数理解析研究所講究録
1014巻 1997年 118-132

Uses of Envelopes for Global and Asymptotic Analysis; geometrical meaning of renormalization group equation

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Abstract

We give a comprehensive review of the renormalization group method for global and asymptotic analysis, putting an emphasis on the relevance to the classical theory of envelopes and the existence of invariant manifolds of the dynamics under consideration. We clarify that an essential point of the method is to convert the problem from solving differential equations to obtaining suitable initial (or boundary) conditions. We mention that the notion of envelopes is also useful for constructing global and asymptotic behavior of wave functions of quantum systems such as the ones with the quartic potential or double-well potential.

1 Introduction

The renormalization group (RG) equations have a peculiar power to improve the global nature of functions obtained in the perturbation theory in quantum field theory [1]: The RG equations represent the fact that a physical quantity $\mathcal{O}(p, \alpha, \mu)$ should not depend on the renormalization point μ,

$$\frac{\partial \mathcal{O}(p, \alpha; \mu)}{\partial \mu} = 0. \tag{1.1}$$

Such a floating renormalization point was first introduced by Gell-Mann and Low in the celebrated paper[1]. We remark that the renormalization point μ plays a role of the initial point and the renormalization condition for physical quantities such as coupling constants at the energy scale μ may be viewed as setting initial values of these quantities.[2]

Recently, the present author has indicated that the RG equation à la Gell-Mann-Low can be identified as the envelope equation[3];. it was shown that the notion of envelopes is useful for improving the perturbative expansions appearing in *quantum field theory*. This identification was realized through an examination of the RG method of Chen, Goldenfeld and Oono for global analysis of differential equations[6]; they applied the RG equation to numerous problems and found that the RG equation gives slow dynamics of the system in question. The new point of their method is to utilize secular terms which usually

appear when perturbation theory is applied to differential equations; this is in contrast to all previous methods, which are formulated on the principle to avoid the appearance of secular terms.

Their method was reformulated on the basis of the classical theory of envelopes[3, 4]. It was demonstrated that owing to the very envelope equation, the functions constructed from the solutions in the perturbation theory certainly satisfies the differential equation in question uniformly up to the order with which local solutions is constructed. It was also shown [5] in a most general setting that the RG method gives a reduction of dynamics and clarified that there is a correspondence between the RG method and the reductive perturbation method[7]: Some interesting examples were also worked out in this method, such as the forced Duffing, the Lotka-Volterra and the Lorenz equations; the first example showed that the method is applicable to not only automonous but also *non*-automonous equations, the second one showed that the method gives phase equations and the last one showed that *center manifolds* of the dynamics can be extracted in the method. It is noteworthy that the notion of *functional self-similarity* (FSS) extracted as the essence of the RG in [2] is only applicable to autonomous equations; see [8] for an application of the notion of FSS for deducing phase equations. The above fact suggests that the notion of envelopes better represents the underlying mathematics of the powerfulness of the RG. It was recognized also that when the unperturbed equation has *neutrally stable* solutions, the RG method works well. It implies that the method is applicable when the dynamical system under consideration has invariant manifolds and useful to extract the manifolds and the dynamics on the manifolds. It was shown that global and asymptotic behavior is obtained even for *discrete* systems by constructing "envelopes" when the system has neutrally stable solution of the unperturbed equations[9].

Subsequently the classical theory of envelopes was applied for getting asymptotic behaviors of wave functions in *quantum mechanics*[10]. This is an optimized perturbation theory in which the perturbation theory is combined with a variational method. The key ingredient is to construct an envelope of a set of perturbative wave functions. This leads to a condition similar to that obtained from the principle of minimal sensitivity[11]. Applications of the method to quantum anharmonic oscillator and the double well potential show that uniformly valid wave functions with correct asymptotic behavior are obtained in the first-order optimized perturbation even for strong couplings.

In this report, we will give only basic ingredients of the RG method based on our formulation putting an emphasis to the relevance to envelopes; for detailed account of the method and various applications, please refer to Ref.'s[3, 4, 5, 9, 10]. In this report, we also clarify that the RG method is most lucidly formulated by noting that the method converts the problem from solving differential equations to obtaining suitable initial (or boundary) conditions as does the usual RG. In this report, we shall not discuss on partial differential equations due to the lack of space. Please refer to [12] as well as [6, 4, 8] for this subject.

2 A short review of the classical theory of envelopes

We here give a brief review of the theory of envelopes. Although the theory can be formulated in higher dimensions[4, 5], we consider here envelope curves, for simplicity.

Let $\{C_\tau\}_\tau$ be a family of curves parametrized by τ in the x-y plane; here C_τ is represented by the equation $F(x,y,\tau)=0$. We suppose that $\{C_\tau\}_\tau$ has the envelope E, which is represented by the equation $G(x,y)=0$. The problem is to obtain $G(x,y)$ from $F(x,y,\tau)$.

Now let E and a curve C_{τ_0} have the common tangent line at $(x,y)=(x_0,y_0)$, i.e., (x_0,y_0) is the point of tangency. Then x_0 and y_0 are functions of τ_0; $x_0=\phi(\tau_0)$, $y_0=\psi(\tau_0)$, and of course $G(x_0,y_0)=0$. Conversely, for each point (x_0,y_0) on E, there exists a parameter τ_0. So we can reduce the problem to get τ_0 as a function of (x_0,y_0); then $G(x,y)$ is obtained as $F(x,y,\tau(x,y))=G(x,y)$.[1] $\tau_0(x_0,y_0)$ can be obtained as follows.

Since the tangent line of E at (x_0,y_0) is perpendicular to the normal direction of $F(x,y,\tau)=0$ at the same point, one has

$$F_x(x_0,y_0,\tau_0)\phi'(\tau_0)+F_y(x_0,y_0,\tau_0)\psi'(\tau_0)=0. \tag{2.1}$$

On the other hand, differentiating $F(x(\tau_0),y(\tau_0),\tau_0)=0$ with respect to τ_0, one also has

$$F_x(x_0,y_0,\tau_0)\phi'(\tau_0)+F_y(x_0,y_0,\tau_0)\psi'(\tau_0)+F_{\tau_0}(x_0,y_0,\tau_0)=0. \tag{2.2}$$

Combining the last two equations, we have

$$F_{\tau_0}(x_0,y_0,\tau_0)\equiv\frac{\partial F(x_0,y_0,\tau_0)}{\partial\tau_0}=0. \tag{2.3}$$

This is the basic equation of the theory of envelopes; we call this type of equation envelope equations and also RG/E equation where RG and E stand for renormalization group and envelope, respectively, becasue the RG equation Eq(1.1) has the same form as Eq.(2.3). One can thus eliminate the parameter τ_0 to get a relation between x_0 and y_0,

$$G(x,y)=F(x,y,\tau_0(x,y))=0, \tag{2.4}$$

with the replacement $(x_0,y_0)\to(x,y)$. $G(x,y)$ is called the discriminant of $F(x,y,t)$.

Comments are in order here: (i) When the family of curves is given by the function $y=f(x,\tau)$, the condition Eq.(2.3) is reduced to $\partial f/\partial\tau_0=0$; the envelope is given by $y=f(x,\tau_0(x))$. (ii) The equation $G(x,y)=0$ may give not only the envelope E but also a set of singularities of the curves $\{C_\tau\}_\tau$. This is because the condition that $\partial F/\partial x=\partial F/\partial y=0$ is also compatible with Eq. (2.3).

[1] Since there is a relation $G(x_0,y_0)=0$ between x_0 and y_0, τ_0 is actually a function of x_0 *or* y_0.

3 The RG method; a simplest example

In this section, using a simplest example we show how the RG method works for obtaining global and asymptotic behavior of solutions of differential equations. We shall present the method so that the reader will readily see that the notion of envelopes is intrinsically related to the method. We shall emphasize that an essential point of the method is tuning the initial condition at an arbitrary time t_0 perturbatively along with solving the perturbative equations successively. One will see that the reasoning for various steps in the procedure and the underlying picture are quite different from the original ones given in [6]. We believe, however, that the present formulation emphasizing the role of initial conditions and the relevance to envelopes of perturbative local solutions straightens the original argument, and is the most comprehensive one.[2]

Let us take the following simplest example to show our method:

$$\frac{d^2x}{dt^2} + \epsilon\frac{dx}{dt} + x = 0, \tag{3.1}$$

where ϵ is supposed to be small. The solution to Eq.(3.1) reads

$$x(t) = \bar{A}\exp(-\frac{\epsilon}{2}t)\sin(\sqrt{1-\frac{\epsilon^2}{4}}t+\bar{\theta}), \tag{3.2}$$

where $\bar{A}$ and $\bar{\theta}$ are constants.

Now, let us obtain the solution around the initial time $t = t_0$ in a perturbative way, expanding x as

$$x(t,t_0) = x_0(t,t_0) + \epsilon x_1(t,t_0) + \epsilon^2 x_2(t,t_0) + ..., \tag{3.3}$$

where $x_n(t,t_0)$ $(n = 0,1,2...)$ satisfy

$$\ddot{x}_0 + x_0 = 0, \quad \ddot{x}_{n+1} + x_{n+1} = -\dot{x}_n. \tag{3.4}$$

The initial condition may be specified by

$$x(t_0,t_0) = W(t_0). \tag{3.5}$$

We suppose that the initial value $W(t_0)$ is always on an exact solution of Eq.(3.1) for any t_0. We also expand the initial value $W(t_0)$;

$$W(t_0) = W_0(t_0) + \epsilon W_1(t_0) + \epsilon^2 W_2(t_0) + ..., \tag{3.6}$$

and the terms $W_i(t_0)$ will be determined so that the perturbative solutions around different initial times t_0 have an envelope. Hence the initial value $W(t)$ thus constructed will give the (approximate but) global solution of the equation.

[2]The following is even a refinment of the argument given in [3, 5] where the fact that the RG method is a theory manipulating initial conditions were not fully recognized.

Let us perform the above program. The lowest solution may be given by

$$x_0(t,t_0) = A(t_0)\sin(t+\theta(t_0)), \tag{3.7}$$

where we have made it explicit that the constants A and θ may depend on the initial time t_0. The initial value $W(t_0)$ as a function of t_0 is specified as

$$W_0(t_0) = x_0(t_0,t_0) = A(t_0)\sin(t_0+\theta(t_0)). \tag{3.8}$$

We remark that Eq.(3.7) is a neutrally stable solution; with the perturbation $\epsilon \neq 0$ the constants A and θ may move slowly. We shall see that the envelope equation gives the equations describing the slow motion of A and θ.

The first order equation now reads $\ddot{x}_1 + x_1 = -A\cos(t+\theta)$, and we choose the solution in the following form,

$$x_1(t,t_0) = -\frac{A}{2}\cdot(t-t_0)\sin(t+\theta), \tag{3.9}$$

which means that the first order initial value $W_1(t_0) = 0$ so that the lowest order value $W_0(t_0)$ approximates the exact value as closely as possible. Similarly, the second order solution may be given by

$$x_2(t) = \frac{A}{8}\{(t-t_0)^2\sin(t+\theta) - (t-t_0)\cos(t+\theta)\}, \tag{3.10}$$

thus $W_2(t_0) = 0$ again for the present linear equation.

It should be noted that the secular terms have appeared in the higher order terms, which are absent in the exact solution and invalidates the perturbation theory for t far from t_0. However, with the very existence of the secular terms, we could make $W_i(t_0)$ $(i = 1, 2)$ vanish and $W(t_0) = W_0(t_0)$ up to the third order.

Collecting the terms, we have

$$\begin{aligned} x(t,t_0) &= A\sin(t+\theta) - \epsilon\frac{A}{2}(t-t_0)\sin(t+\theta) \\ &\quad +\epsilon^2\frac{A}{8}\{(t-t_0)^2\sin(t+\theta) - (t-t_0)\cos(t+\theta)\}, \end{aligned} \tag{3.11}$$

and more importantly

$$W(t_0) = W_0(t_0) = A(t_0)\sin(t_0+\theta(t_0)) \tag{3.12}$$

up to $O(\epsilon^3)$. We remark that $W(t_0)$ describing the solution is parametrized by possibly slowly moving variable $A(t_0)$ and $\phi(t_0) \equiv t_0 + \theta(t_0)$ in a definite way.

Now we have a family of curves $\{C_{t_0}\}_{t_0}$ given by functions $\{x(t,t_0)\}_{t_0}$ parametrized with t_0. They are all on the exact curve $W(t)$ at $t = t_0$ up to $O(\epsilon^3)$, but only valid locally

for t near t_0. So it is conceivable that the envelope E of $\{C_{t_0}\}_{t_0}$ which contacts with each local solution at $t = t_0$ will give a global solution. Thus the envelope function $x_{_E}(t)$ coincides with $W(t)$;

$$x_{_E}(t) = x(t,t) = W(t). \tag{3.13}$$

Our task is actually to determine $A(t_0)$ and $\theta(t_0)$ as functions of t_0 so that the family of the local solutions has an envelope. According to the classical theory of envelopes given in the previous section, the above program can be achieved by imposing that the envelope equation

$$\frac{dx(t,t_0)}{dt_0} = 0, \tag{3.14}$$

gives the solution $t_0 = t$. From Eq.'s (3.11) and (3.14), we have

$$\frac{dA}{dt_0} + \epsilon A = 0, \quad \frac{d\theta}{dt_0} + \frac{\epsilon^2}{8} = 0, \tag{3.15}$$

where we have used the fact that dA/dt is $O(\epsilon)$ and neglected the terms of $O(\epsilon^3)$. Solving the equations, we have

$$A(t_0) = \bar{A}\mathrm{e}^{-\epsilon t_0/2}, \quad \theta(t_0) = -\frac{\epsilon^2}{8}t_0 + \bar{\theta}, \tag{3.16}$$

where $\bar{A}$ and $\bar{\theta}$ are constant numbers. Thus we get

$$x_{_E}(t) = x(t,t) = W_0(t) = \bar{A}\exp(-\frac{\epsilon}{2}t)\sin((1-\frac{\epsilon^2}{8})t + \bar{\theta}), \tag{3.17}$$

up to $O(\epsilon^3)$. Noting that $\sqrt{1-\epsilon^2/4} = 1 - \epsilon^2/8 + O(\epsilon^4)$, one finds that the resultant envelope function $x_{_E}(t) = W_0(t)$ is an approximate but *global* solution to Eq.(3.1); see Eq. (3.2).

4 Nonlinear equations

In this section, we treat a couple of examples of systems of ODE's with nonlinearity to show how the RG method works[5]. The examples are the Lotka-Volterra[13] and the Lorenz[14] equation. We shall derive the time dependence of the solution to the Lotka-Volterra equation explicitly; a phase equation will be derived by our method. The Lorenz equation is an example with three degrees of freedom, which shows a bifurcation. We shall give the center manifolds of this equation around the first bifurcation point.

4.1 Lotka-Volterra equation

The Lotka-Volterra equation reads[13];

$$\dot{x} = ax - \epsilon xy, \quad \dot{y} = -by + \epsilon' xy, \tag{4.1}$$

where the constants a, b, ϵ and ϵ' are assumed to be positive. It is well known that the equation has the conserved quantity, i.e.,

$$b \ln |x| + a \ln |y| - (\epsilon' x + \epsilon y) = \text{const.}. \tag{4.2}$$

The fixed points are given by $(x = 0, y = 0)$ and $(x = b/\epsilon', y = a/\epsilon)$. Shifting and scaling the variables by

$$x = (b + \epsilon\xi)/\epsilon', \quad y = a/\epsilon + \eta, \tag{4.3}$$

we get the reduced equation given by the system

$$\left(\frac{d}{dt} - L_0\right)\mathbf{u} = -\epsilon\xi\eta \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{4.4}$$

where

$$\mathbf{u} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad L_0 = \begin{pmatrix} 0 & -b \\ a & 0 \end{pmatrix}. \tag{4.5}$$

The eigen value equation

$$L_0\mathbf{U} = \lambda_0\mathbf{U} \tag{4.6}$$

has the solution

$$\lambda_0 = \pm i\sqrt{ab} \equiv \pm i\omega, \quad \mathbf{U} = \begin{pmatrix} 1 \\ \mp i\frac{\omega}{b} \end{pmatrix}. \tag{4.7}$$

Let us try to extract the global behavior of the solution around the fixed point. Our strategy is the following: We suppose that we are on the exact solution at $t = t_0$ where t_0 is arbitrary; we denote the initial value by $\mathbf{W}(t_0)$. We also suppose that we can apply perturbation theory for the solution at least in the small neighborhood of $t = t_0$. We expand the variable in a Taylor series of ϵ;

$$\mathbf{u} = \mathbf{u}_0 + \epsilon\mathbf{u}_1 + \epsilon^2\mathbf{u}_2 + \cdots, \tag{4.8}$$

with $\mathbf{u}_i = {}^t(\xi_i, \eta_i)$. An essential point of our method is to expand the initial value, too;

$$\mathbf{W}(t_0) = \mathbf{W}_0(t_0) + \epsilon\mathbf{W}_1(t_0) + \epsilon^2\mathbf{W}_2(t_0) + \cdots. \tag{4.9}$$

Our central task is to extract the initial value as a function of t_0 so that the resulting local solutions starting from the different initial points at $t = t_0$ and , say, $t = t_0 + \Delta t$

are continued smoothly. This condition is found to be nothing but the one that the local solutions have an envelope. In actual calculations, it is also important to use the fact that the functional form of the initial values can be reduced from the general solution of the differential equations in a perturbative way; in this procedure, only independent functions modulo to secular terms are retained.

The lowest term satisfies the equation

$$\left(\frac{d}{dt} - L_0\right)\mathbf{u}_0 = \mathbf{0}, \tag{4.10}$$

which yields the solution

$$\mathbf{u}_0(t; t_0) = \mathcal{A}(t_0)\mathrm{e}^{i\omega t}\mathbf{U} + \text{c.c.}. \tag{4.11}$$

Notice that $\mathcal{A}$ is a complex number, so one may parametrize it as

$$\mathcal{A}(t_0) = A(t_0)/2i \cdot \exp(i\theta(t_0)). \tag{4.12}$$

The solution implies that the initial condition is given by

$$\mathbf{u}_0(t_0; t_0) = \mathcal{A}(t_0)\mathrm{e}^{i\omega t_0}\mathbf{U} + \text{c.c.}. \tag{4.13}$$

It means that in the lowest approximation the solution is parametrized by a complex function $\mathcal{A}(t_0)$ or a pair of real functions, $A(t_0)$ and $\phi(t_0) \equiv \omega t_0 + \theta(t_0)$. With a small perturbation, we expect that A and θ will move slowly.

Noting that $\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \alpha\mathbf{U} + \text{c.c.}$, with $\alpha = (1 - ib/\omega)/2$, one finds that the first order term satisfies the equation

$$\left(\frac{d}{dt} - L_0\right)\mathbf{u}_1 = \frac{\omega}{b}\left[i\mathcal{A}^2\mathrm{e}^{2i\omega t}(\alpha\mathbf{U} + \text{c.c.}) + \text{c.c.}\right], \tag{4.14}$$

the solution to which may be given by

$$\mathbf{u}_1 = \frac{1}{b}\left[\mathcal{A}^2(\alpha\mathbf{U} + \frac{\alpha^*}{3}\mathbf{U}^*)\mathrm{e}^{2i\omega t} + \text{c.c.}\right]. \tag{4.15}$$

Thus the initial value $\mathbf{W}_1(t_0)$ in this order is given $\mathbf{u}_1(t_0, t_0)$.

Similarly, the second order solution may be given by

$$\begin{aligned}\mathbf{u}_2 \;=\; & \left[\frac{b - i\omega}{3b^2}|\mathcal{A}|^2\mathcal{A}\left\{\alpha(t - t_0 + i\frac{\alpha^*}{2\omega})\mathbf{U} + \frac{\alpha^*}{2i\omega}\mathbf{U}^*\right\}\mathrm{e}^{i\omega t}\right. \\ & \left. + \frac{b + i\omega}{4b^2 i\omega}\mathcal{A}^3(2\alpha\mathbf{U} + \alpha^*\mathbf{U}^*)\mathrm{e}^{3i\omega t}\right] + \text{c.c.}.\end{aligned} \tag{4.16}$$

Here we have a secular term proportional to the unperturbed solution. Since we want to make the lowest initial value as close as the exact one, we demand that as many as

possible terms in the higher order vanish at $t = t_0$. Thus adding unperturbative solutions, we make the secular term (of the upper component) vanishes at $t = t_0$.[3] Thus neglecting higher order terms, we have $\mathbf{u}(t, t_0) = \mathbf{u}_0 + \epsilon \mathbf{u}_1 + \epsilon^2 \mathbf{u}_2$, and $\mathbf{W}(t_0) = \mathbf{u}(t_0, t_0)$. We impose that the solutions $\mathbf{u}(t, t_0)$ and $\mathbf{u}(t, t_0 + \Delta t)$ give the same value at t. By taking a limit $\Delta t \to 0$, we have the envelope equation;

$$\frac{d\mathbf{u}}{dt_0} = \mathbf{0}, \tag{4.17}$$

with $t_0 = t$. This gives the equation for $\mathcal{A}(t)$ as

$$\frac{d\mathcal{A}}{dt} = -i\epsilon^2 \frac{\omega^2 + b^2}{6\omega b^2} |\mathcal{A}|^2 \mathcal{A}. \tag{4.18}$$

In terms of $A(t)$ and $\theta(t)$, we have

$$A(t) = \text{const.}, \quad \theta(t) = -\frac{\epsilon^2 A^2}{24}(1 + \frac{b^2}{\omega^2})\omega t + \bar{\theta}_0, \tag{4.19}$$

with $\bar{\theta}_0$ being a constant. Owing to the prefactor i in r.h.s. of Eq. (4.18), the absolute value of the amplitude A becomes independent of t, while the phase θ has a t-dependence. The envelope function is given by

$$\mathbf{u}_E(t) = \begin{pmatrix} \xi_E(t) \\ \eta_E(t) \end{pmatrix} = \mathbf{u}(t, t) = \mathbf{W}(t). \tag{4.20}$$

In terms of the components, one has

$$\begin{aligned} \xi_E &= A \sin\Theta(t) - \epsilon \frac{A^2}{6\omega}(\sin 2\Theta(t) + \frac{2\omega}{b}\cos 2\Theta(t)) \\ &\quad - \frac{\epsilon^2 A^3}{32}\frac{3\omega^2 - b^2}{\omega^2 b^2}(\sin 3\Theta(t) - \frac{4\omega b}{3\omega^2 - b^2}\cos 3\Theta(t)), \\ \eta_E &= -\frac{\omega}{b}\Big[\Big(A - \frac{\epsilon^2 A^3}{24}\frac{b^2 - \omega^2}{b^2\omega^2}\Big)\cos\Theta(t) - \frac{\epsilon^2 A^3}{12b\omega}\sin\Theta(t) \\ &\quad + \epsilon\frac{A^2}{2b}\Big(\sin 2\Theta(t) - \frac{2b}{3\omega}\cos 2\Theta(t)\Big) - \frac{\epsilon^2 A^3}{8b\omega}\Big(\sin 3\Theta(t) - \frac{3b^2 - \omega^2}{4b^2\omega^2}\cos 3\Theta(t)\Big)\Big], \end{aligned} \tag{4.21}$$

where

$$\Theta(t) \equiv \tilde{\omega} t + \bar{\theta}_0, \quad \tilde{\omega} \equiv \{1 - \frac{\epsilon^2 A^2}{24}(1 + \frac{b^2}{\omega^2})\}\omega. \tag{4.22}$$

One sees that the angular frequency is shifted.

We can identify $\mathbf{u}_E(t) = (\xi_E(t), \eta_E(t)) = \mathbf{W}(t)$ as an approximate solution to Eq.(4.4) by construction. We see that $\mathbf{u}_E(t)$ is an approximate but uniformly valid solution to the

[3] Although we can make other terms also vanish at $t = t_0$[15], the resulting dynamics of A and θ become more complicated than the present choice. In the theory of reduction of dynamics, one usually prefers simpler dynamics[16].

equation up to $O(\epsilon^3)$. We remark that the resultant trajectory is closed in conformity with the conservation law Eq. (4.2).

"Explicit solutions" of two-pieces of Lotka-Volterra equation were considered by Frame [17]; however, his main concern was on extracting the period of the solutions in an average method. we are not aware of any other work than ours which gives an explicit form of the solution as given here.

4.2 The Lorenz model

The Lorenz model[14] for the thermal convection is given by

$$\begin{aligned} \dot{\xi} &= \sigma(-\xi+\eta), \\ \dot{\eta} &= r\xi - \eta - \xi\zeta, \\ \dot{\zeta} &= \xi\eta - b\zeta. \end{aligned} \tag{4.23}$$

The steady states are give by

$$\text{(A)}\ \ (\xi,\eta,\zeta) = (0,0,0), \quad \text{(B)}\ \ (\xi,\eta,\zeta) = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1). \tag{4.24}$$

The linear stability analysis[18] shows that the origin is stable for $0 < r < 1$ but unstable for $r > 1$, while the latter steady states (B) are stable for $1 < r < \sigma(\sigma+b+3)/(\sigma-b-1) \equiv r_c$ but unstable for $r > r_c$. In this report, we examine the non-linear stability around the origin for $r \sim 1$ and extract a center manifold for the dynamics in this region.

We put

$$r = 1+\mu \quad \text{and} \quad \mu = \chi\epsilon^2, \quad \chi = \mathrm{sgn}\mu. \tag{4.25}$$

We expand the quantities as Taylor series of ϵ:

$$\mathbf{u} \equiv \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \epsilon\mathbf{u}_1 + \epsilon^2\mathbf{u}_2 + \epsilon^3\mathbf{u}_3 + \cdots, \tag{4.26}$$

where $\mathbf{u}_i = {}^t(\xi_i, \eta_i, \zeta_i)$ $(i = 1,2,3,\ldots)$. We also expand the initial value at $t = t_0$;

$$\mathbf{u}(t_0,t_0) = \mathbf{W}(t_0) = \epsilon\mathbf{W}_1 + \epsilon^2\mathbf{W}_2 + \epsilon^3\mathbf{W}_3 + \cdots. \tag{4.27}$$

The first order equation reads

$$\left(\frac{d}{dt} - L_0\right)\mathbf{u}_1 = \mathbf{0}, \tag{4.28}$$

where

$$L_0 = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}, \tag{4.29}$$

the eigenvalues of which are found to be

$$\lambda_1 = 0, \quad \lambda_2 = -\sigma - 1, \quad \lambda_3 = -b. \tag{4.30}$$

The respective eigenvectors are

$$\mathbf{U}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{U}_2 = \begin{pmatrix} \sigma \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{U}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{4.31}$$

We are interested in the *asymptotic state* as $t \to \infty$. In this asymptotic region, one may take only the neutrally stable solution

$$\mathbf{u}_1(t; t_0) = A(t_0)\mathbf{U}_1, \tag{4.32}$$

because the other terms proportional to $\mathbf{U}_{2,3}$ will decay out at a sufficiently large time. Here we have made it explicit that the solution may depend on the initial time t_0. Eq.(4.32) implies that we have taken the initial condition that

$$\mathbf{W}(t_0) \simeq \mathbf{W}_1(t_0) = A(t_0)\mathbf{U}_1. \tag{4.33}$$

In terms of the components,

$$\xi_1(t) = A(t_0), \quad \eta_1(t) = A(t_0), \quad \zeta_1(t) = 0. \tag{4.34}$$

In another word, the motion of $\mathbf{u}_1(t)$ is confined or reduced to the one-dimensional manifold ${}^t(A, A, 0)$, although A is a constant in this approximation. One expects that the small perturbation with $\epsilon \neq 0$ will give rise to a slow motion of A as well as a modification of the slow manifold.

The second order equation reads

$$\left(\frac{d}{dt} - L_0\right)\mathbf{u}_2 = \begin{pmatrix} 0 \\ -\xi_1\zeta_1 \\ \xi_1\eta_1 \end{pmatrix} = A^2\mathbf{U}_3, \tag{4.35}$$

which may yield

$$\mathbf{u}_2(t) = \frac{A^2}{b}\mathbf{U}_3, \tag{4.36}$$

or in terms of the components

$$\xi_2 = \eta_2 = 0, \quad \zeta_2 = \frac{A^2}{b}. \tag{4.37}$$

Here we have retained only functions independent of the ones appearing in the lowest approximation. Then the third order equation is given by

$$\Big(\frac{d}{dt}-L_0\Big)\mathbf{u}_3=\begin{pmatrix}0\\ -\chi\xi_1-\xi_2\zeta_1-\xi_1\zeta_2\\ \xi_2\eta_1+\xi_1\eta_2\end{pmatrix}=\frac{1}{1+\sigma}(\chi A-\frac{1}{b}A^3)(\sigma\mathbf{U}_1-\mathbf{U}_2), \tag{4.38}$$

which may yield

$$\mathbf{u}_3=\frac{1}{1+\sigma}(\chi A-\frac{1}{b}A^3)\{\sigma(t-t_0+\frac{1}{1+\sigma})\mathbf{U}_1-\frac{1}{1+\sigma}\mathbf{U}_2\}. \tag{4.39}$$

Here we have again retained functions which have not appeared before except for the term with which the secular terms vanishes at $t=t_0$. Of course, one may have other choices for the independent functions, but it is found that the present "minimal" choice gives the simplest dynamics for the amplitude A.

Thus collecting all the terms, one has

$$\begin{aligned}\mathbf{u}(t;t_0) &= \epsilon A(t_0)\mathbf{U}_1+\frac{\epsilon^2}{b}A(t_0)^2\mathbf{U}_3\\ &\quad+\frac{\epsilon^3}{1+\sigma}(\chi A(t_0)-\frac{1}{b}A(t_0)^3)\{\sigma(t-t_0+\frac{1}{1+\sigma})\mathbf{U}_1-\frac{1}{1+\sigma}\mathbf{U}_2\},\end{aligned} \tag{4.40}$$

up to $O(\epsilon^4)$. Accordingly, the initial value reads

$$\begin{aligned}\mathbf{W}(t_0)=\mathbf{u}(t_0,t_0) &= \epsilon A(t_0)\mathbf{U}_1+\frac{\epsilon^2}{b}A(t_0)^2\mathbf{U}_3\\ &\quad+\frac{\epsilon^3}{(1+\sigma)^2}(\chi A(t_0)-\frac{1}{b}A(t_0)^3)\{\sigma\mathbf{U}_1-\mathbf{U}_2\}.\end{aligned} \tag{4.41}$$

Demanding that the solutions at different initial times are continued smoothly, we have the RG/E equation, which reads

$$\begin{aligned}\mathbf{0} &= \frac{d\mathbf{u}}{dt_0}\Big|_{t_0=t},\\ &= \epsilon\frac{dA}{dt}\mathbf{U}_1+2\frac{\epsilon^2}{b}A\frac{dA}{dt}\mathbf{U}_3-\frac{\sigma}{1+\sigma}\epsilon^3(\chi A-\frac{1}{b}A^3)\mathbf{U}_1,\end{aligned} \tag{4.42}$$

up to $O(\epsilon^4)$. Noting that one may self-consistently assume that $dA/dt=O(\epsilon^2)$, we have the amplitude equation

$$\frac{dA}{dt}=\epsilon^2\frac{\sigma}{1+\sigma}(\chi A(t)-\frac{1}{b}A(t)^3). \tag{4.43}$$

With this $A(t)$, the envelope function is given by the initial value

$$\begin{aligned}\mathbf{u}_E(t) &= \mathbf{u}(t;t_0=t),\\ &= \mathbf{W}(t),\\ &= \epsilon A(t)\mathbf{U}_1+\frac{\epsilon^2}{b}A(t)^2\mathbf{U}_3+\frac{\epsilon^3}{(1+\sigma)^2}(\chi A(t)-\frac{1}{b}A(t)^3)(\sigma\mathbf{U}_1-\mathbf{U}_2),\end{aligned} \tag{4.44}$$

or

$$\xi_E(t) = \epsilon A(t), \quad \eta_E(t) = \epsilon A(t) + \frac{\epsilon^3}{1+\sigma}(\chi A(t) - \frac{1}{b}A(t)^3), \quad \zeta_E(t) = \frac{\epsilon^2}{b}A(t)^2. \tag{4.45}$$

We see that the initial value is obtained as the envelope of the local solutions and becomes a global solution to the Lorenz model.

A remark is in order here; Eq.(4.45) shows that the slow manifold which may be identified with a center manifold[18] is given by

$$\eta = (1+\epsilon^2\frac{\chi}{1+\sigma})\xi - \frac{1}{b(1+\sigma)}\xi^3, \quad \zeta = \frac{1}{b}\xi^2. \tag{4.46}$$

Notice that the invariant manifold is modified with the perturbation and also the slow dynamics Eq.(4.43) on the manifold is obtained by our method. One thus sees that the RG method is a powerful tool to extract center manifolds in a concrete form. It is worth mentioning that since the RG method utilizes neutrally stable solutions as the unperturbed ones, it is rather natural that the RG method can extract center manifolds when exist[7].

5 The basis of the RG method for systems

In this section, we give an account of our method in a general setting for ordinary equations as a summary.

Let $\mathbf{X} = {}^t(X_1, X_2, \cdots, X_n)$ and $\mathbf{F}(\mathbf{X}, t; \epsilon) = {}^t(F_1(\mathbf{X}, t; \epsilon), F_2(\mathbf{X}, t; \epsilon), \cdots, F_n(\mathbf{X}, t; \epsilon))$, and $\mathbf{X}$ satisfy the equation

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}, t; \epsilon), \tag{5.1}$$

with the initial condition $\mathbf{X}(t_0) = \mathbf{W}$, where t_0 is arbitrary. We remark that the initial value $\mathbf{W}$ may be dependent on t_0, i.e., $\mathbf{W} = \mathbf{W}(t_0)$. We also write the solution $\mathbf{X}$ as $\mathbf{X}(t; t_0)$ so that the initial-time dependence is explicit.

Let us try to have the perturbation solution of Eq.(5.1) around $t = t_0$ by expanding

$$\mathbf{X}(t; t_0) = \mathbf{X}_0(t; t_0) + \epsilon\mathbf{X}_1(t; t_0) + \epsilon^2\mathbf{X}_2(t; t_0)\cdots. \tag{5.2}$$

We also expand the initial value as

$$\mathbf{X}(t_0, t_0) = \mathbf{W}(t_0) = \mathbf{W}_0(t_0) + \epsilon\mathbf{W}_1(t_0)) + \epsilon^2\mathbf{W}_2(t_0) + \cdots. \tag{5.3}$$

In fact, $\mathbf{X}_i(t_0, t_0) = \mathbf{W}_i(t_0)$. We suppose that an approximate solution $\tilde{\mathbf{X}} = \tilde{\mathbf{X}}(t; t_0)$ to the equation up to $O(\epsilon^p)$ is obtained. It implies that the initial value $\mathbf{W}(t_0)$ at $t = t_0$ coincides with an exact solution up to $O(\epsilon^p)$. We also have

$$\frac{d\tilde{\mathbf{X}}(t; t_0, \tilde{\mathbf{W}}(t_0))}{dt} = \mathbf{F}(\tilde{\mathbf{X}}(t; t_0, \tilde{\mathbf{W}}(t_0)), t; \epsilon) + O(\epsilon^p). \tag{5.4}$$

One may say that now we have a family of the orbits given by the functions $\tilde{\mathbf{X}}(t;t_0,\tilde{\mathbf{W}}(t_0))$ with t_0 parameterizing the orbits. We see that the envelope E of the family of the orbits which contacts with each curve at $t=t_0$ will give an approximate but global solution of the equation. Thus the envelope function is nothing but the initial value as a function of the initial time

$$\mathbf{X}_E(t)=\tilde{\mathbf{X}}(t;t,\mathbf{W}(t))=\tilde{\mathbf{W}}(t). \tag{5.5}$$

The construction of E is performed as follows: We impose that the RG/E equation

$$\frac{d\tilde{\mathbf{X}}}{dt_0}=\mathbf{0} \tag{5.6}$$

gives the solution $t_0=t$, from which the dynamics of the initial value $\mathbf{W}(t)$ is obtained. Eq.(5.6) may give equations as many as n which are independent of each other. In the applications to describe asymptotic behavior of solutions, the equation is usually reduced to a low-dimensional equation.

In accord with the above relation, one can easily show that $\mathbf{X}_E(t)=\tilde{\mathbf{W}}(t)$ satisfies the original equation uniformly up to $O(\epsilon^p)$. In fact, $\forall t_0$, one has

$$\begin{aligned}\left.\frac{d\mathbf{X}_E}{dt}\right|_{t=t_0} &= \left.\frac{d\tilde{\mathbf{X}}(t;t_0,\mathbf{W}(t_0))}{dt}\right|_{t=t_0}+\left.\frac{d\tilde{\mathbf{X}}(t;t_0,\mathbf{W}(t_0))}{dt_0}\right|_{t=t_0},\\ &= \left.\frac{d\tilde{\mathbf{X}}(t;t_0,\mathbf{W}(t_0))}{dt}\right|_{t=t_0},\\ &= \mathbf{F}(\mathbf{X}_E(t_0),t_0;\epsilon)+O(\epsilon^p),\end{aligned} \tag{5.7}$$

where Eq.(5.6) has been used in the last equality.

6 Concluding remarks

We have described the perturbative RG method for global and asymptotic analysis. We have emphasized its relevance to the classical theory of envelopes and that the method concerns with the initial or boundary values of differential equations. It should be remarked that the notion of envelopes is also useful for improvement of perturbation series in quantum filed theory[3] and for obtaining asymptotic behavior of wave functions in quantum mechanics [10].

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