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# THE OVSIENKO PRODUCT REVISED

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## §1. DEFORMATION QUANTIZATION OF LAURANT POLYNOMIALS

It is well-known that the Moyal bracket gives an unique deformation quantization up to equivalence of the canonical phase space $(\mathbb{R}^{2n}, \sum_{i=1}^{2n} dx^i \wedge dy^i)$. By presenting an interesting deformation quantization, of the space of Laurant polynomials, Ovsienko [3] askes the equivalences of deformation quantizations of this algebras. In this paper, we discuss on this question and show that under suitable conditions, deformation quantizations are equivalent. We also study the stepwise equivalence between the Moyal product and the Ovsienko product for the space of Laurant polynomials.

Let $\mathcal{F} = C^\infty(\mathbb{R})[y, y^{-1}]$ be the space of Laurant polynomials with forms

$$F = \sum_{m=-N}^{N} f_m(x) y^m \quad , \quad f_m \in C^\infty(\mathbb{R}). \tag{1}$$

Endowing with the Poisson bracket given by the formula:

$$\{F, G\}_1 = \frac{\partial F}{\partial y} \cdot \frac{\partial G}{\partial x} - \frac{\partial F}{\partial x} \cdot \frac{\partial G}{\partial y} \quad , \tag{2}$$

we see that $\mathcal{F}$ is an Poisson algebra.

In this paper, we discuss on the equivalence of deformation quantizations of $\mathcal{F}$. Parallel to the usual deformation quantization [1], we redefine:

**Definition 1.** Deformation quantization of $\mathcal{F}$ is an operation

$$\mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{F}[[\nu]], \quad F * G = \sum \nu^r R_r(F, G) \tag{3}$$

where $\nu$ is a formal parameter satisfying the following:

(D1) (3) is extended to $\mathcal{F}[[\nu]] \times \mathcal{F}[[\nu]]$ as a formally associative product.
(D2) $R_0(F, G) = FG$, $R_1(F, G) = \frac{1}{2}\{F, G\}_1$.
For a deformation quantization, we impose the following condition;
(D3) For any $m, n \in \mathbb{Z}$ and $f, g \in C^\infty(\mathbb{R})$,

$$R_r(fy^m, gy^n) = R_r^{m,n}(f, g) y^{m+n-r} \tag{4}$$

for every $r$ and $R_r^{m,n}(\ ,\ )$ are bidifferential operators on $C^\infty(\mathbb{R}) \times C^\infty(\mathbb{R})$.

A deformation quantization of $\mathcal{F}$ with (D3) is called *homogeneous type*. In this paper, we are concerned with deformation quantizations of $\mathcal{F}$ of homogeneous type. We see that there are interesting examples of deformation quantizations of $\mathcal{F}$ of homogeneous type; the Moyal products and the Ovsienko product.

A well-known example of deformation quantization of the algebra $\mathcal{F}$ is given by the Moyal product:

$$F *_M G = \sum \nu^r C_r(F, G) \tag{5}$$

where

$$C_r(F, G) = \frac{1}{2^r r!} \sum_{i=1}^{r} (-1)^i \cdot \binom{r}{i} \cdot \frac{\partial^r F}{\partial y^{r-i} \partial x^i} \cdot \frac{\partial^r G}{\partial y^i \partial x^{r-i}} \quad . \tag{6}$$

The Moyal product (5) gives a non trivial deformation quantization of $\mathcal{F}$, which is denoted by $(\mathcal{F}[[\nu]], *_M)$.

**Definition 2 ([2]).** Two deformation quantizations $(\mathcal{F}[[h]], *_1)$ and $(\mathcal{F}[[\nu]], *_2)$ of $\mathcal{F}$ of homogeneous type are *full homogeneously equivalent* if there exists a series

$$T = Id + \sum_{r=1} \nu^r T_r, \tag{7}$$

where $T_r$ is a linear operation on $\mathcal{F}$ satisfying for any $m \in \mathbb{Z}$ and any $f \in C^\infty(\mathbb{R})$,

$$T_r(f y^m) = T_r(f) y^{m-r} \tag{8}$$

and $T_r(f)$ are differential operators such that

$$F *_1 G = T^{-1}(TF *_2 TG). \tag{9}$$

Furthermore, these are called *even homogeneously equivalent* if there exists a series

$$T = Id + \sum_{r:even} \nu^r T_r, \tag{7'}$$

where $T_r$ is a linear operation on $\mathcal{F}$ satisfying for any $m \in \mathbb{Z}$ and any $f \in C^\infty(\mathbb{R})$,

$$T_r(f y^m) = T_r(f) y^{m-r} \tag{8'}$$

and $T_r(f)$ are differential operators such that

$$F *_1 G = T^{-1}(TF *_2 TG). \tag{9'}$$

First purpose of this paper is to show that deformation quantizations of $\mathcal{F}$ of homogeneous type are homogeneously equivalent to $(\mathcal{F}[[\nu]], *_M)$. Let $(\mathcal{F}[[\nu]], *)$ be a deformation quantization of $\mathcal{F}$ of homogeneous type with the form (3). Set for any $m,n \in \mathbb{Z}$ and $f,g \in C^\infty(\mathbb{R})$:

$$R_r(fy^m, gy^n) = \sum_{i,j<\infty} R^{m,n}_{r,ij}(x) f^{(i)} g^{(j)} y^{m+n-r} \tag{10}$$

where $f^{(i)} = \frac{d^i}{dx^i} f$, and $R^{m,n}_{r,ij}(x) \in C^\infty(\mathbb{R})$.

(D2) is equivalent to

$$R^{m,n}_{0,ij}(x) = \begin{cases} 1 & i,j=0 \\ 0 & otherwise \end{cases} \tag{11}$$

$$R^{m,n}_{1,ij}(x) = \begin{cases} m & i=0, j=1 \\ -n & i=1, j=0 \\ 0 & otherwise \end{cases} \tag{12}$$

**Theorem A.** *Let $(\mathcal{F}[[\nu]], *)$ be a deformation quantization of $\mathcal{F}$ of homogeneous type. Then, $(\mathcal{F}[[\nu]], *)$ is full homogeneously equivalent to $(\mathcal{F}[[\nu]], *_M)$.*

By Theorem A, any deformation quantization of $\mathcal{F}$ of homogeneous type is homogeneously equivalent to the Moyal product. Ovsienko [3] presented an interesting example of deformation quantization of $\mathcal{F}$ by using the transvectance (See in §4), where we call it *Ovsienko product.* In particular, Ovsienko product is homogeneously equivalent to the Moyal product.

Second part of this paper is to study the equivalence between of the deformation quantizations gives by the Moyal product and the Rankin-Cohen bracket from the view point of stepwise constructions of equivalences.

On the algebra $\mathcal{F}$, we define two deformation quantizations $(\mathcal{F}[[\nu]], *_M)$ and $(\mathcal{F}[[\nu]], *_O)$, where $*_M$ is Moyal bracket given by (8) and $*_O$ is given by the transvectance(see §4 for details). Comparing these products, we have the following, which is found through the communication with Ovsienko:

**Theorem B.** *$(\mathcal{F}[[\nu]], *_M)$ and $(\mathcal{F}[[\nu]], *_O)$ are not even homogeneously equivalent.*

## §2. Ovsienko product

By using the Rankin-Cohen product (Transvectance), Ovsienko [3] introduced a defromation quantization of $\mathcal{F}$ which has different flavor from the Moyal product. Let us recall this product:

For $F, G \in \mathcal{F}$, $F = f(x)y^m$, $G = g(x)y^n$, $m,n \in \mathbb{Z}$, we set

$$F *_O G = \sum_{k=0}^{\infty} \frac{\nu^k}{k! 2^{2k}} J^{m,n}_k(f,g) y^{m+n-k} \tag{19}$$

This product $*_O$ gives a deformation quantization of $\mathcal{F}$ (cf.[O]). where

$$J^{m,n}_k(f,g) = \sum_{i+j=k} (-1)^i \binom{k}{i} \frac{(2m-i)!}{(2m-k)!} \frac{(2n-j)!}{(2n-k)!} f^{(i)} g^{(j)}. \tag{20}$$

Note that $[x,y]_{*_O} = -\nu$, and

$$[f(x),y]_{*_O} = -\nu f'(x).$$

One of the feature of Rankin-Cohen product is that the product formula of $(fy^m) *_O (gy^n)$ is given by differential operators of constant coefficients. Hence we see for instance

$$x *_O (fy^{-m}) = xfy^{-m} + \sum_{k\geq 1} \frac{\nu^k}{2^{2k}}(2m+k-1)f^{(k-1)}y^{-(m+k)}. \tag{21}$$

Using (21), we compute the $*$-exponential $e_*^{itx}$ in the form:

$$e_{*_O}^{itx} = \sum_{m\geq 0} c_m(it\nu)^m e^{itx}y^{-m} \tag{22}$$

Inserting (22) to the differential equation $\frac{d}{dt}e_{*_O}^{itx} = ix *_O e_{*_O}^{itx}$, we have

$$c_{m+1} = \sum_{l=0}^{m} \frac{(-1)^{m-l}(m+l)}{2^{2(m-l+1)}(m+1)}c_l, \quad c_0 = 1. \tag{23}$$

Note that $c_1 = 0$. Using (23), we write the intertwiner:

$$f_*(x) = f(x) + \sum_{m\geq 2} \nu^m c_m f^{(m)}(x)y^{-m}. \tag{24}$$

Thus, $f_*(x)$ is given in terms of a series of differential operators with constant coefficients.

Since $[x,y]_{*_O} = -\nu$, $[f_*(x),y]_{*_O} = -\nu(f'(x))_* = -\nu(f_*)'(x)$. We also have

$$[f_*y^m, g_*y^n]_{*_O} = (mfg' - nfg')_{*_O}y^{m+n-1}. \tag{25}$$

Thus, we have

**Corollary 4.** *The deformation quantization $(\mathcal{F}[[\nu]], *_O)$ is equivalent to $(\mathcal{F}[[\nu]], *_M)$.* ∎

Cohen-Manin-Zagier [2] generalized Rankin-Cohen bracket from the point of conjugate-automorphic ΦDOp's. It also give a generalization of the deformation quantization given by (19): For integers $n,k,l \geq 0$, set coefficients $t_n(k,l)$ by

$$t_k(m,n) = \left(\frac{-1}{4}\right)^k \sum_{j\geq 0} \binom{k}{2j} \frac{\binom{-\frac{1}{2}}{j}\binom{-\frac{3}{2}}{j}\binom{\frac{3}{2}}{j}}{\binom{-k-\frac{1}{2}}{j}\binom{-n-\frac{1}{2}}{j}\binom{k+m+n-\frac{3}{2}}{j}}, \tag{26}$$

where we put $\binom{w}{e} = w(w-1)\cdots(w-e+1)/e!, e \geq 0, e! = 0$. For example, we have $t_0 = 1$, and $t_1 = \frac{-1}{4}$.

We set for $F = y^m f, G = y^n g \in \mathcal{F}, m,n \in \mathbb{Z}$:

$$F \hat{*} G = \sum_{n=0} \nu^k t_k(m,n) J_k^{m,n}(f,g). \tag{27}$$

By the result by Cohen-Zagien-Manin [2], (27) gives a deformation quantization of $\mathcal{F}$. It is easily seen that the product (27) satisfies the condition (D1-3).

If we replace $\nu$ to $\frac{1}{2}\nu'$, recalling the bracket (27) again. We have

**Corollary C.** *The deformation quantization $(\mathcal{F}[[\nu]], \hat{*})$ is equivalent to $(\mathcal{F}[[\nu]], *_M)$.* ∎

For the proof of Theorem B, we investigate the homogeneously equivalence between $(\mathcal{F}[[\nu]], *_M)$ and $(\mathcal{F}[[\nu]], *_O)$ carefully by direct computations.

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