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An Inverse Assignment Problem
ある逆割り当て問題について

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Abstract

In this paper we focus our attention on the famous "Plane Assignment Problem" in Beckmann's *Dynamic Programming of Economic Decisions* and develop a further theory of the assignment problem. Formulating the problem into an optimal (main) stopping problem, we propose a new inversion of the stopping problem. By exchange of objective function and constraint function together with replacement of optimizer min by Max, we introduce an inverse assignment problem, which is also an optimal stopping problem. We establish several inverse theorems between main and inverse stopping problems. We also analyze the finite-stage (nonstopping) problems and specify the enveloping relation to the stopping problems. Detailed numerical solutions for both problems are specified.

1 Introduction

In this paper we devote ourselves exclusively to the study of the so called Plane Assignment Problem which has its origin in Beckmann and Laderman [1]. The Plane Aassignment Problem is one of the most typical resourse allocation problems [4]. For its simple structure and elegant economic interpretation, this problem has several approaches, for instance, linear programming, integer programming, combinatorial programming, and dynamic programming (see [8]). Beckmann [2] illustrates heuristically the principle of optimality [3] through the problem.

In this paper we develop a further inverse theory of the assignment problem. We formulate the problem into an optimal stopping problem, which we call main stopping problem. We propose an inversion of the stopping problem. By exchange of objective function and constraint function together with replacement of optimizer min by Max ([5],[6],[7]), we introduce an inverse assignment problem, which is called an inverse stopping problem. An inverse relation between main and inverse stopping problems is condensed into four inverse theorems; Weak Inverse Theorem, Strong Inverse Theorem, Strict Inverse Theorem and Inverse Stopping Time Theorem. We specify detailed numerical optimal solutions for both problems.

In Section 2 we formulate Beckmann's Plane Assignment Problem into an optimal stopping problem in a deterministic sense. We show the monotonicity of optimal value function and derive the recursive equation.

In Section 3 we introduce its inverse problem, show the monotonicity of its optimum value function, and derive the recursive equation for the inverse problem. The three inverse theorems are established.

In Section 4 we discuss both main and inverse nonstopping (finite-stage) problems. We give both problems their optimal solutions and show inverse relations. An enveloping property between stopping and nonstopping problems is shown for both main and inverse problems, respectively. Further an inverse relation between both optimal stopping times is established.

In Section 5 we illustrate detailed numerical solutions both for Beckmann's Assignment Problem and for its inverse problem.

2 Main Stopping Problem

We begin to consider the Beckmann's Plane Assignment Problem [2] in the following quatation:

> **Example** [BECKMAN/LADERMAN [1](1956)]: Plane Assignment.
>
> As an illustration of the principle of optimality consider the problem of finding the best combination of two indivisible resources to meet a given demand.
>
> Let the demand be a number of passengers and the resources to be two types of planes
>
> | Plane | Capacity | Cost |
> |---|---|---|
> | DC 3 | 38 | 1.0 |
> | DC 6 | 58 | 1.4 |
>
> Let the cost of operating a DC 6 on a given flight be 1.4 times that of running a DC 3. For any number n of passengers up to $n = 200$ it is desired to find the cheapest combination of planes that will carry them.
>
> From 1 to 38 passengerss are carried most cheaply by one DC 3; from 39 to 58 passengers by one DC 6.
>
> To decide which is the cheapest cost of trnasporting 59 passengers we denote the minimum cost of transporting m passengers by $v(m)$ and have the recursive relation
>
> $$v(m) = \min[1.4 + v(m-58),\ 1.0 + v(m-38)] \quad \text{in particular}$$
>
> $$v(59) = \min[1.4 + v(1),\ 1.0 + v(21)]$$

where "min" means the smaller of the two values in the brackets.

Since $v(1) = v(21) = 1.0$ one has $v(59) = 2.0$. And so on.

Now let us formulate Beckmann's Assignment Problem into a stopping problem and analyze it.

Throughout the paper, we use the *cost function* $f : \{1, 2, \ldots, 38, \ldots, 58\} \to \{1.0, 1.4\}$ defined by

$$f(1) = f(2) = \cdots = f(38) := 1.0, \quad f(39) = \cdots = f(58) := 1.4 \tag{1}$$

Thus the assignment problem is formulated into the following minimization problem :

$$\begin{array}{lll} & \text{minimize} & f(x_1) + f(x_2) + \cdots + f(x_t) \\ \text{MSP}(200) & \text{subject to} & \text{(i)} \;\; x_1 + x_2 + \cdots + x_t = 200 \\ & & \text{(ii)} \;\; 1 \le x_n \le 58 \quad 1 \le n \le t \\ & & \text{(iii)} \;\; 1 \le t \le 200 \end{array} \tag{2}$$

The condition (iii) means when to stop assigning. Thus the *deterministic* variable t is considered as a stopping time. This is the main reason why we call MSP(200) a main *stopping problem.* Of course, the problem is a problem of finding not only an optimal stopping time t but also an optimal assignment itself $(x_1, \ldots, x_t)$, which together yields the minimum cost.

Let $v(200)$ be the minimum value. In general, let $v(m)$ be the minimum value of MSP(m) with the right-hand side parameter m in place of 200, where m ranges on the set of natural numbers $N = \{1, 2, \ldots, 200, \ldots\}$. Let $< 1.0\,, \infty >$ be the set of discrete real numbers $1.0, 1.1, \ldots$ with step-size 0.1:

$$< 1.0\,, \infty > = \{1.0, 1.1, \ldots, 5.2, \ldots\}.$$

Note that the set $< 1.0\,, \infty >$ contains all the possible values that the optimal value function v takes.

First we have the monotonicity of optimum value function $v(\cdot)$ as follows:

LEMMA 2. 1 *The minimum value function* $v : N \to < 1.0\,, \infty >$ *is nondecreasing, and it goes to* ∞ *as so does* m.

Second we have the following recursive equation.

THEOREM 2. 1

$$v(m) = \min[1.4 + v(m - 58),\ 1.0 + v(m - 38)] \quad m = 59, 60, \ldots. \tag{3}$$

$$v(1) = v(2) = \cdots = v(38) = 1.0, \quad v(39) = v(40) = \cdots = v(58) = 1.4 \tag{4}$$

Let us define the *optimal pocicy* $\pi^* : N \rightarrow \{38, 58\}$ by

$$\pi^*(m) = \begin{cases} 58 \\ 38 \end{cases} \text{if} \begin{cases} 1.4 + v(m-58) \\ 1.0 + v(m-38) \end{cases} \text{attains the minimum in (3)} \quad (5)$$

where

$$\pi^*(1) = \cdots = \pi^*(38) = 38, \quad \pi^*(39) = \cdots = \pi^*(58) = 58. \quad (6)$$

In the last section, Table 1 shows an *optimal solution* - a pair of optimal value and optimal policy - for MPS $\{v(\cdot), \pi^*(\cdot)\}$. Figure 1 illustrates that an successive application of optimal policy $\pi^*(\cdot)$ from the given initial state $m = 200$ generates an *optimal decision tree* for the given Beckmann's problem. In summary, the optimal decision tree states that the cheapest cost 5.2 of transporting 200 passengers is attained by use of a combination of one DC 3 and three DC 6.

3 Inverse Stopping Problem

In this section, as an inverse problem, we consider the following maximization problem :

$$\text{ISP}(5.2) \qquad \begin{array}{ll} \text{Maximize} & x_1 + x_2 + \cdots + x_t \\ \text{subject to} & \text{(i)}' \;\; f(x_1) + f(x_2) + \cdots + f(x_t) \leq 5.2 \\ & \text{(ii)} \;\; 1 \leq x_n \leq 58 \quad 1 \leq n \leq t \\ & \text{(iii)} \;\; t \geq 1. \end{array} \quad (7)$$

This is also a stopping problem. Thus we call this problem Inverse Stopping Problem.

Let $u(5.2)$ be the maximum value. In general, let $u(c)$ be the minimum value of ISP(c) with the right-hand side parameter c in place of 5.2, where c ranges on the set of discrete real numbers $< 1.0, \infty >= \{1.0, 1.1, 1.2, \ldots\}$. Then we have the monotonicity of optimum value function $u(\cdot)$ as follows:

LEMMA 3. 1 *The maximum value function* $u :< 1.0, \infty >\rightarrow N$ *is nondecreasing, and it goes to* ∞ *as so does* c.

We have also the following recursive equation.

THEOREM 3. 1

$$u(c) = \text{Max}[58 + u(c-1.4),\, 38 + u(c-1.0)] \quad c = 1.5, 1.6, \ldots. \quad (8)$$

$$u(0.1) = u(0.2) = \cdots = u(0.9) = 0,$$
$$u(1.0) = u(1.1) = \cdots = u(1.3) = 38, \quad u(1.4) = 58 \quad (9)$$

We define the *optimal pocicy* $\hat{\sigma} :< 1.0, \infty >\rightarrow \{38, 58\}$ by

$$\hat{\sigma}(c) = \begin{cases} 58 \\ 38 \end{cases} \text{if} \begin{cases} 58 + u(c-1.4) \\ 38 + u(c-1.0) \end{cases} \text{attains the maximum in (8)} \quad (10)$$

wehere

$$\hat{\sigma}(1.0) = \cdots = \hat{\sigma}(1.3) = 38, \quad \hat{\sigma}(1.4) = 58. \tag{11}$$

The optimal solution for IPS $\{u(\cdot), \hat{\sigma}(\cdot)\}$ is shown in Table 2 in Section 5. Figure 2 illustrates that an successive application of optimal policy $\hat{\sigma}(\cdot)$ from the given initial total cost $c = 5.2$ generates an optimal decision tree for the inverse problem. The optimal decision tree states that the maximum total number of passengers 212 for the total cost 5.2 or less is also attained by use of the combination of one DC 3 and three DC 6.

Furthermore, we have the following inverse relationship between Main and Inverse Stopping Problems:

THEOREM 3. 2 *(Weak Inverse Theorem I)*

$$\text{(i)} \quad v(u(c)) \leq c \qquad c \in < 1.0, \infty > \tag{12}$$

$$\text{(ii)} \quad u(v(m)) \geq m \qquad m \in N. \tag{13}$$

It is verified in Tables 2 and 1 that Eqs. (12),(13) hold, respectively.

Let $w : X \to Y$ be a nondecreasing function, where X, Y are nonempty discrete subsets in one-dimensional Euclidean space R^1. Then we define two kinds of its inverse function as follows: One is the *upper-semi inverse function* $w^{-1} : Y \to X$

$$w^{-1}(y) := \min\{ x \in X \mid w(x) \geq y \}. \tag{14}$$

The other is the *lower-semi inverse function* $w_{-1} : Y \to X$

$$w_{-1}(y) := \text{Max}\{ x \in X \mid w(x) \leq y \}. \tag{15}$$

We say that a value $y \in Y$ is *attainable* if there exists some $x \in X$ satisfying $w(x) = y$. Then we have the following properties.

LEMMA 3. 2

$$w_{-1}(y) \geq w^{-1}(y) \quad \textit{for attainable } y \in Y \tag{16}$$

$$w_{-1}(y) < w^{-1}(y) \quad \textit{for nonattainable } y \in Y. \tag{17}$$

Furthermore, for any nonattainable $y \in Y$, both $w_{-1}(y)$ and $w^{-1}(y)$ take two adjacent (neighbouring) values in X.

Moreover, we have a rather strict inverse relations as follows:

THEOREM 3. 3 *(Strong Inverse Theorem I)*

$$\text{(i)}' \quad v_{-1}(c) = u(c) \qquad c \in < 1.0, \infty > \tag{18}$$

$$\text{(ii)}' \quad u^{-1}(m) = v(m) \qquad m \in N. \tag{19}$$

As for Eqs. (18),(19) see Tables 2 and 1, respectively. Further, one pair of optimal value function and optimal policy characterizes the other pair as follows:

THEOREM 3. 4 *(Strict Inverse Theorem I)*

$$\text{(iii)}\quad \hat{\sigma}(c) = \pi^*(v_{-1}(c)) \qquad c \in < 1.0\,, \infty > \tag{20}$$

$$\text{(iv)}\quad \pi^*(m) = \hat{\sigma}(u^{-1}(m)) \qquad m \in N. \tag{21}$$

Table 1. Optimal Solution for MSP and Composite Solution

no. of passengers m	minimum cost $v(m)$	optimal policy $\pi^*(m)$	composite function $u(v(m))$	u.s. inverse function $u^{-1}(m)$	composite policy $\hat{\sigma}(u^{-1}(m))$
1	1.0	38	38	1.0	38
⋮	⋮	⋮	⋮	⋮	⋮
38	1.0	38	38	1.0	38
39	1.4	58	58	1.4	58
⋮	⋮	⋮	⋮	⋮	⋮
58	1.4	58	58	1.4	58
59	2.0	38	76	2.0	38
⋮	⋮	⋮	⋮	⋮	⋮
76	2.0	38	76	2.0	38
77	2.4	38 or 58	96	2.4	38 or 58
⋮	⋮	⋮	⋮	⋮	⋮
96	2.4	38 or 58	96	2.4	38 or 58
97	2.8	58	116	2.8	58
⋮	⋮	⋮	⋮	⋮	⋮
116	2.8	58	116	2.8	58
117	3.4	38 or 58	134	3.4	38 or 58
⋮	⋮	⋮	⋮	⋮	⋮
134	3.4	38 or 58	134	3.4	38 or 58
135	3.8	38 or 58	154	3.8	38 or 58
⋮	⋮	⋮	⋮	⋮	⋮
154	3.8	38 or 58	154	3.8	38 or 58
155	4.2	58	174	4.2	58
⋮	⋮	⋮	⋮	⋮	⋮
174	4.2	58	174	4.2	58
175	4.8	38 or 58	192	4.8	38 or 58
⋮	⋮	⋮	⋮	⋮	⋮
192	4.8	38 or 58	192	4.8	38 or 58
193	5.2	38 or 58	212	5.2	38 or 58
⋮	⋮	⋮	⋮	⋮	⋮
200	5.2	38 or 58	212	5.2	38 or 58

Symbolically we write

$$\hat{\sigma} = \pi^* \circ v_{-1} \text{ on } < 1.0, \infty >, \qquad \pi^* = \hat{\sigma} \circ u^{-1} \text{ on } N \tag{22}$$

, where $\circ$ is the composition operator between functions.
As for Eqs. (20),(21) see Tables 2 and 1, respectively.

Table 2. Optimal Solution for ISP and Composite Solution

given cost c	maximum no. of passengers $u(c)$	optimal policy $\hat{\sigma}(c)$	composite function $v(u(c))$	l.s. inverse function $v_{-1}(c)$	composite policy $\pi^*(v_{-1}(c))$
1.0	38	38	1.0	38	38
⋮	⋮	⋮	⋮	⋮	⋮
1.3	38	38	1.0	38	38
1.4	58	58	1.4	58	58
⋮	⋮	⋮	⋮	⋮	⋮
1.9	58	58	1.4	58	58
2.0	76	38	2.0	76	38
⋮	⋮	⋮	⋮	⋮	⋮
2.3	76	38	2.0	76	38
2.4	96	38 or 58	2.4	96	38 or 58
⋮	⋮	⋮	⋮	⋮	⋮
2.7	96	38 or 58	2.4	96	38 or 58
2.8	116	58	2.8	116	58
⋮	⋮	⋮	⋮	⋮	⋮
3.3	116	58	2.8	116	58
3.4	134	38 or 58	3.4	134	38 or 58
⋮	⋮	⋮	⋮	⋮	⋮
3.7	134	38 or 58	3.4	134	38 or 58
3.8	154	38 or 58	3.8	154	38 or 58
⋮	⋮	⋮	⋮	⋮	⋮
4.1	154	38 or 58	3.8	154	38 or 58
4.2	174	58	4.2	174	58
⋮	⋮	⋮	⋮	⋮	⋮
4.7	174	58	4.2	174	58
4.8	192	38 or 58	4.8	192	38 or 58
⋮	⋮	⋮	⋮	⋮	⋮
5.1	192	38 or 58	4.8	192	38 or 58
5.2	212	38 or 58	5.2	212	38 or 58

4 Nonstopping Problems

In this section we consider the nonstopping problems. Let n be any given total number of planes. Then two problems arise. One is a main problem. For any given total number of passengers m, we consider the problem of finding the minimum total cost of carrying m passengers by n planes. The other is its inverse problem. For any given total cost of operating c, we consider the problem of finding the maximum total number of passengeres that n planes carry for not more than the total cost c. Since all the results in the following are proved in a similar way as in Sections 2 and 3, the proof is omitted in this section.

4.1 Main Problems

For any $n \in N$, let us define the following two discrete intervals;

$$N_n := \{n, n+1, \ldots, 58n\} \tag{23}$$
$$C_n := \{1.0n, 1.0n+0.1, \ldots, 1.4n+0.5\}. \tag{24}$$

Then the interval N_n contains all the possible total numbers of passengers n planes can carry. The interval C_n does all the possible total costs for which or less n planes can carry.

Given two positive integers n, m satisfing $m \in N_n$, we consider the problem of dividing m into n possible natural numbers between 1 and 58 and minimizing the summed value measured through the cost function f:

$$\text{NMP}(m;n) \qquad \begin{array}{ll} \text{minimize} & f(x_1)+f(x_2)+\cdots+f(x_n) \\ \text{subject to} & \text{(i)}\ \ x_1+x_2+\cdots+x_n = m \\ & \text{(ii)}\ \ 1 \le x_i \le 58 \quad 1 \le i \le n. \end{array} \tag{25}$$

Let $v_n(m)$ be the minimum value. Then we have the following double-monotone property and recursive equation:

LEMMA 4. 1 *(i) The minimum value function $v_n : N_n \to C_n$ is nondecreasing :*

$$v_n(m) \le v_n(m+1) \qquad m,\ m+1 \in N_n. \tag{26}$$

(ii) The sequence of minimum functions $\{v_n\}_{n\ge1}$ is nondecreasing:

$$v_n(m) \le v_{n+1}(m) \qquad m \in N_n \cap N_{n+1}. \tag{27}$$

THEOREM 4. 1

$$v_1(m) = f(m) \quad m \in N_1 \tag{28}$$
$$v_{n+1}(m) = \min_{x:*}[f(x)+v_n(m-x)] \qquad m \in N_{n+1},\ \ n \ge 1 \tag{29}$$

where $x : *$ means that the minimization is taken for all x satisfying

$$1 \le x \le 58, \quad m-x \in N_n. \tag{30}$$

In particular, when $\{m-38,\ m-58\} \subset N_n$, Eq.(29) reduces

$$v_{n+1}(m) = \min[1.0+v_n(m-38),\ 1.4+v_n(m-58)]. \tag{31}$$

Let $\pi_{n+1}^*(m) \subset \{1, 2, \ldots, 58\}$ be the set of all minimizers for (29), where

$$\pi_1^*(1) = \cdots = \pi_1^*(38) = 38, \quad \pi_1^*(39) = \cdots = \pi_1^*(58) = 58. \tag{32}$$

Then the point-to-set valued function $\pi_n^* : N_n \to \{1, 2, \ldots, 58\}$ is called *n-th optimal decision function.* We call the sequence of optimal decision functions $\pi^* = \{\pi_1^*, \pi_2^*, \ldots, \pi_n^*, \ldots\}$ an *optimal policy* for Nonstopping Main Problem NMP$(m; n)$.

Further we have the following relation between Stopping Problem and Nonstopping Problem:

THEOREM 4. 2 *(Main Envelope Theorem)*

$$v(m) = \min_{n|m \in N_n} v_n(m) \qquad m \in N. \tag{33}$$

Let $t^*(m)$ be the first positive integer n such that $v(m) = v_n(m)$. Then t^* is the optimal stopping time for MSP:

$$v(m) = v_{t^*}(m) \quad m \in N. \tag{34}$$

As for Eqs. (33),(34), see Table 3.

4.2 Inverse Problems

We consider the inverse problem of NMP$(m; n)$ as follows:

$$\text{NIP}(c; n) \qquad \begin{array}{ll} \text{Maximize} & x_1 + x_2 + \cdots + x_n \\ \text{subject to} & \text{(i)}' \;\; f(x_1) + f(x_2) + \cdots + f(x_n) \le c \\ & \text{(ii)} \;\; 1 \le x_i \le 58 \quad 1 \le i \le n \end{array} \tag{35}$$

where $c \in C_n, \;\; n \ge 1$. Let $u_n(c)$ be the maximum value. Then the maximum value functions enjoy the following double-monotone property and recursive equation:

LEMMA 4. 2 *(i) The maximum value function $u_n : C_n \to N_n$ is nondecreasing :*

$$u_n(c) \le u_n(c + 0.1) \qquad c, \; c + 0.1 \in C_n. \tag{36}$$

(ii) The sequence of maximum functions $\{u_n\}_{n \ge 1}$ is nonincreasing:

$$u_n(c) \le u_{n+1}(c) \qquad c \in C_n \cap C_{n+1}. \tag{37}$$

THEOREM 4. 3

$$u_1(1.0) = u_1(1.1) = \cdots = u_1(1.3) = 38, \;\; u_1(1.4) = \cdots = u_1(1.9) = 58 \tag{38}$$

$$u_{n+1}(c) = \operatorname*{Max}_{x:**}[x + u_n(c - f(x))] \qquad c \in C_{n+1} \tag{39}$$

where $x : **$ denotes that the maximization is taken for al1 x satisfying

$$1 \le x \le 58, \;\; c - f(x) \in C_n. \tag{40}$$

In particular, when $\{c - 1.0, \; c - 1.4\} \subset C_n$, Eq.(39) reduces

$$u_{n+1}(c) = \text{Max}[38 + u_n(c - 1.0), \; 58 + u_n(c - 1.4)]. \tag{41}$$

Let $\hat{\sigma}_{n+1}(c) \subset \{1, 2, \ldots, 58\}$ be the set of all maximizers for (39), where

$$\hat{\sigma}_1(1.0) = \cdots = \hat{\sigma}_1(1.3) = 38, \quad \hat{\sigma}_1(1.4) = \cdots = \hat{\sigma}_1(1.9) = 58. \tag{42}$$

Then the point-to-set valued function $\hat{\sigma}_n : C_n \to \{1, 2, \ldots, 58\}$ is an *n-th optimal decision function.* Thus the sequence of optimal decision functions $\hat{\sigma} = \{\hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_n, \ldots\}$ is an *optimal policy* for Nonstopping Inverse Problem NIP$(c; n)$.

We have the following enveloping relation.

THEOREM 4. 4 *(Inverse Envelope Theorem)*

$$u(c) = \operatorname*{Max}_{n|c \in C_n} u_n(c) \qquad c \in < 1.0\,, \infty > . \tag{43}$$

Let $\hat{t}(c)$ be the first positive integer n such that $u(c) = u_n(c)$. Then $\hat{t}$ is the optimal stopping time for ISP:

$$u(c) = u_{\hat{t}}(c) \qquad c \in < 1.0\,, \infty > . \tag{44}$$

As for Eqs. (43),(44), see Table 4. Furthermore, we have three corresponding inverse theorems for Nonstopping Problems.

THEOREM 4. 5 *(Weak Inverse Theorem II) For* $n \geq 1$

$$\text{(i)} \quad v_n(u_n(c)) \leq c \qquad c \in C_n \tag{45}$$

$$\text{(ii)} \quad u_n(v_n(m)) \geq m \qquad m \in N_n. \tag{46}$$

THEOREM 4. 6 *(Strong Inverse Theorem II) For* $n \geq 1$

$$\text{(i)}' \quad (v_n)_{-1}(c) = u_n(c) \qquad c \in C_n \tag{47}$$

$$\text{(ii)}' \quad (u_n)^{-1}(m) = v_n(m) \qquad m \in N_n. \tag{48}$$

THEOREM 4. 7 *(Strict Inverse Theorem II) For* $n \geq 1$

$$\text{(iii)}' \quad \hat{\sigma}_n(c) = \pi_n^*((v_n)_{-1}(c)) \qquad c \in C_n \tag{49}$$

$$\text{(iv)}' \quad \pi_n^*(m) = \hat{\sigma}_n((u_n)^{-1}(m)) \qquad m \in N_n. \tag{50}$$

Symbolically we have

$$\hat{\sigma}_n = \pi_n^* \circ (v_n)_{-1} \text{ on } C_n, \qquad \pi_n^* = \hat{\sigma}_n \circ (u_n)^{-1} \text{ on } N_n. \tag{51}$$

Further both optimal stopping times are characterized in the following inverse sense:

THEOREM 4. 8 *(Inverse Stopping Time Theorem)*

$$\hat{t} = t^* \circ v_{-1} \quad on \quad < 1.0\,, \infty >, \qquad t^* = \hat{t} \circ u^{-1} \quad on \ N. \tag{52}$$

Finally we we specify Tables 4 and 5, which illustrate optimal value functions, optimal policies and optimal stopping times for the main and inverse nonstopping problems, respectively. Further the forementioned relations are also shown in tables. The specification verifies that all the results in both Inverse Theorems and Envelope Theorems are valid.

Table 3. Envelope Property and Optimal Stopping Time for MPS

m	$v(m)$	$v_1(m)$	$v_2(m)$	$v_3(m)$	$v_4(m)$	$\cdots$	$t^*(m)$
1	1.0	1.0					1
2	1.0	1.0	2.0				1
3	1.0	1.0	2.0	3.0			1
4	1.0	1.0	2.0	3.0	4.0		1
⋮	⋮	⋮	⋮	⋮	⋮	$\cdots$	⋮
38	1.0	1.0	2.0	3.0	4.0	$\cdots$	1
39	1.4	1.4	2.0	3.0	4.0	$\cdots$	1
⋮	⋮	⋮	⋮	⋮	⋮	$\cdots$	⋮
58	1.4	1.4	2.0	3.0	4.0	$\cdots$	1
59	2.0		2.0	3.0	4.0	$\cdots$	2
⋮	⋮		⋮	⋮	⋮	$\cdots$	⋮
76	2.0		2.0	3.0	4.0	$\cdots$	2
77	2.4		2.4	3.0	4.0	$\cdots$	2
⋮	⋮		⋮	⋮	⋮	$\cdots$	⋮
96	2.4		2.4	3.0	4.0	$\cdots$	2
97	2.8		2.8	3.0	4.0	$\cdots$	2
⋮	⋮		⋮	⋮	⋮	$\cdots$	⋮
114	2.8		2.8	3.0	4.0	$\cdots$	2
115	2.8		2.8	3.4	4.0	$\cdots$	2
116	2.8		2.8	3.4	4.0	$\cdots$	2
117	3.4			3.4	4.0	$\cdots$	3
⋮	⋮			⋮	⋮	$\cdots$	⋮
134	3.4			3.4	4.0	$\cdots$	3
135	3.8			3.8	4.0	$\cdots$	3
⋮	⋮			⋮	⋮	$\cdots$	⋮
152	3.8			3.8	4.0	$\cdots$	3
153	3.8			3.8	4.4	$\cdots$	3
154	3.8			3.8	4.4	$\cdots$	3
155	4.2			4.2	4.4	$\cdots$	3
⋮	⋮			⋮	⋮	$\cdots$	⋮
172	4.2			4.2	4.4	$\cdots$	3
173	4.2			4.2	4.8	$\cdots$	3
174	4.2			4.2	4.8	$\cdots$	3
175	4.8				4.8	$\cdots$	4
⋮	⋮				⋮	$\cdots$	⋮
192	4.8				4.8	$\cdots$	4
193	5.2				5.2	$\cdots$	4
⋮	⋮				⋮	$\cdots$	⋮
200	5.2				5.2	$\cdots$	4

Table 4. Envelope Property and Optimal Stopping Time for IPS

c	$u(c)$	$u_1(c)$	$u_2(c)$	$u_3(c)$	$u_4(c)$	$\cdots$	$\hat{t}(c)$
1.0	38	38					1
⋮	⋮	⋮					⋮
1.3	38	38					1
1.4	58	58					1
⋮	⋮	⋮					⋮
1.9	58	58					1
2.0	76		76				2
⋮	⋮		⋮				⋮
2.3	76		76				2
2.4	96		96				2
⋮	⋮		⋮				⋮
2.7	96		96				2
2.8	116		116				2
2.9	116		116				2
3.0	116		116	114			2
⋮	⋮		⋮	⋮			⋮
3.3	116		116	114			2
3.4	134			134			3
⋮	⋮			⋮			⋮
3.7	134			134			3
3.8	154			154			3
3.9	154			154			3
4.0	154			154	152		3
4.1	154			154	152		3
4.2	174			174	152		3
4.3	174			174	152		3
4.4	174			174	172		3
⋮	⋮			⋮	⋮		⋮
4.7	174			174	172		3
4.8	192				192		4
⋮	⋮				⋮		⋮
5.0	192				192	$\cdots$	4
5.1	192				192	$\cdots$	4
5.2	212				212	$\cdots$	4

References

[1] M.J. Beckmann and J. Laderman, A bound on the use of inefficient indivisible units, Naval Research Logistics Quartely **3**(1956), 245-252

[2] M.J. Beckmann, *Dynamic Programming of Economic Decisions*, Springer, NY, 1968.

[3] R.E. Bellman, *Dynamic Programming*, Princeton Univ. Press, NJ, 1957.

[4] T. Ibaraki and N. Katoh, *Resource Allocation Problems: Algorithmic Approaches*, MIT Press, MA, 1988.

[5] S. Iwamoto, Inverse theorem in dynamic programming I, *J. Math. Anal. Appl.* **58**(1977), 113-134.

[6] S. Iwamoto, Inverse theorem in dynamic programming II, *J. Math. Anal. Appl.* **58**(1977), 249-279.

[7] S. Iwamoto, Inverse theorem in dynamic programming III, *J. Math. Anal. Appl.* **58**(1977), 439-448.

[8] S. Iwamoto, *Theory of Dynamic Programs* (in Japanese), Kyushu Univ. Press, Fukuoka, 1987.