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MASLOV FORM ON THE GROUP GENERATED BY INVERTIBLE FOURIER INTEGRAL OPERATORS

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1. INTRODUCTION

It is well-known that there are many examples of the infinite dimensional Fréchet-Lie groups. For instance, suppose that M is a compact manifold with symplectic (or contact) structure Ω. Then it is known that the group $Diff_{\Omega}(M)$ of all diffeomorphisms on M preserving the structure Ω is an infinite dimensional Fréchet-Lie group. Moreover, the group $(FIO)^0(N)$ generated by the invertible Fourier integral operators of order 0 on compact Riemannian manifold N is also an infinite dimensional Fréchet-Lie group (cf.[Om], [OMY], [OMYK], [ARS]). From the physical point of view, the group $Diff_{\Omega}(M)$ (resp. $(FIO)^0(N)$) gives the framework of the dynamics of classical (resp. quantum) mechanics, that is, the fundamental solution of the Hamiltonian equation (resp. Schrödinger eqaution) is 1-parameter group in $Diff_{\Omega}(M)$ (resp. $(FIO)^0(N)$). Furthermore, the group $(FIO)^0(N)$ can be viewed as the quantized group of $Diff_{\Omega}(M)$.

On the other hand, as mentioned in [Ma] and [Ar], the geometrical structure Ω induces the notion of the Lagrangian-Grassmannian variety, Lagrangian submanifold and Maslov form.

In order to define Maslov form, we fix a Lagrangian submanifold, and Maslov form is defined as a closed 1-form on the Lagrangian submanifold.

Key words and phrases. Maslov form, Symplectic topology, Quantization, Infinite dimensional Lie group, Fourier integral operator, Oscillatory integral, etc..

The purpose of this work is to define Maslov form on $Diff_{\Omega}(T^*N)$ and $(FIO)^0(N)$ by regarding a diffeomorphism $\varphi \in Diff_{\Omega}(T^*N)$ as a Lagrangian submanifold, that is, we regard the Lagrangian submanifold as a variable on $Diff_{\Omega}(T^*N)$. Furthermore, as seen in §4, this form is essentially determined by the determinant of the "complex part" for the push-forward $d\varphi$ (cf.[Mi2]).

In this article, we restricted our concern to the groups of all contact diffeomorphisms on unit cosphere bundle on compact Riemannian manifold and the group generated by invertible Fourier integral operators. By a similar way, we can define Maslov forms on the group of all contact diffeomorphisms on the odd dimensional sphere S^n, and the group generated by invertible oscillatory integral transformations (cf. [Mi1], [Mi3]).

2. Preliminaries

2.1. Examples of infinite dimensional Lie group. We recall some examples of infinite dimensional Lie groups (cf.[Om], [OMY], [OMYK], [ARS]). First we assume that N is an orientable compact Riemannian manifold. In this article, we treat the following groups:

- The group of contact transformations on unit cosphere bundle:

(2.1)
$$Diff_{\theta}(S^*N) = \big\{\hat{\varphi} : \text{ diffeomorphism} \;\big|\; \hat{\varphi}^*\theta = f_{\hat{\varphi}} \cdot \theta \text{ (where} f_{\hat{\varphi}} \text{ is a non-varnishing } C^{\infty}\text{-function on } S^*N \text{ depending on } \varphi)\big\}.$$

- The group of homogeneous symplectic diffeomorphisms:

(2.2)
$$Diff_{\Theta}^{(1)}(T_{\star}^*N) = \big\{\varphi \in Diff(T_{\star}^*N) \;\big|\; \varphi^*\Theta = \Theta, \; \varphi(x, r\xi) = \big(\varphi^{(\bar{x})}(x,\xi), r\varphi^{(\bar{\xi})}(x,\xi)\big)(\forall r \neq 0)\big\},$$

where $T_{\star}^*N = T^*N - N$.

- The group of invertible Fourier integral operators:

(2.3)

$$(FIO)^0(N) = \text{generated by } \{F(a,\phi) : \text{ Fourier integral operator on } N \mid a(x, r\xi)(\sim a_0(x,\xi) + a_{-1}(x,\xi)r^{-1} + \cdots \doteqdot 1) \text{ is an amplitude function, } \phi \text{ is a phase function determined by } \varphi\},$$

where φ stands for a homogeneous symplectic diffeomorphism on the punctured cotangent bundle $T^*_\star N$.

The group $Diff_\theta(S^*N)$ can be identified with $Diff^{(1)}_\Theta(T^*_\star N)$ using the following mapping:

$$i : Diff_\theta(S^*N) \ni \hat{\varphi} \mapsto \varphi \in Diff^{(1)}_\Theta(T^*_\star N), \tag{2.4}$$

where

$$\varphi(x, r\xi) = \left(\hat{\varphi}^{(\bar{x})}(x,\xi), \frac{r}{f_{\hat{\varphi}}(x,\xi)}\hat{\varphi}^{(\bar{\xi})}(x,\xi)\right), \tag{2.5}$$

$r \in (0,\infty)$, $(x,\xi) \in S^*N$.

On the other hand there exists a mapping $\tilde{\pi}$ of $(FIO)^0(N)$ onto the identity component $Diff^{(1)}_\Theta(T^*_\star N)_0$.

(2.6)

$$\tilde{\pi} : (FIO)^0(N) \ni F(a,\phi) \mapsto WF(F(a,\phi)) = \varphi^{-1} \in Diff^{(1)}_\Theta(T^*_\star N)_0,$$

where $WF(F(a,\phi))$ is the wave front set of the distribution kernel of Fourier integral operator $F(a,\phi)$.

2.2. Summary of Maslov form. We review the definition of Maslov form briefly (cf.[Ar]).

Let (V,h) be an n-dimensional Hermitian space with Hermitian inner product h, and $g(u,v) = \text{Re } h(u,v)$, $\sigma(u,v) = \text{Im } h(u,v)$. By fixing an orthonormal basis $(e_1, \cdots, e_n)$, we can identify (V,h) with $(\mathbb{C}^n, h)$, where $h(z,z') = \sum_{i=1}^n z_i \cdot \bar{z}'_i$ for $z = (z_1, \cdots, z_n)$, $z' = (z'_1, \cdots, z'_n) \in$

$\mathbb{C}^n$. Let $\Lambda(n)$ be the Lagrangian-Grassmannian manifold of symplectic space $(\mathbb{C}^n, \sigma)$:

(2.7)
$$\Lambda(n) = \{\lambda : \text{ subspace of } \mathbb{C}^n \,\big|\, \dim_{\mathbb{R}} \lambda = n,\ \sigma(z, z') = 0\ (\forall z,\ z' \in \lambda)\}.$$

It is well-known that the unitary group $U(n)$ acts on $\Lambda(n)$ transitively, and also $\Lambda(n) = U(n)/O(n)$ (cf.[Ar]). Let $\lambda_{im} = \{ix | x \in \mathbb{R}^n\} \in \Lambda(n)$. Then, for any $\lambda \in \Lambda(n)$, there exists $U_\lambda \in U(n)$ satisfying $\lambda = U_\lambda \lambda_{im}$. Using this U_λ, we can define mappings W of $\Lambda(n)$ into $U(n)$ and Det^2 of $\Lambda(n)$ into S^1 as follows:

$$W(\lambda) = U_\lambda {}^t U_\lambda, \qquad \mathrm{Det}^2(\lambda) = \det W(\lambda). \tag{2.8}$$

Next, let L be a Lagrangian submanifold of $\mathbb{C}^n$, and let ι be the inclusion mapping. Then $\iota_*(T_pL)$ can be regarded as a Lagrangian subspace of $\mathbb{C}^n$ $(\forall p \in L)$. For any $p \in L$, define $\tau : L \to \Lambda(n)$ by $\tau(p) = \iota_*(T_pL)$. Maslov form m_L of L is given by

$$m_L = \left(\mathrm{Det}^2 \circ \tau\right)^* \left(\frac{1}{2\pi\sqrt{-1}} \frac{dz}{z}\right), \tag{2.9}$$

where $z \in \mathbb{C}, \quad |z| = 1$.

Next we recall the construction of the generating function of Lagrangian submanifold L of $\mathbb{C}^n$. For example, let p_0 be a point of L such that $T_{p_0}L$ transversely intersects $\lambda_{Re} = \{\xi | \xi \in \mathbb{R}^n\}$. Then there is a neighborhood V of p_0 in L parameterized by the variable $x \in \lambda_{Re}$, i.e. $L|_V = \{(x, \xi(x)) | x \in U \subset \lambda_{im}\}$. On the other hand, the restriction of standard canonical 1-form θ to L is a closed 1-form. Thus, we have a local potential function S of $\theta|_V$ as follows:

$$S(x) = \int_{p_0}^{p} \theta, \quad \text{where} \quad p_0 = (0, \xi(0)),\ p = (x, \xi(x)). \tag{2.10}$$

Hence we have

$$L|_V = \{(x, \frac{\partial S(x)}{\partial x}) | x \in U\}. \tag{2.11}$$

We shall refer to the function S as the generating function of L around p_0. Furthermore, it is well-known in [Ar] that

$$W(\tau(p)) = \frac{E - \sqrt{-1}\partial_x\partial_x S(x)|_{x(p)}}{E + \sqrt{-1}\partial_x\partial_x S(x)|_{x(p)}}, \tag{2.12}$$

where E is the $n \times n$-identity matrix.

3. Notations

3.1. Complex part. First of all, we mention the notion of the "complex part" of matrix. Let $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\varphi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be $2n \times 2n$-matrix and j be the identification mapping:

$$j : \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \to A + \sqrt{-1}B.$$

Using these notations, we define the *complex part* of matrix as follows:

$$\begin{aligned} \mathfrak{C}(\varphi) &= \frac{1}{2}\left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} + J\begin{pmatrix} A & B \\ C & D \end{pmatrix}{}^tJ\right\} \\ &= \frac{1}{2}\begin{pmatrix} A+D & B-C \\ -(B-C) & A+D \end{pmatrix} \end{aligned} \tag{3.1}$$

Using the above notation, we easily have:

Lemma 3.1. (1) $J\mathfrak{C}(\varphi) = \mathfrak{C}(\varphi)J$.

(2) *If* $U \in U(n)$, $\mathfrak{C}(U) = U$.

(3) *If* H *is a real Hermitian(symmetric),* ${}^t\mathfrak{C}(H) = \mathfrak{C}(\varphi)$.

(4) *If* H *is a real Hermitian symplectic,* $JH = H^{-1}J$.

(5) *If* H *is a real Hermitian symplectic and* U *is a unitary matrix,* $\mathfrak{C}(UH) = \frac{1}{2}U(H + H^{-1})$.

(6) *If* $\varphi \in Sp(n, \mathbb{R})$, $\mathfrak{C}(\varphi)$ *is a regular matrix.*

(7) *If we take the polar decomposition* $\varphi = UH$, $\mathfrak{C}(\varphi) = \frac{1}{2^n}\det U \prod_{i=1}^n(\lambda_i + \lambda_i^{-1})^2$, *where* λ_i, λ_i^{-1} *are eigenvalues of the symplectic matrix* φ.

Remark. The complex part depends on the choice of symplectic coordinate.

3.2. Symplectic normal coordinate of cotangent bundle of orientable Riemannian manifold. Let N be an orientable Riemannian manifold and U_λ be an open covering of N. Suppose that $e_{i,\lambda}$ $(i = 1, \cdots, n)$ is an orthonormal frame on U_λ. Then the dual frame $e^i_\lambda = e^*_{i,\lambda}$ $(i = 1, \cdots, n)$ is an orthonormal frame of cotangent bundle (T^*N, π, N). Using this frame, we can define *symplectic normal coordinate of cotangent bundle around* $p = (x_0, \xi)$ as follows:

(3.2)

$$(X^1, \cdots, X^n; \Xi_1, \cdots, \Xi_n) \mapsto \big(\exp_{x_0}(\textstyle\sum X^i e_i(x_0)); {}^t(d\exp_{x_0})^{-1}_\xi \sum \Xi_i e^i(x_0)\big).$$

For any symplectic diffeomorphism φ, we denote the push-forward $d\varphi_p$ as

$$\begin{pmatrix} \partial X'/\partial X & \partial X'/\partial \Xi \\ \partial \Xi'/\partial X & \partial \Xi'/\partial \Xi \end{pmatrix}\Bigg|_p, \tag{3.3}$$

where

(3.4)

$$\begin{aligned}&(X^{1\prime}, \cdots, X^{n\prime}; \Xi'_1, \cdots, \Xi'_n)\\ &\qquad \mapsto \big(\exp_{x_0}(\textstyle\sum X^{i\prime} e_i(\pi(\varphi(p))); {}^t(d\exp_{x_0})^{-1}_\xi \sum \Xi'_i e^i(\pi(\varphi(p)))\big)\end{aligned}$$

is a symplectic orthonormal coordinate around $\varphi(p)$.

4. Definition of $\mathfrak{M}$-form

Using these notation we define *Maslov function* on $Diff_\theta(S^*N)$ as follows: Fix a reference point $p \in S^*N$.

Definition 4.1.

$$\Phi_p(\varphi) = \det(-2j \circ \mathfrak{C}(d\varphi|_p)), \tag{4.1}$$

where

$$j : \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \to A + \sqrt{-1}B.$$

Then we have the following:

Proposition 4.2.

$$\Phi_p(\varphi) = \det\Big[-\frac{\partial \Xi'}{\partial \Xi} - \frac{\partial X'}{\partial X} - \sqrt{-1}\Big\{\frac{\partial X'}{\partial \Xi} - \frac{\partial \Xi'}{\partial X}\Big\}\Big]_p. \tag{4.2}$$

Note that this function is well defined as a C^∞-function on $Diff_\theta(S^*N)$. Although this function is determined by using symplectic orthonormal coordinate at the point p, it is independent of the choice of normal coordinates.

Using the above function(4.2), we define the following closed 1-form:

$$m_{p,c} = \frac{1}{\pi} d_\varphi \arg \Phi_p. \tag{4.3}$$

We call this closed 1-form as $\mathfrak{M}$*-form.* Furthermore, using the following mapping(cf. (2.6)):

$$\tilde{\pi} : (FIO)^0(N) \to Diff_\theta(S^*N), \tag{4.4}$$

we define the following closed 1-form

$$m_{p,q} = \tilde{\pi}^* m_{p,c}. \tag{4.5}$$

Remarks Let p' be another reference point of S^*N, $\gamma(s)$ is a smooth curve from p to p' and φ_t is a curve in $Diff_\theta(S^*N)$. Then $\varphi_t(\gamma(s))$ gives a homotope between $\varphi_t(p)$ and $\varphi_t(p')$. If we do not fix the point $p \in S^*N$, then we have a function Φ of $S^*N \times Diff_\theta(S^*N)$ into $\mathbb{C}$.

Proposition 4.3. *Suppose that $N = S^n$. Set $P = \sqrt{-\Delta + (\frac{n-1}{2})^2}$, and Φ_P is a solution of the Schrödinger equation*

$$\frac{d}{dt}\Phi_P = -\sqrt{-1}P\Phi_P, \quad \Phi_P(0) = Id. \tag{4.6}$$

Then

$$\int_{\Phi_P} m_q \neq 0, \quad \int_{\tilde{\pi}(\Phi_P)} m_c \neq 0. \tag{4.7}$$

As a result, if $N = S^n$ then

$$\pi_1(Diff_\theta(S^*S^n)) \neq 0, \quad \pi_1((FIO)^0(S^n) \neq 0. \tag{4.8}$$

Proof. In fact, the fundamental solution Φ_p of (4.6) gives a closed curve of $(FIO)^0(S^n)$, and $\tilde{\pi}(\Phi_p)$ is the geodesic flow on S^n. By direct computation, we see (4.7). □

5. The relation between Maslov form and $\mathfrak{M}$-form

In order to define Maslov form on the infinite dimensional Lie groups by the same way as usual Maslov form on the Lagrangian submanifolds, we need the following diagram:

$$\begin{array}{ccccccc}
(FIO)^0(N) & \xrightarrow{\tilde{\pi}} & Diff_\Theta^{(1)}(T^*_\star N) & & & & \\
\Cup & & \Cup & & & & \\
F(a,\phi) & \mapsto & WF(F(a,\phi)) = \varphi^{-1} & & & & \\
 & \xrightarrow{\tilde{\tau}} & \Lambda(2n) = U(2n)/O(2n) & \xrightarrow{W} & U(2n) & \xrightarrow{\det} & U(1) \\
 & & \Cup & & \Cup & & \Cup \\
 & \mapsto & \lambda = U_\lambda \lambda_{im} & \mapsto & U_\lambda\ {}^tU_\lambda & \mapsto & (\det U_\lambda)^2
\end{array},$$

where $\tilde{\tau}(\varphi)$ is the tangent space of the graph of φ at the reference point $(p, \varphi(p))$ and $\Lambda(2n)$ is the Lagrangian-Grassmaniann manifold.

Note that, in general, the canonical graph of symplectic diffeomorphism on symplectic manifold (M,ω) is a Lagrangian submanifold in $(M \times M, \omega \ominus \omega)$.

We call $(\det \circ W \circ \tilde{\tau})^*(d\theta)$ (resp. $(\det \circ W \circ \tilde{\tau} \circ \tilde{\pi})^*(d\theta)$) as Maslov form on $Diff_\theta(S^*N)$ (resp. $(FIO)^0(N)$) (cf. (2.8)). Then we have the following:

Proposition 5.1.

(5.1) $\quad m_{p,c} = (\det \circ W \circ \tilde{\tau})^*(d\theta), \quad m_{p,q} = (\det \circ W \circ \tilde{\tau} \circ \tilde{\pi})^*(d\theta).$

Proof. We use the notations prepared in §3.2. Also ∂ denotes the derivative at p.

Let φ be an element of $Diff_\theta(S^*N)$. Set a system $\mathbb{H}$ of functions as follows:

$$(5.2)\quad \begin{aligned} H^i(\varphi, X, \Xi, X', \Xi') &= X'^{,i} - \Xi'_i(X, \Xi) \quad (i = 1, \cdots, n), \\ H_i(\varphi, X, \Xi, X', \Xi') &= \Xi'_i - X'^{,i}(X, \Xi) \quad (i = 1, \cdots, n), \end{aligned}$$

where $\varphi(X,\Xi) = (X'^{,i}(X,\Xi), \Xi'_i(X,\Xi))_{i=1,\cdots,n}$ is the symplectic diffeomorphism on $T^*_\star N$.Then, the level surface of $\mathbb{H} = 0$ coincides with the graph of symplectic diffeomorphism φ. If $T_p\mathrm{Graph}(\varphi)$ is transversal to $\lambda_{Re} = \{(0,\Xi,X',0)|\Xi,X' \in \mathbb{R}^n\}$, then any point of a neighborhood of p of the Graph(φ) is parameterized by the variable (X,Ξ') of $U \subset \lambda_{im}$. In this case $\lambda_{im} = \{(X,0,0,\Xi') \mid X,\ \Xi' \in \mathbb{R}^n\}$, since we now regard $\mathbb{R}^{4n}$ as a symplectic space with canonical structure $\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. Then we have the generating function $S(X,\Xi')$ of φ around p.

Since $\mathbb{H}\big(X,\Xi',\partial_{(X,\Xi')}S(X,\Xi')\big) = 0$, we have

(5.3)
$$\partial^2_{(X,\Xi')}S(X,\Xi') = -\partial_{(X,\Xi')}\mathbb{H}\cdot\partial_{(X',\Xi)}\mathbb{H}^{-1}\big(X,\Xi',\partial_{(X,\Xi')}S(X,\Xi')\big).$$

Substituting this equality into (2.12), then we get the desired conclusion.

If the $T_p\mathrm{Graph}(\varphi)$ is not transversal to λ_{Re}, then we can show the proposition using Legendre transformation (see [Yo], [Fu] for details). □

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