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Some questions about the index of quantized contact transformations

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1 Introduction

If M_1 and M_2 are compact differentiable manifolds, a contact diffeomorphism ϕ between their cosphere bundles gives rise to a class $C(\phi)$ of Fredholm operators, called *Fourier integral operators* or *quantized contact transformations* between the Hilbert spaces of L^2 functions (or, more invariantly, half densities) on M_1 and M_2. The question of whether there is a unitary operator in this class was raised in [20], where such operators were used to approximately intertwine the laplacians on riemannian manifolds with symplectically equivalent geodesic flows. It was shown there that the existence of the unitary operator was equivalent to the vanishing of the index of operators in $C(\phi)$, and the problem of finding a topological formula for the index of the operators in $C(\phi)$ was posed. A conjecture for such a formula was made by M. Atiyah in a conversation with the author at some time in the mid-1970's. Little progress has been made since then, partly because it

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is hard to produce examples where the index even has a chance of being non-zero.

Recent developments in analysis and symplectic geometry have suggested generalizations of this index problem to settings where non-zero indices are known to exist, and technical advances in analysis seem to have brought a solution within reach. This talk will give an overview of the problem and describe prospects for its solution in the context of Epstein's relative index for CR structures [7]. Work of Guillemin [11] using analysis on Grauert tubes implies that our original index problem can be set in this context.

Much of this paper is speculative in nature. It is in part a report on ongoing discussions (in person and by electronic mail) with David Borthwick, Ana Cannas da Silva, Charles Epstein, Victor Guillemin, and Steven Zelditch. I would like to thank all of them for their contributions to this project. In addition, I have received helpful advice from Michael Christ, Peter Gilkey, Ian Grojnowski, Janos Kollar, Richard Melrose, Gregory Sankaran, Bernard Shiffman, and Sidney Webster.

2 Polarizations of contact manifolds

In this section, we will see how the index problem for Fourier integral operators can be considered as a version of the question "how does the quantum Hilbert space depend on the polarization?" which is central to the theory of geometric quantization. First of all, we will recall how the notions of geometric quantization are transplanted to contact manifolds from their usual symplectic setting. This discussion is very much inspired by the work of Boutet de Monvel and Guillemin [5].

Let Y be a contact manifold, $C \subset TY$ the contact distribution. The bracket of sections of C determines a natural nondegenerate 2-form Ω on C with values in the normal line bundle TY/C. A **polarization** of Y is defined to be a complex subbundle $\mathcal{J}$ of the complexification $C_{\mathbf{C}}$ such that:

- (the natural complex extension of) Ω is zero on $\mathcal{J}$;
- $\dim \mathcal{J} = \frac{1}{2} \dim C_{\mathbf{C}}$;
- $[\Gamma(\mathcal{J}), \Gamma(\mathcal{J})] \subseteq \Gamma(\mathcal{J})$.

One should add a further condition relating $\mathcal{J}$ and $\overline{\mathcal{J}}$, analogous to that in the symplectic case, but it will be automatically satisfied in the two extreme cases which will interest us in this paper.

The "quantum Hilbert space" associated to the polarization $\mathcal{J}$ is obtained by taking the space of smooth functions on Y which are annihilated by all sections of $\mathcal{J}$, and then taking its closure $H_{\mathcal{J}}$ in $L^2(Y)$ (defined with the aid of a chosen volume element on Y). A fundamental problem in geometric quantization theory is to relate the Hilbert spaces arising from different polarizations of the same contact manifold. In our setting, these spaces are infinite-dimensional, but we can define the "difference between the dimensions" of two such spaces as the index of the orthogonal projection operator (in $L^2(Y)$) from one space to the other. We will call this index the **relative index** of the two polarizations. We will see that, in many cases, the projection operator is Fredholm, so that the relative index is finite , and we will propose a topological formula for computing it.

Our basic idea is to associate to each polarization $\mathcal{J}_i$ of a compact contact manifold Y some "filling" of Y, i.e. some compact manifold X_i having Y as its boundary. The relative index of two polarizations, defined provisionally as the index of the orthogonal projection from one quantum Hilbert space to the other, should then be the index of a Dirac operator on the manifold obtained by gluing the two fillings along Y. This is our **gluing conjecture**.

3 Complex polarizations

A polarization $\mathcal{J}$ is called a **complex polarization** if $\mathcal{J}$ and $\overline{\mathcal{J}}$ are complementary subbundles. Such polarizations are also known as (nondegenerate) **CR** (or Cauchy-Riemann) **structures**.[1] These complex polarizations almost complex structures J on the vector bundle C by the rule $\mathcal{J} = \{x - iJx | x \in C\}$. The condition $[\Gamma(\mathcal{J}), \Gamma(\mathcal{J})] \subseteq \Gamma(\mathcal{J})$ is the usual integrability condition for CR structures.

For a complex polarization of CR type, the smooth functions annihilated by the sections of $\mathcal{J}$ are generally known as CR functions. Their closure H_J in $L^2(Y)$ is essentially

[1]For the most general CR structures, $C \subset TY$ may be any distribution of codimension 1, not necessarily contact.

independent of the choice of volume element on Y and is called the **Hardy space** of the CR structure. The orthogonal projection onto this quantum Hilbert space does depend on the volume element and is known as the **Szego projector**.

An important supplementary condition on complex polarizations is strict **pseudoconvexity**, which is definiteness of the TY/C-valued **Levi form** on C defined by $(x,y) \mapsto \Omega(Jx,y)$. As in the symplectic case, the vanishing of Ω on $\mathcal{J}$ means that this form is symmetric and J-invariant. It is usual to suppose further that the normal bundle TY/C has a prescribed orientation, in which case it makes sense to require that the Levi form be *positive* definite; in the negative case, we speak of strict **pseudoconcavity**. Following standard terminology in the symplectic case, we will call a strictly pseudoconvex complex polarization a **positive** polarization.

We note that the space of adapted complex structures on a symplectic vector space, i.e. those for which the form $(x,y) \mapsto \Omega(Jx,y)$ is positive definite and symmetric, is contractible. Any two such almost complex structures are related by a transformation which preserves Ω (which is therefore unitary); furthermore, this transformation can be chosen in a "natural" way if one uses the riemannian geometry of the symmetric space $Sp(2n-2)/U(n-1)$ to select the geodesic connecting the two structures and then lift it to the symplectic group.

A CR structure is called **embeddable** if there are enough CR functions to realize Y as the pseudoconvex boundary of a compact normal (possibly singular) Stein domain X_J (which is then uniquely determined by J). In dimension at least 5, all strictly pseudoconvex CR structures are embeddable [3] , but in dimension 3 this is a real restriction. The importance of X_J is that the smooth CR functions on Y are precisely the boundary values of holomorphic functions on X_J. We refer to [12] for a general treatment of geometry and analysis on CR manifolds.

Epstein [7] has shown that, if J_1 and J_2 are embeddable CR structures on Y, then the orthogonal projection from H_{J_1} to H_{J_2} is a Fredholm operator whose homotopy class is independent of the choice of smooth measure on Y.[2] The **relative index** of J_1 and J_2 is thus finite in this situation. Surprisingly, perhaps, the index is not always conserved

[2] Actually, the cited papers only prove this statement when Y is 3-dimensional, but the methods should extend to the general case.

under deformations of J_1 and J_2.

For a positive polarization, the filling used for computing relative indices will be taken to be the Stein domain mentioned above. If the Stein domains X_{J_1} and X_{J_2} determined by a pair of embeddable CR structures J_1 and J_2 on Y are nonsingular, these manifolds can be glued together along their common boundary to form a closed manifold X. Although the complex structures on X_{J_1} and X_{J_2} do not match along Y, it is possible, using the natural isomorphism between the vector bundle complex structures mentioned above, to endow X with a natural (up to homotopy) stable almost complex structure and hence with a Dirac operator D^+ which restricts away from a neighborhood of Y to the "rolled-up Dolbeault complexes" (see [9]) on X_{J_1} and X_{J_2}. Our gluing conjecture then states that the relative index of J_1 and J_2 is equal to index of D_+. We will see in Section 7 we will see how to extend the conjecture to the singular case.

4 Real polarizations

$\mathcal{J}$ is a **real polarization** if $\mathcal{J} = \overline{\mathcal{J}}$. This means that $\mathcal{J}$ is the complexification of the tangent distribution of a foliation of Y by legendrian submanifolds. **Fibrating** real polarizations are those for which this foliation is a fibration. Cosphere bundles foliated by their fibres are examples of this type. In fact, Pang [16] proves that these are the only examples with compact, simply connected leaves. The quantum Hilbert space associated to S^*M with its polarization by fibres is just $L^2(M)$. A filling in this special case is constructed as follows. Choose a riemannian (or finslerian) metric on M, let D^*M be the unit disc bundle in the cotangent bundle, and identify the cosphere bundle with its boundary, so that the cotangent disc bundle becomes the filling.

Given a contact transformation ϕ between cosphere bundles S^*M_1 and S^*M_2, we may use it to *identify* both bundles with a single contact manifold Y, which then inherits a pair of real polarizations. The quantum Hilbert spaces for these polarizations are $L^2(M_1)$ and $L^2(M_2)$, but the operator between then obtained by orthogonal projection in $L^2(Y)$ is not in the class of Fourier integral operators $C(\phi)$ associated with ϕ but is rather a **Radon integral** operator associated with the double fibration $M_1 \leftarrow Y \rightarrow M_2$. This operator, defined by pulling back by one fibration followed by integration over the fibres of the other, is indeed a Fourier integral operator, but its associated canonical relation is too big: it contains at least the "unoriented" version of ϕ consisting of the graph of ϕ together with that of $\xi \mapsto -\phi(-\xi)$, and is even larger except in "clean" cases.

We should not, therefore, define the relative index of two real polarizations to be the index of the orthogonal projection between their quantum Hilbert spaces. Instead, we must use an indirect method, such as that described in the next section.

5 The Guillemin transform

In order to realize Fourier integral operators as intertwining operators between real polarizations, we follow an idea of Zelditch and relate them through polarizations of CR type. The groundwork for this argument has been laid by Guillemin [11] in the following way. If M is a compact manifold of dimension n, we choose a real analytic structure on M (which is essentially unique). According to Grauert [10], M can be embedded as a totally real submanifold of a complex n-manifold $M_{\mathbf{C}}$ with strictly pseudoconvex boundary Y. Like any hypersurface in a complex manifold, Y inherits a CR structure which in this pseudoconvex case determines a contact structure on Y. The analysis of Guillemin shows that the **Grauert tube** $M_{\mathbf{C}}$ can be identified with a cotangent disc bundle D^*M for some riemannian metric on M in such a way that the contact structure Y arising from $M_{\mathbf{C}}$ agrees with the one arising from the identification of Y with S^*M.

Y thus has two polarizations, one positive and one real. We will call these polarizations **affiliated** with one another. The corresponding fillings are diffeomorphic, but one carries the structure of a Stein manifold while the other is symplectic. Guillemin shows that the projection operator between the quantum Hilbert spaces for these two polarizations (holomorphic functions on M_C in one case, and all functions on M in the other) is an elliptic Fourier integral operator with complex phase and hence a Fredholm operator. We will call this operator a **Guillemin transform** for M and denote its index by i_M. Guillemin shows that this index is independent of all the choices made in its construction and is therefore an invariant of the differentiable manifold M. Recently, Epstein and Melrose [8] have shown that this index is always zero. In fact, they show that the transform is an isomorphism for sufficiently small tubes. (This result can also be seen as verifying a special case of our gluing conjecture, since the two fillings are topologically equivalent.)

Now for the idea of Zelditch [21]. If ϕ is a contact transformation between S^*M_1 and S^*M_2, we use it to identify these two cosphere bundles with a common manifold Y as before, but now we consider *four* polarizations on Y. In order, these are:

- $\mathcal{L}_1 =$ the real polarization by fibres over M_1;
- $\mathcal{J}_1 =$ the positive polarization as the boundary of $M_{1,\mathbf{C}}$;
- $\mathcal{J}_2 =$ the positive polarization as the boundary of $M_{2,\mathbf{C}}$;
- $\mathcal{L}_2 =$ the real polarization by fibres over M_2.

Zelditch observes that the successive composition of the orthogonal projections operators between the quantum Hilbert spaces of these polarizations *is* a Fourier integral operator in the class $C(\phi)$, so that its relative index can be computed as a relative index of Epstein type between the two complex polarizations plus the difference of Guillemin indices i_{M_1} and i_{M_2}. As we noted above, the Guillemin indices are zero. Thus, the index problem for Fourier integral operators is reduced to the relative index problem for CR structures.

In general, to define the relative index between two polarizations, we replace any which are real by affiliated positive polarizations.

6 Extension to vector bundles

The standard index theorems for pseudodifferential and Toeplitz operators are most interesting when applied to operators on sections of vector bundles rather than just on scalar functions. The same should be true for Fourier integral operators and their variants. In this section, we will propose a setup for an extension of our conjectures to vector bundles, and we will see that the conjecture reduces to known theorems in the pseudodifferential case.

Our starting data will now be a vector bundle F over the contact manifold Y, together with polarizations $\mathcal{J}_j$ of Y corresponding to fillings X_j. In order to extend the vector bundle over the fillings in an appropriate way, we need a condition of compatibility with the polarizations. In both the real and complex cases, the condition will be "constancy of the fibres along the leaves."

In the real case, where X_j is a cotangent disc bundle D^*M_j and $\mathcal{J}_j$ is the polarization by fibres of the cosphere bundle, the fibres of F should be identical over all the points of each fibre, which means that Y should be the pullback to S^*M_j of a vector bundle V_j over M_j. In this case, we can also pull back V_j to the filling X_j to give an extension of F to a bundle whose fibres are constant along the leaves of the polarization of the symplectic manifold X_j by fibres of the cotangent bundle.

In the complex case, we interpret "constancy along the leaves of a polarization" as the existence of a flat connection along the corresponding distribution. When J_j is a CR structure on Y, this leads directly to the condition that the bundle F should be a **holomorphic** vector bundle in the sense of Tanaka [18] (called an **almost CR** vector bundle by Webster [19]). In this situation, we will further assume that F extends to a holomorphic vector bundle E_j over the Stein filling X_j, and that the CR sections of F are the boundary values of holomorphic sections of E_j.[3] The simplest example of this setup occurs when F is a trivial bundle, in which case we are simply dealing with $\mathbb{C}^N$-valued functions which are CR on Y and holomorphic on X_j.

Once we have lifted the polarizations $\mathcal{J}_j$ on Y to the vector bundle F as described above, we can identify a space of smooth sections which are "parallel in the direction of the polarization," and then form their L^2 closure, using a volume element on Y and a hermitian structure on F, obtaining a space which we will again call H_j. The index of the orthogonal projection from one space to the other is again well defined in many cases and could be called the relative index of the two lifted polarizations. (When a polarization is real, we replace it by an affiliated positive one before computing the index.) As before, we conjecture that this relative index is equal to the index of a Dirac operator on the glued manifold X. This time, the operator is a twisted Dirac operator, obtained by tensoring with the vector bundle over X obtained by gluing the bundles E_j by using their identifications with F over the common boundary Y.

We recover standard index theorems for Toeplitz and pseudodifferential operators by

[3] It would be interesting to have verifiable hypotheses guaranteeing the existence of such extensions.

choosing the polarizations $\mathcal{J}_J$ (and hence the fillings X_j) to be equal to one another, but by allowing two different lifts of the polarizations to F. For instance, if we are given a bundle automorphism σ of F, we can define one lift to be the pullback of the second by σ. In this case, if π denotes the orthogonal projection onto H (which does not depend on j in this case), the operator which gives the relative index $\pi\sigma\pi : H \mapsto H$. When $\mathcal{J}$ is positive, this operator is just the Toeplitz operator whose symbol is σ, and our conjecture for the index reduces to the index formula of Boutet de Monvel [4].

When the polarizations are both real, with $X_j = D^*M_j$, σ is the symbol of a pseudodifferential operator P between sections of vector bundles V_1 and V_2 over M, The index of our glued twisted Dirac operator is now the Atiyah-Singer topological index of P, but the operator obtained from σ by the projection process described above is *not* P; rather, it is simply the multiplication operator by the bundle map from V_1 to V_2 given by integrating σ over the fibres of the cosphere bundle Y. To get the operator P, we must use affiliated polarizations as described in Section 5 and use the results of [11].

7 Singular fillings

There are several ways to approach the problem of singular fillings. One is to resolve the singularities and then add a correction term to account for the nontrivial pseudoconvex (but no longer Stein) filling. We will present here an alternative approach which appears to be more conceptual in nature. It still uses resolution of singularities, for the moment, but only to show that a certain index is well defined, not to define it.

As usual, we consider polarized contact manifolds Y of either of two types–cosphere bundles and embeddable CR manifolds. In the first case, the filling will be the corresponding disk bundle in a cotangent bundle; in the second, the filling will be the (possibly singular) Stein domain having Y as its strictly pseudoconvex boundary.

Let X_1 and X_2 be fillings of Y corresponding to polarizations $\mathcal{J}_1 =$ and $\mathcal{J}_2 =$. We may glue X_1 to X_2 along Y to get a new object X, but the nature of X depends on the nature of X_1 and X_2. If X_1 and X_2 are both either symplectic or are nonsingular Stein varieties, they can be considered as almost complex manifolds and hence X becomes a stable almost complex manifold. The index of the corresponding Dirac operator is therefore well defined.

If either X_1 and X_2 is possibly singular, we will resort to the following construction. According to Theorem 8.1 of [14] (see [6] for related results), each X_j can be completed by adding a nonsingular complex manifold Q_j with strictly *pseudoconcave* boundary Y to make a (possibly singular) projective variety Z_j.[4] For such a variety, we *define* the "index of its Dirac operator", denoted simply by index(Z_j) to be the Euler characteristic of its cohomology with values in the sheaf $\mathcal{O}$ of germs of holomorphic functions. This is a good definition because, if Z_j happens to be singular, this Euler characteristic equals the Euler characteristic for the Dolbeault cohomology on forms of type $(0, q)$, which is in turn equal to the index of the Dirac operator given by the rolled-up Dolbeault complex.

If Q_1 and Q_2 were isomorphic, it would be reasonable to define the relative index of X_1 and X_2 to be the difference of the indices of the Z_j. In general, account for the difference between Q_1 and Q_2 in the following way. Glue Q_1 and Q_2 along their common boundary Y to form a smooth manifold Q. The complex structures on the pieces glue to give a stable almost complex structure on Y for which the natural orientation agrees with that on Q_2 but is opposite to the orientation of Q_1. We now define the topological relative index of X_1 and X_2 to be index(Z_2) − index(Z_1) − index(Q), where the last index is the index of the Dirac operator on Q associated with its almost complex structure.

Since the "caps" Q_1 and Q_2 are not unique, we have to check that our relative index is well-defined. This can be done by an argument which we will not give here. It uses the cobordism invariance of the index and resolution of singularities. (We hope that the latter may be replaced by a localization argument for the index of a singular variety.)

Our conjecture is that this expression index(Z_2) − index(Z_1) − index(Q) plays the role of the index of the object X obtained by gluing X_1 and X_2 along Y, and hence is equal to the relative index of X_1 and X_2. Using Riemann-Roch theory, it is not hard to verify that the conjecture gives the correct relative index for the pairs of CR structures on a circle bundle over a Riemann surface of genus 2 as considered in [7].

Remark It would interesting to define the index of X directly. As a geometric object, X can be thought of as consisting of two ends which are (possibly singular) complex varieties, joined by a band on which there is a stable almost complex structure. The

[4] I learned about this result in a talk by G. Matić on the paper [15], where I also learned about gluing complex and symplectic manifolds along contact boundaries!

Dirac operator of the band agrees on the overlap with the rolled up Dolbeault complex on the smooth parts of the ends.

It is tempting to try to define the index of the glued object as the Euler characteristic of an object in a derived category of sheaves on X, obtained by gluing the sheaf $\mathcal{O}$ on the holomorphic ends to the (very short) complex of sheaves given by the Dirac operator on the band, using the techniques in [13]. Unfortunately, these sheaves are not quite quasi-isomorphic on the overlap of the two regions—it is only the alternating sums of their cohomologies which agree in some sense there. Perhaps suitable holomorphic vector fields near Y could be used, in the spirit of [1], to surmount this problem.

8 Holomorphic vs. Dirac indices: a proof strategy

Our strategy for proving the gluing conjecture for the relative index of CR structures is to reduce the problem to related known results about Dirac operators. If D^+ is a Dirac operator between sections of Clifford bundles E^+ and E^- over a filling of the compact manifold Y, then a famous result of Seeley [17] implies that the orthogonal projection (the so-called **Calderon projector**) from $L^2(Y)$ to the **Cauchy data space** of boundary values of solutions of $D^+u = 0$ is a pseudodifferential operator of classical type (i.e. with symbol an asymptotic sum of homogeneous terms) whose principal symbol is a projection operator on the pullback of E^+ to S^*Y. Given a pair of such operators with Calderon projectors having the same principal symbols, the orthogonal projection operator between their Cauchy data spaces is shown to be a Fredholm operator by Booß-Bavnbek and Wojciechowski [2], who prove the following "gluing theorem" (originally conjectured by Bojarski) for the index of this operator, which we call the **relative index** of the two Dirac operators. (In general, it depends on the boundary isomorphism as well as the operators.)

Theorem. *Let D_1^+ and D_2^+ be Dirac operators on compact manifolds X_1 and X_2 having the common boundary Y, with isomorphisms over Y between the domain and range Clifford bundles, such that their Calderon projectors have the same principal symbol with respect to the domain isomorphism. Then the relative index of D_1^+ and D_2^+ is equal to the index of a Dirac operator on the glued manifold $X = X_1 \cup_Y X_2$ obtained by gluing the* *bundles and operators over X_1 and X_2 via the isomorphisms over Y.*

The Dirac operators to which we wish to apply the theorem above are the Dolbeault-Dirac operators on the Stein fillings (assumed nonsingular) X_1 and X_2 associated with a pair of positive polarizations on the contact manifold Y. More precisely, we assume that these fillings are equipped with Kähler metrics (for instance those obtained from embeddings in some $\mathbb{C}^N$), and we consider on each the operator $D^+ = \overline{\partial} + \overline{\partial}^* : \Omega^{0,even} \to \Omega^{0,odd}$ between the even and odd parts of the Dolbeault resolution of the sheaf of holomorphic functions. "Rolling up" the Dolbeault complex by replacing its usual $\mathbb{Z}$ grading by a $\mathbb{Z}_2$ grading has the result of replacing the rather delicate Dirichlet problem for the $\overline{\partial}$ operator by a much more robust problem, to which the gluing result above may be applied.

The isomorphism over Y between the domain and range bundles for D_1^+ and D_2^+ is obtained from an isomorphism between the restrictions to Y of the complex vector bundles TX_1 and TX_2. This isomorphism is in turn obtained from the natural isomorphism between the two induced almost complex structures on the fixed contact distribution C, as described in Section 3 above.

The problem is now reduced to the following conjecture, in some sense a relative version of the result in the compact case that the dimension of the space of holomorphic sections of a line bundle without higher cohomology is equal to the index of a rolled-up (twisted) Dolbeault complex.

Conjecture. *Let X_1 and X_2 be a nonsingular Stein fillings of a contact manifold Y. Then the relative index of X_1 and X_2 defined by the boundary values of their spaces of holomorphic functions is equal to the relative index of the Dirac operators D_1^+ and D_2^+.*

Some evidence in favor of this conjecture comes from the case where the complex dimension of X_j is 2. In this case, the Cauchy data space for the Dirac operator can be written as the direct sum (but not an orthogonal one!) of the Cauchy data space for the holomorphic functions and a subspace isomorphic to that for the harmonic forms of type $(0,2)$. The latter space is independent of the CR structure, since the Dirichlet problem for the laplacian can be solved for any Cauchy data. Thus, in considering the relative indices for X_1 and X_2, it ought to be possible to "cancel" the contributions coming from the harmonic forms.

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