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Some isomorphism theorems of cohomology groups for completely integrable connections

お茶の水女子大学理学部数学科　真島　秀行
(Hideyuki Majima, Ochanomizu University)

1 Introduction

About 14 years ago, the author proved an isomorphism theorem between the cohomology group of complex of global meromorphic sections derived from a completely integrable connection and the cohomology group of kernel sheaf with values in the sheaf of functions asymptotically developable to the formal seies 0 for the connection ([5]). Recently, several rescerchers, who are interested in the intersection theory for differential equations with singular points, pushed him to prove the C^{∞} version (cf. [1] , [2]). In this paper, firstly, we give a short review of the isomorphism theorem in asymptotic analysis and some examples with concrete calculation of basis for the cohomology groups. Secondly, we explain the C^{∞} version.

2 Isomorphism Theorem in Asymptotic Analysis

Let M be a complex manifold and let H be a divisor on M at most normal crossing singularities. we denote by $\Omega^{p}(*H)$ the sheaf of germs of meromorphic p-forms which are holomorphic in $M-H$ and have poles on H and denote by $\mathcal{S}$ a locally free sheaf of $\mathcal{O}$-modules of rank m on M. Put $\mathcal{S}\Omega^{p}(*H)=\Omega^{p}(*H)\otimes_{\mathcal{O}}\mathcal{S}$ for $p=0,\ldots,n$. For $p=0$, instead of $\mathcal{S}\Omega^{0}(*H)$, we use frequently $\mathcal{S}(*H)$ of which the restriction to U, $\mathcal{S}(*H)|_{U}$ is isomorphic to

$$(\mathcal{O}(*H))^{m}|_{U}=(\mathbf{C}^{m}\otimes_{C}\mathcal{O}(*H))|_{U}$$

and the isomorphism is denoted by g_{U}.

Let ∇ be a connection on $\mathcal{S}\Omega^{0}(*H)$: ∇ is an additive mapping of $\mathcal{S}\Omega^{0}(*H)$ into $\mathcal{S}\Omega^{1}(*H)$ satisfying "Leipnitz' rule"

$$\nabla(f\cdot u)=u\otimes df+f\cdot\nabla(u),$$

for all sections $f\in\Omega^{0}(*H)(U)$ and $u\in\mathcal{S}\Omega^{1}(*H)(U)$. We suppose that the connection is integrable, that is, the composite mapping

$$\nabla_{2}:\mathcal{S}\Omega^{0}(*H)\longrightarrow\mathcal{S}\Omega^{1}(*H)\longrightarrow\mathcal{S}\Omega^{2}(*H),$$

is a zero mapping.

If we take adequately an open covering $\{U_k\}_k$ on M, then to give the connection ∇ means as follows: for each U_k, the mapping

$$g_{U_k} \circ \nabla \circ g_{U_k}^{-1} : \Omega^0(*H)(U_k)^m \longrightarrow \Omega^1(*H)(U_k)^m,$$

is induced by a mapping

$$\nabla_k : \Omega^0(*H)(U_k)^m \longrightarrow \Omega^1(*H)(U_k)^m,$$

which is represented by $(d + \Omega_k)$ under a generator system

$$< e_{k,1}, \cdots, e_{k,m} >$$

of $(\mathcal{O}(U_k))^m$, i.e.

$$\nabla_k(< e_{k,1}, ..., e_{k,m} > u) =< e_{k,1}, ..., e_{k,m} > (du + \Omega_k u)$$

where Ω_k is an m-by-m matrix of meromorphic 1-forms on U_k at most with poles in $U_k \cap H$.

Let $x_1, \cdots, x_n$ be holomorphic local coordinates on U_k and suppose

$$U_k \cap H = \{(x_1, \cdots, x_n) \,|\, x_1 \cdots x_{n''} = 0\},$$

then Ω_k is of the form

$$\Omega_k = \sum_{i=1}^{n''} x^{-p_i} x_i^{-1} A_i(x) dx_i + \sum_{i=n''+1}^{n} x^{-p_i} A_i(x) dx_i,$$

where $p_i = (p_{i1}, \cdots, p_{in''}, 0, ..., 0) \in \mathbf{N}^n$ and $A_i(x)$ is an m-by-m matrix of holomorphic functions in U_k for $i = 1, \cdots, n$.

The connection ∇ is integrable if and only if, for k, $d\Omega_k + \Omega_k \wedge \Omega_k = 0$. For any k, k', denote by $g_{kk'}$ the isomorphism

$$g_{kk'} : (O(U_k \cap U_{k'}))^m \longrightarrow (O(U_k \cap U_{k'}))^m,$$

induced by the isomorphism

$$g_{U_k} g_{U_{k'}}^{-1} : (O|_{U_k \cap U_{k'}})^m \longrightarrow (O|_{U_k \cap U_{k'}})^m.$$

Then, by using the generator systems, $g_{kk'}$ is represented by $G_{kk'}$ a matrix of elements in $O(U_k \cap U_{k'})$, i.e.

$$g_{kk'} < e_{k,1}, \cdots, e_{k,m} >=< e_{k,1}, \cdots, e_{k,m} > G_{kk'},$$

and

$$\Omega_k = G_{kk'}^{-1} dG_{kk'} + G_{kk'}^{-1} \Omega_k G_{kk'},$$

in $U_k \cap U_{k'}$.

Denote by M^- the real blow-up of M along H and denote by pr the natural projection from M^- to M. Let $\mathcal{A}^-$ be the sheaf of germs of functions strongly asymototically developable, and let $\mathcal{A}'^-$ and $\mathcal{A}_0^-$ be the sheaves of germs of functions strongly asymototically developable to $\mathcal{O}_{M|\hat{H}}$ and to 0, respectively, over the real blow-up M^-. Define the locally free $\mathcal{A}^-$ (resp. $\mathcal{A}'^-$)-sheaf $\mathcal{S}^-\Omega^p(*H)$ (resp. $\mathcal{S}'^-\Omega^p(*H)$) over the real blow-up M^- by $\mathcal{S}^-\Omega^p(*H) = \mathcal{A}^- \otimes_{pr^*\mathcal{O}} pr^*\mathcal{S}\Omega^p(*H)$ (resp. $\mathcal{S}'^-\Omega^p(*H) = \mathcal{A}'^- \otimes_{pr^*\mathcal{O}} pr^*\mathcal{S}^-\Omega^p(*H)$, and the locally free $\mathcal{A}_0^-$-sheaf $\mathcal{S}_0^-\Omega^p$ by $\mathcal{S}_0^-\Omega^p = \mathcal{A}_0^- \otimes_{pr^*\mathcal{O}} pr^*\mathcal{S}\Omega^p(*H)$ for $p = 0, \cdots, n$. Then, by a natural way, we obtain integrable connections

$$\nabla^- : \mathcal{S}^-(*H) \longrightarrow \mathcal{S}^-\Omega^1(*H),$$

$$\nabla'^- : \mathcal{S}'^-(*H) \longrightarrow \mathcal{S}'^-\Omega^1(*H),$$

and

$$\nabla_0^- : \mathcal{S}_0^- \longrightarrow \mathcal{S}_0^-\Omega^1(*H).$$

For simplicity, we use also ∇ instead of ∇^-, ∇'^- and ∇_0. By the integrability, we can consider the complexes of sheaves

$$\mathcal{S}^-(*H) \xrightarrow{\nabla} \mathcal{S}^-\Omega^1(*H) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{S}^-\Omega^n(*H) \xrightarrow{\nabla} 0$$

$$\mathcal{S}'^-(*H) \xrightarrow{\nabla} \mathcal{S}'^-\Omega^1(*H) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{S}'^-\Omega^n(*H) \xrightarrow{\nabla} 0$$

$$\mathcal{S}_0^- \xrightarrow{\nabla} \mathcal{S}_0^-\Omega^1(*H) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{S}_0^-\Omega^n(*H) \xrightarrow{\nabla} 0.$$

Suppose here that ∇ satisfies the following condition: for any point $p \in H$, under the local representation of ∇,

(H.1) $p_i = 0$ and $A_i(0)$ has no eigenvalue of integer for all $i \in [1, n]$,

or

(H.2)$p_{ii} > 0$ and $A_i(0)$ is invertible for all $i \in [1, n'']$ or $p_i = 0$ and $A_i(0)$ has no eigenvalue of integer for all $i \in [1, n'']$.

Then, we can assert

Theorem 1. If the assumption (H.1) is satisfied for any point in H, then the above three sequences are exact. If (H.1) or (H.2) is satisfied for any point in H, then the above sequences are exact except the second.

Moreover, we consider the complex $(\Gamma(M^-, \mathcal{S}^-\Omega^\bullet(*H)), \nabla)$ of global sections:

$$\mathcal{S}^-(*H)(M^-)^m \xrightarrow{\nabla} \mathcal{S}^-\Omega^1(*H)(M^-)^m \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{S}^-\Omega^1(*H)(M^-)^m \xrightarrow{\nabla} 0.$$

Then, we can prove

Theorem 2. If $H^1(M, \mathcal{S}) = 0$ and if (H.1) or (H.2) is satisfied for any point in H, then the following isomorphim is valid:

$$H^1(\Gamma(M^-, \mathcal{S}^-\Omega^\bullet(*H)), \nabla) \cong H^1(M^-, \mathcal{K}er\nabla_0^-),$$

where $\mathcal{K}er\nabla_0$ denote the sheaf of solutions of $\nabla_0{}^-$.

Note that we have the natural isomorphism by the projection pr

$$H^1(\Gamma(M,\Omega^\bullet(*H)),\nabla) \cong H^1(\Gamma(M^-,\mathcal{S}^-\Omega^\bullet(*H))$$

and we can rewrite the theorem as

Theorem 2'. If $H^1(M,\mathcal{S}) = 0$ and if (H.1) or (H.2) is satisfied for any point in H, then the following isomorphim is valid:

$$H^1(\Gamma(M,\mathcal{S}\Omega^\bullet(*H)),\nabla) \cong H^1(M^-,\mathcal{K}er\nabla_0^-).$$

Example. Consider the case where $M = \mathbf{P}^1_{\mathbf{C}}$, $H = \{\infty\}$ and $\nabla = d + x^{r-1}\Lambda$. We can find the basis of $H^1(\Gamma(M,\Omega^\bullet(*H)),\nabla)$:

$$H^1(\Gamma(M,\Omega^\bullet(*H)),\nabla) = \mathbf{C} < [dx],\cdots,[x^{r-2}dx] > .$$

On the other hand, we can find the basis of $H^1(M^-,\mathcal{K}er\nabla_0^-)$ in the following manner. Let $\{U_k \mid k = 1,\cdots,r\}$ be the covering of $M - H$, where

$$U_k = \{\, x \in \mathbf{C} \mid |x| \geq R, \frac{(4k-5)\pi}{2r} < \arg x < \frac{(4k+1)\pi}{2r}\} \cup \{\, x \in \mathbf{C} \mid |x| < R\}$$

for $k = 1,\cdots,r$. We put $U_{r+1} = U_1$ and for $k = 1,\cdots,r$, define 1-cocycles $\{f^{(k)}_{j,j+1}\}$ $(j = 1,\cdots,r)$ by

$$f^{(k)}_{j,j+1}(x) = \begin{cases} \exp(-\frac{1}{r}x^r), & (x \in U_j \cap U_{j+1})\,(j = k) \\ 0, & (x \in U_j \cap U_{j+1})\,(j \neq k) \end{cases}$$

Then, we have

$$< [\{f^{(k)}_{j,j+1}\}_{j=1,\cdots,r}],\ k = 1,\cdots,r >$$

as a basis of $H^1(M^-,\mathcal{K}er\nabla_0^-)$.

3 Isomorphism Theorem in C^∞ case

We restrict here to treat the case of one variable. We give a C^∞ version of isomorphism theorem of cohomology group. Let M, H, ∇ be as above. Let $\mathcal{P}_0^{(j,h)}$ be the sheaf of germs of $C^\infty(j,h)$–forms infinitely flat on H over M. Consider the following double complex of sheaves:

$$\begin{array}{ccc} \mathcal{P}_0^{(0,0)} & \xrightarrow{\bar\partial} & \mathcal{P}_0^{(0,1)} \\ \nabla\downarrow & & \nabla\downarrow \\ \mathcal{P}_0^{(1,0)} & \xrightarrow{\bar\partial} & \mathcal{P}_0^{(1,1)} \end{array}$$

and the complex of global sections

$$\begin{array}{ccc} \mathcal{P}_0^{(0,0)}(M) & \xrightarrow{\bar\partial} & \mathcal{P}_0^{(0,1)}(M) \\ \nabla\downarrow & & \nabla\downarrow \\ \mathcal{P}_0^{(1,0)}(M) & \xrightarrow{\bar\partial} & \mathcal{P}_0^{(1,1)}(M) \end{array}$$

and the associated simple complex

$$GC^\infty K^\cdot : \mathcal{P}_0^{(0,0)}(M) \xrightarrow{\nabla_\omega+\bar\partial} \mathcal{P}_0^{(0,1)}(M) \oplus \mathcal{P}_0^{(1,0)}(M) \xrightarrow{\nabla_\omega+\bar\partial} \mathcal{P}_0^{(1,1)}(M) \longrightarrow 0.$$

Then, we know the following lemma formally due to Malgrange ([6]).

Lemma 3. We have the following isomorphism for $j = 0, 1, 2$:

$$H^j(M^-, \mathcal{K}er\nabla_0^-) \cong H^j(GC^\infty K^\cdot)$$

By Theorem 2' and Lemma 3, we can derive the

Theorem 4. We have the following isomorphism:

$$H^1(\Gamma(M, \Omega^\bullet(*H)), \nabla) \cong H^1(GC^\infty K^\cdot).$$

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