(\mathcal{S})$ be the set of all measures of finite support with the fixed barycenter φ. Then three pairs of complexities are

$$T^{\mathcal{S}}(\varphi\,;\Lambda^*) \equiv \sup\left\{\int_{\mathcal{S}} S(\Lambda^*\omega, \Lambda^*\varphi)\,d\mu\,;\ \mu \in M_\varphi(\mathcal{S})\right\}$$
$$C_T^{\mathcal{S}}(\varphi) \equiv T^{\mathcal{S}}(\varphi\,;\ id)$$
$$I^{\mathcal{S}}(\varphi\,;\Lambda^*) \equiv \sup\left\{S\left(\int_{\mathcal{S}} \omega\otimes\Lambda^*\omega d\mu,\ \varphi\otimes\Lambda^*\varphi\right);\ \mu \in M_\varphi(\mathcal{S})\right\}$$
$$C_I^{\mathcal{S}}(\varphi) = I^{\mathcal{S}}(\varphi\,;\ id)$$
$$J^{\mathcal{S}}(\varphi\,;\Lambda^*) \equiv \sup\left\{\int_{\mathcal{S}} S(\Lambda^*\omega, \Lambda^*\varphi)\,d\mu_f;\ \mu_f \in F_\varphi(\mathcal{S})\right\}$$
$$C_J^{\mathcal{S}}(\varphi) \equiv J^{\mathcal{S}}(\varphi\,;\ id)\,.$$

Based on the above complexities, we explain the formulation of quantum dynamical complexity (QDC) [18].

Let θ (resp. $\bar{\theta}$) be a stationary automorphism of $\mathcal{A}$ (resp. $\bar{\mathcal{A}}$); $\varphi\circ\theta = \varphi$, and Λ be a covariant CP map (i.e., $\Lambda\circ\theta = \bar{\theta}\circ\Lambda$) from $\bar{\mathcal{A}}$ to $\mathcal{A}$. $\mathcal{B}_k$ (resp. $\bar{\mathcal{B}}_k$) is a finite subalgebra of $\mathcal{A}$ (resp. $\bar{\mathcal{A}}$). Moreover, let α_k (resp. $\bar{\alpha}_k$) be a CP unital map from $\mathcal{B}_k$ (resp. $\bar{\mathcal{B}}_k$) to $\mathcal{A}$ (resp. $\bar{\mathcal{A}}$) and α^M and $\bar{\alpha}_\Lambda^N$ are given by

$$\alpha^M = (\alpha_1,\, \alpha_2,\, \cdots, \alpha_M)\,,$$
$$\bar{\alpha}_\Lambda^N = (\Lambda\circ\bar{\alpha}_1,\, \Lambda\circ\bar{\alpha}_2,\, \cdots, \Lambda\circ\bar{\alpha}_N)$$

The two compound states for α^M and $\bar{\alpha}_\Lambda^N$ with respect to $\mu \in M_\varphi(\mathcal{S})$ are defined such as

$$\Phi_\mu^{\mathcal{S}}\left(\alpha^M\right) = \int_{\mathcal{S}} \overset{M}{\underset{m=1}{\otimes}} \alpha_m^* \omega d\mu,$$

$$\Phi_\mu^{\mathcal{S}}\left(\alpha^M \cup \bar{\alpha}_\Lambda^N\right) = \int_{\mathcal{S}} \overset{M}{\underset{m=1}{\otimes}} \alpha_m^* \omega \overset{N}{\underset{n=1}{\otimes}} \bar{\alpha}_n^* \Lambda^* \omega d\mu.$$

By using the above compound states, three transmitted complexities [24] are defined by

$$\begin{aligned}
&T_\varphi^{\mathcal{S}}\left(\alpha^M, \bar{\alpha}_\Lambda^N\right) \\
&\equiv \sup\{\int_{\mathcal{S}} S(\overset{M}{\underset{m=1}{\otimes}} \alpha_m^* \omega \overset{N}{\underset{n=1}{\otimes}} \bar{\alpha}_n^* \Lambda^* \omega,\ \Phi_\mu^{\mathcal{S}}(\alpha^M) \otimes \Phi_\mu^{\mathcal{S}}(\bar{\alpha}_\Lambda^N)) d\mu;\ \mu \in M_\varphi(S)\} \\
&I_\varphi^{\mathcal{S}}(\alpha^M, \bar{\alpha}_\Lambda^N) \equiv \sup\{S(\Phi_\mu^{\mathcal{S}}(\alpha^M \cup \bar{\alpha}_\Lambda^N),\ \Phi_\mu^{\mathcal{S}}(\alpha^M) \otimes \Phi_\mu^{\mathcal{S}}(\bar{\alpha}_\Lambda^N))\ ;\ \mu \in M_\varphi(\mathcal{S})\} \\
&J_\varphi^{\mathcal{S}}(\alpha^M, \bar{\alpha}_\Lambda^N) \\
&\equiv \sup\{\int_{\mathcal{S}} S(\overset{M}{\underset{m=1}{\otimes}} \alpha_m^* \omega \overset{N}{\underset{n=1}{\otimes}} \bar{\alpha}_n^* \Lambda^* \omega,\ \Phi_\mu^{\mathcal{S}}(\alpha^M) \otimes \Phi_\mu^{\mathcal{S}}(\bar{\alpha}_\Lambda^N)) d\mu_f;\ \mu_f \in F_\varphi(\mathcal{S})\}
\end{aligned}$$

When $\mathcal{B}_k = \bar{\mathcal{B}}_k = \mathcal{B}$, $\mathcal{A} = \bar{\mathcal{A}}$, $\theta = \bar{\theta}$, $\alpha_k = \theta^{k-1} \circ \alpha = \bar{\alpha}_k$, where α is a unital CP map from $\mathcal{A}_0$ to $\mathcal{A}$, the mean transmitted complexities are

$$\tilde{T}_\varphi^{\mathcal{S}}\left(\theta, \alpha, \Lambda^*\right) \equiv \limsup_{N \to \infty} \frac{1}{N} T_\varphi^{\mathcal{S}}\left(\alpha^N, \bar{\alpha}_\Lambda^N\right),$$

$$\tilde{T}_\varphi^{\mathcal{S}}\left(\theta, \Lambda^*\right) \equiv \sup_\alpha \tilde{T}_\varphi^{\mathcal{S}}\left(\theta, \alpha, \Lambda^*\right).$$

Same for $\tilde{I}_\varphi^{\mathcal{S}}, \tilde{J}_\varphi^{\mathcal{S}}$. These quantities have the similar properties of the CNT entropy [15,24].

§3. Formulation by QMC

Another formulation of the dynamical entropy is due to quantum Markov chain [4].

Let $\mathcal{A}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, φ be a state on $\mathcal{A}$ and $\mathcal{A}_0 = M_d$ ($d \times d$ matrix algebra). Take the transition expectation $\mathcal{E}_\gamma : \mathcal{A}_0 \otimes \mathcal{A} \to \mathcal{A}$ of Accardi [1,2] such that

$$\mathcal{E}_\gamma(\tilde{A}) = \sum_i \gamma_i A_{ii} \gamma_i$$

where $\tilde{A} = \sum_{i,j} e_{ij} \otimes A_{ij} \in \mathcal{A}_0 \otimes \mathcal{A}$ and $\gamma = \{\gamma_j\}$ is a finite partition of unity $I \in \mathcal{A}$. Quantum Markov chain is defined by $\psi \equiv \{\varphi, \mathcal{E}_{\gamma,\theta}\} \in \Sigma(\overset{\infty}{\underset{1}{\otimes}} \mathcal{A}_0)$ such that

$$\psi(j_1(A_1) \cdots j_n(A_n)) \equiv \varphi(\mathcal{E}_{\gamma,\theta}(A_1 \otimes \mathcal{E}_{\gamma,\theta}(A_2 \otimes \cdots \otimes A_{n-1} \mathcal{E}_{\gamma,\theta}(A_n \otimes I) \cdots)))$$

where $\mathcal{E}_{\gamma,\theta} = \theta \circ \mathcal{E}_\gamma$, $\theta \in Aut(\mathcal{A})$ and j_k is the embeding from $\mathcal{A}_0$ to $\overset{\infty}{\underset{1}{\otimes}} \mathcal{A}_0$ such as $j_k(A) = I \otimes \cdots \otimes I \otimes \underset{k-th}{A} \otimes I \cdots$.

Suppose that φ has a unique density operator ρ such as $\varphi(A) = tr\rho A$, for any $A \in \mathcal{A}$. Let define ψ_n the state on $\overset{n}{\underset{1}{\otimes}} \mathcal{A}_0$ expressed as

$$\psi_n(A_1 \otimes \cdots \otimes A_n) = \psi(j_1(A_1) \cdots j_n(A_n)).$$

The density operator ξ_n of ψ_n is given by

$$\xi_n \equiv \sum_{i_1} \cdots \sum_{i_n} tr_{\mathcal{A}}(\theta^n(\gamma_{i_n}) \cdots \gamma_{i_1} \rho \gamma_{i_1} \cdots \theta^n(\gamma_{i_n}))) e_{i_1 i_1} \otimes \cdots \otimes e_{i_n i_n}.$$

Put

$$P_{i_n \cdots i_1} = tr_{\mathcal{A}}(\theta^n(\gamma_{i_n}) \cdots \gamma_{i_1} \rho \gamma_{i_1} \cdots \theta^n(\gamma_{i_n}))).$$

The mean dynamical entropy [4] through QMC is defined by

$$\begin{aligned}\tilde{S}_{\varphi}(\theta; \gamma) &\equiv \limsup_{n \to \infty} \frac{1}{n} \Big(-tr \xi_n \log \xi_n \Big), \\ &= \limsup_{n \to \infty} \frac{1}{n} \Big(- \sum_{i_1, \cdots, i_n} P_{i_n \cdots i_1} \log P_{i_n \cdots i_1} \Big).\end{aligned}$$

When $P_{i_n \cdots i_1}$ satisfies the Markov property, the above equality is written by

$$\tilde{S}_{\varphi}(\theta; \gamma) = - \sum_{i_1, i_2} P(i_2 | i_1) P(i_1) \log P(i_2 | i_1).$$

The dynamical entropy through QMC with respect to θ and a von Neumann subalgebra $\mathcal{B}$ of $\mathcal{A}$ is

$$\tilde{S}_{\varphi}(\theta; \mathcal{B}) \equiv \sup \{ \tilde{S}_{\varphi}(\theta; \gamma); \gamma \subset \mathcal{B} \}.$$

§4. Formulation by AF

Let $(\mathcal{A}, \varphi, \theta)$ be a C*-dynamical system, where $\mathcal{A}$ is a C*-algebra, θ is an automorphism on $\mathcal{A}$ and φ is a θ-invariant state. Let $\mathcal{B}$ be a unital *-subalgebra of $\mathcal{A}$. A set $\gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_k\}$ of elements of $\mathcal{B}$ is called a finite operational partition of unity of size k if γ satisfies the following conditions:

$$\sum_{i=1}^{k} \gamma_i^* \gamma_i = I \tag{4.1}$$

An operation $\circ$ is defined by

$$\gamma \circ \xi \equiv \{ \gamma_i \xi_j ; \ i = 1, 2, \cdots, k, \ j = 1, 2, \cdots, l \}$$

for any partitions $\gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_k\}$ and $\xi = \{\xi_1, \xi_2, \cdots, \xi_l\}$. For any partition γ of size k, a $k \times k$ density matrix $\rho[\gamma] = (\rho[\gamma]_{i,j})$ is given by

$$\rho[\gamma]_{i,j} = \varphi(\gamma_j^* \gamma_i).$$

Then the dynamical entropy $\tilde{H}_\varphi(\theta, \mathcal{B}, \gamma)$ with respect to the partition γ and the shift θ is defined by

$$\tilde{H}_\varphi(\theta, \mathcal{B}, \gamma) = \limsup_{n\to\infty} \frac{1}{n} S(\rho[\theta^{n-1}(\gamma) \circ \cdots \circ \theta(\gamma) \circ \gamma]). \tag{4.2}$$

The dynamical entropy $\tilde{H}_\varphi(\theta, \mathcal{B})$ is obtained by taking supremum over operational partition of unity in $\mathcal{B}$ as

$$\tilde{H}_\varphi(\theta, \mathcal{B}) = \sup\left\{\tilde{H}_\gamma(\mathcal{A}_0, \theta, \varphi); \gamma \in \mathcal{B}\right\}. \tag{4.3}$$

§5. Relations Among Four Formulations

In tis section, we discuss the relations among the above three formulations.

The $\mathcal{S}$ - mixing entropy in GQS introduced in [21] is

$$S^{\mathcal{S}}(\varphi) = \inf\{H(\mu) \ ; \ \mu \in M_\varphi(\mathcal{S})\},$$

where $H(\mu)$ is given by

$$H(\mu) = \sup\left\{ -\sum_{A_k \in \tilde{A}} \mu(A_k) \log \mu(A_k) \ : \ \tilde{A} \in P(\mathcal{S}) \right\}$$

and $P(\mathcal{S})$ is the set of all finite partitions of $\mathcal{S}$.

The following theorem [15,24] shows the relation between the CNT formulation and the formulation by the complexity.

Theorem 4.1 Under the above settings, we have the following relations:
(1) $0 \le I^{\mathcal{S}}(\varphi\,;\Lambda^*) \le T^{\mathcal{S}}(\varphi\,;\Lambda^*) \le J^{\mathcal{S}}(\varphi\,;\Lambda^*)$
(2) $C_I^{\Sigma}(\varphi) = C_T^{\Sigma}(\varphi) = C_J^{\Sigma}(\varphi) = S^{\Sigma}(\varphi) = H_\varphi(\mathcal{A})$
(3) $\mathcal{A} = \bar{\mathcal{A}} = B(\mathcal{H})$, for any density operator ρ

$$0 \le I^{\mathcal{S}}(\rho\,;\Lambda^*) = T^{\mathcal{S}}(\rho;\Lambda^*) \le J^{\mathcal{S}}(\rho\,;\Lambda^*)$$

Since there exists a model showing $S^{I(\alpha)}(\varphi) \ge H_\varphi(\mathcal{A}_\alpha)$, $S^{\mathcal{S}}(\varphi)$ distinguishes states more sharply than $H_\varphi(\mathcal{A})$, where $\mathcal{A}_\alpha = \{A \in \mathcal{A};\ \alpha(A) = A\}$.

Furthermore we have the following results [25].

(1) When $\mathcal{A}_n, \mathcal{A}$ are the abelian C*-algebras and α_k is an embedding map,

$$T^{\Sigma}(\mu; \alpha^M) = S_\mu^{\text{classical}}\left(\bigvee_{m=1}^{M} \tilde{A}_m\right)$$

$$I^{\Sigma}(\mu; \alpha^M, \bar{\alpha}^N) = I_\mu^{\text{classical}}\left(\bigvee_{m=1}^{M} \tilde{A}_m, \bigvee_{n=1}^{N} \tilde{B}_n\right)$$

are satisfied for any finite partitions $\tilde{A}_m, \tilde{B}_n$ on the probability space ($\Omega = \text{spec}(\mathcal{A})$, $\mathcal{F}$, μ).

(2) When Λ is the restriction of $\mathcal{A}$ to a subalgebra $\mathcal{M}$ of $\mathcal{A}$; $\Lambda = |\mathcal{M}$,

$$H_\varphi(\mathcal{M}) = J^\Sigma(\varphi; |\mathcal{M}) = J_\varphi^\Sigma(id; |\mathcal{M}).$$

Moreover when

$$\mathcal{N} \subset \mathcal{A}_0, \mathcal{A} = \overset{\mathbb{N}}{\otimes} \mathcal{A}_0, \theta \in Aut(\mathcal{A});$$

$$\alpha^N \equiv (\alpha, \theta \circ \alpha, \cdots; \theta^{N-1} \circ \alpha);$$

$$\alpha = \bar{\alpha}; \mathcal{A}_0 \to \mathcal{A} \text{ embedding};$$

$$\mathcal{N}_N \equiv \overset{N}{\underset{1}{\otimes}} \mathcal{N},$$

we have

$$\tilde{H}_\varphi(\theta; \mathcal{N}) = \tilde{J}_\varphi^\Sigma(\theta; \mathcal{N}) = \limsup_{N \to \infty} \frac{1}{N} J_\varphi^\Sigma(\alpha^N; |\mathcal{N}_N).$$

We have the same results for $\tilde{T}_\varphi^{\mathcal{S}}(\theta), \tilde{I}_\varphi^{\mathcal{S}}(\theta)$.

We show the relation between the formulation by complexity and that by QMC.

Under the same settings in § 3, we define a map $\mathcal{E}_{(n)}^*$ from $\Sigma(\mathcal{A})$ to $\Sigma((\overset{n}{\underset{1}{\otimes}} \mathcal{A}_0) \otimes \mathcal{A})$ by

$$\mathcal{E}_{(n,\gamma)}^*(\varphi)(A_1 \otimes \cdots \otimes A_n \otimes I) = \varphi(\mathcal{E}_{\gamma,\theta}(A_1 \otimes \mathcal{E}_{\gamma,\theta}(A_2 \otimes \cdots \otimes A_{n-1}\mathcal{E}_{\gamma,\theta}(A_n \otimes I) \cdots)))$$

for any $A_1 \otimes \cdots \otimes A_n \otimes I \in (\overset{n}{\underset{1}{\otimes}} \mathcal{A}_0) \otimes \mathcal{A}$. Take a map $E_{(n)}^*$ from $\Sigma((\overset{n}{\underset{1}{\otimes}} \mathcal{A}_0) \otimes \mathcal{A})$ to $\Sigma(\overset{n}{\underset{1}{\otimes}} \mathcal{A}_0)$ such that

$$(E_{(n)}^*\omega)(Q) = \omega(Q \otimes I), \quad \forall Q \in \overset{n}{\underset{1}{\otimes}} \mathcal{A}_0, \quad \forall \omega \in \Sigma((\overset{n}{\underset{1}{\otimes}} \mathcal{A}_0) \otimes \mathcal{A}).$$

Then a channel $\Gamma_{(n,\gamma)}^*$ from $\Sigma(\mathcal{A})$ to $\Sigma(\overset{n}{\underset{1}{\otimes}} \mathcal{A}_0)$ is given by

$$\Gamma_{(n)}^* \equiv E_{(n)}^* \circ \mathcal{E}_{(n,\gamma)}^*$$

so that $\Gamma_{(n,\gamma)}^*(\varphi) = \psi_n$ and

$$\tilde{S}_\varphi(\theta; \gamma) = \limsup_{n \to \infty} \frac{1}{n} S(\Gamma_{(n,\gamma)}^* \varphi).$$

Therefore we have

$$\tilde{S}_\varphi(\theta; \gamma) = \tilde{C}_I^\Sigma(\Gamma_{(\gamma)}^* \varphi) (= \limsup_{n \to \infty} \frac{1}{n} C_I^\Sigma(\Gamma_{(n,\gamma)}^* \varphi))$$

We briefly show the relation between the formulation by complexity and that by AF.

Let $(\mathcal{A}, \theta, \varphi)$ be a C*-dynamical system and $\gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_k\}$ be a finite operational partition of unity of size k.

We define a channel $\Xi^*_{(m,\gamma)}$ from $\Sigma(\mathcal{A})$ to $\Sigma(\mathcal{A})$ by

$$\Xi^*_{(m,\gamma)}(\varphi)(A) = \varphi\Big([\theta^{m-1}(\gamma) \circ \cdots \circ \theta(\gamma) \circ \gamma]A\Big)$$

for any $\varphi \in \Sigma(\mathcal{A})$ and any $A \in \mathcal{A}$. The dynamical entropy by AF is given by

$$\tilde{H}_\varphi(\theta, \mathcal{B}, \gamma) = \limsup_{m \to \infty} \frac{1}{m} S(\Xi^*_{(m,\gamma)}\varphi).$$

Therefore we have

$$\tilde{H}_\varphi(\theta, \mathcal{B}, \gamma) = \tilde{C}^\Sigma_I(\Xi^*_{(\gamma)}\varphi)(= \limsup_{m \to \infty} \frac{1}{m} C^\Sigma_I(\Xi^*_{(m,\gamma)}\varphi)).$$

In any case, the formulation by the entropic complexities contains other formulations, moreover it opens other possibility to classify the dynamical systems more fine [25].

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