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Continuity properties and exponential integrability for Riesz potentials of functions in Orlicz classes

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1 Introduction

In this paper we study continuity properties and exponential integrability for Riesz potentials of order α, $0<\alpha<n$, of a nonnegative measurable function f on R^n, which is defined by

$$U_\alpha f(x)=\int_{R^n}|x-y|^{\alpha-n}f(y)dy.$$

Here it is natural to assume that $U_\alpha f\not\equiv\infty$, which is equivalent to

$$\int_{R^n}(1+|y|)^{\alpha-n}f(y)dy<\infty. \tag{1.1}$$

To obtain general results, we treat functions f satisfying an Orlicz condition with weight ω of the form

$$\int_{R^n}\Phi_p(f(y))\omega(|y|)dy<\infty. \tag{1.2}$$

Here $\Phi_p(r)$ is a positive monotone function on the interval $(0,\infty)$ with the following properties:

(φ1) $\Phi_p(r)$ is of the form $r^p\varphi(r)$, where $1\le p<\infty$ and φ is a positive monotone function on the interval $(0,\infty)$; set $\varphi(0)=\lim_{r\to 0}\varphi(r)$.

(φ2) φ is of logarithmic type, that is, there exists $A_1>0$ such that

$$A_1^{-1}\varphi(r)\le\varphi(r^2)\le A_1\varphi(r)\qquad\text{whenever } r>0.$$

(ω1) ω satisfies the doubling condition; that is, there exists $A_2>0$ such that

$$A_2^{-1}\omega(r)\le\omega(2r)\le A_2\omega(r)\qquad\text{whenever } r>0.$$

Riesz potentials may not, in general, be continuous at any point of R^n. But, it is known (see [11]) that if $p>1$ and

$$\int_0^1\left[r^{n-\alpha p}\varphi(r^{-1})\right]^{-1/(p-1)}r^{-1}dr<\infty, \tag{1.3}$$

then $U_\alpha f$ is continuous everywhere on R^n; in case $\alpha p > n$, (1.3) holds by condition (φ2) and the continuity also follows from well-known Sobolev's theorem. In case $\alpha p = n$, the functions

$$[\log(e+r)]^\delta, \quad [\log(e+r)]^{p-1}[\log(e+\log(e+r))]^\delta, \cdots$$

satisfy (1.3) if and only if $\delta > p-1$.

For simplicity, let $\omega(r) = r^\beta$, where $-n < \beta \le \alpha p - n$, and ℓ be the nonnegative integer such that $\ell \le \alpha - (n+\beta)/p < \ell + 1$. In this case, we treat functions f satisfying

$$\int_{R^n} \Phi_p(f(y))|y|^\beta dy < \infty. \tag{1.4}$$

In Section 3, we shall show that if (1.3) holds, then there exists a polynomial P_ℓ such that

$$\lim_{x\to 0} [K(|x|)]^{-1}[U_\alpha f(x) - P_\ell(x)] = 0 \tag{1.5}$$

for any function f satisfying (1.1) and (1.4), where

$$K(r) = \begin{cases} [r^{n-\alpha p+\beta}\varphi(r^{-1})]^{-1/p} & \text{in case } \ell < \alpha - (n+\beta)/p < \ell+1 \\ & \quad \text{and } n - \alpha p < 0, \\ r^{-\beta/p}\left(\int_0^r [\varphi(t^{-1})]^{-p'/p} t^{-1} dt\right)^{1/p'} & \text{in case } \ell < \alpha - (n+\beta)/p < \ell+1 \\ & \quad \text{and } n - \alpha p = 0, \\ r^\ell \left(\int_0^r [\varphi(t^{-1})]^{-p'/p} t^{-1} dt\right)^{1/p'} & \text{in case } \ell = \alpha - (n+\beta)/p. \end{cases}$$

Since $\lim_{r\to 0} r^{-\ell}K(r) = 0$, (1.5) implies that $U_\alpha f$ is ℓ times differentiable at the origin.

Let $R_\alpha(x) = |x|^{\alpha-n}$ and consider the remainder term of Taylor's expansion:

$$R_{\alpha,\ell}(x,y) = R_\alpha(x-y) - \sum_{|\mu|\le\ell} \frac{x^\mu}{\mu!}[(D^\mu R_\alpha)(-y)].$$

Then $U_\alpha f(x) - P_\ell(x)$ will be written as

$$U_{\alpha,\ell} f(x) = \int_{R^n} R_{\alpha,\ell}(x,y) f(y) dy,$$

provided

$$\int_{B(0,1)} |y|^{\alpha-n-\ell} f(y) dy < \infty; \tag{1.6}$$

here, we may assume a condition weaker than (1.1):

$$\int_{R^n - B(0,1)} |y|^{\alpha-n-\ell-1} f(y) dy < \infty, \tag{1.7}$$

where $B(x,r)$ denotes the open ball centered at x with radius $r > 0$.

Recently Edmunds and Krbec studied almost Lipschitz continuity for Bessel potentials of order $n/p+1$ of functions f satisfying

$$\int_{R^n} f(y)^p[\log(e+f(y))]^{-\sigma}dy < \infty$$

for some $\sigma > 0$. Letting $J_\alpha f$ denote the Bessel potential of f of order α (see Meyers [8] and Stein [19]), they showed in [6, Theorem 3.1] that

$$J_{n/p+1}f(x) - J_{n/p+1}f(0) = O(|x||\log|x||^{(p-1+\sigma)/p}) \qquad \text{as } x \to 0, \tag{1.8}$$

which gives an extension of the result by Brézis-Wainger [3] in case $\sigma = 0$. These results are based on general theorems for Orlicz-Sobolev spaces (see Adams [1] and Rao-Ren [16]). In Section 4, we study differentiability properties for Riesz potentials of order α. In fact, if $\ell < \alpha - (n/p)$ or if $\ell = \alpha - (n/p)$ and $\sigma < 1-p$, then we show that $U_\alpha f$ has differentials of order ℓ which satisfy Hölder type condition

$$D^\mu U_\alpha f(x+h) - D^\mu U_\alpha f(x) = O(\kappa(|h|)) \qquad \text{as } h \to 0$$

with a suitable function κ, where $D^\mu = (\partial/\partial x)^\mu$ is a partial differential operator of order $\ell = |\mu|$. For example, if $\alpha = (n/p) + \ell + 1$ and $\varphi(r) = [\log(e+r)]^{-\sigma}$ for $\sigma > 1-p$, then we can take

$$\kappa(r) = r[\log(1/r)]^{(p-1+\sigma)/p},$$

and our result gives the above mentioned result by Edmunds and Krbec.

If (1.3) does not hold, then the potential may not be continuous anywhere, and Mizuta ([12]) studied the fine limits of $U_\alpha f$, that is,

$$\lim_{x\to 0, x\in R^n - E} U_\alpha f(x) = U_\alpha f(0)$$

with an exceptional set E which is thin at 0 in a certain sense (see also Adams-Meyers [2] and Meyers [10]). To evaluate the size of exceptional sets, for a set $E \subset R^n$ and an open set $G \subset R^n$, we consider the relative Orlicz capacity

$$C_{\alpha,\Phi_p}(E;G) = \inf_g \int_G \Phi_p(g(y))dy, \qquad E \subset G,$$

where the infimum is taken over all nonnegative measurable functions g on G such that $U_\alpha g(x) \geq 1$ for every $x \in E$ (cf. Meyers [8] and Mizuta [12]). For simplicity, we write $C_{\alpha,\Phi_p}(E) = 0$ if $C_{\alpha,\Phi_p}(E\cap G;G) = 0$ for every bounded open set G. If a property holds except for a set E with $C_{\alpha,\Phi_p}(E) = 0$, then we say that the property holds C_{α,Φ_p}-quasi everywhere. In Section 5, we extend the result by Mizuta [12] and in fact show that if f satisfies (1.1) and (1.4), then there exist a set $E \subset R^n$ and a polynomial P_ℓ such that

$$\lim_{x\to 0, x\in R^n - E} [\kappa(|x|)]^{-1}[U_\alpha f(x) - P_\ell(x)] = 0$$

and

$$\sum_{j=1}^{\infty} 2^{j(n-\alpha p)}[\varphi(2^j)]^{-1} C_{\alpha,\Phi_p}(E_j; B_j) < \infty,$$

where $E_j = \{x \in E : 2^{-j} \le |x| < 2^{-j+1}\}$, $B_j = \{x : 2^{-j-1} < |x| < 2^{-j+2}\}$ and

$$\kappa(r) = r^{\ell} \left(\int_0^r [t^{n-\alpha p+\beta+\ell p}\varphi(t^{-1})]^{-1/(p-1)} t^{-1} dt \right)^{1-1/p}.$$

Note here that

$$C_{\alpha,\Phi_p}(A_j; B_j) \sim 2^{-j(n-\alpha p)}\varphi(2^j), \qquad A_j = B(0, 2^{-j+1}) - B(0, 2^{-j})$$

(cf. [12, Lemma 7.3]), and our definition of thinness differs from that of Adams-Meyers [2]. If in addition (1.3) holds, then the exceptional set E is empty and the above fine limit is seen to be replaced by the usual limit similar to (1.5).

In Section 6, we are concerned with the existence of radial limits. We shall show that if f satisfies (1.1) and (1.4), then there exist a set $E^* \subset \partial B(0,1)$ and a polynomial P_ℓ such that $C_{\alpha,\Phi_p}(E^*) = 0$ and

$$\lim_{r\to 0} r^{(n-\alpha p+\beta)/p}[U_\alpha f(r\xi) - P_\ell(r\xi)] = 0 \qquad \text{for any } \xi \in \partial B(0,1) - E^*.$$

In Section 7, we deal with L^q-mean limits for Taylor's expansion of Riesz potentials $U_\alpha f$

$$\lim_{r\to 0} r^{-\ell} \left(r^{-n} \int_{B(x_0,r)} |U_\alpha f(x) - P_{x_0}(x)|^q \, dx \right)^{1/q} = 0 \tag{1.9}$$

for functions f satisfying

$$\int_{R^n} \Phi_p(f(y))dy < \infty$$

and for $0 < q < \infty$ satisfying $1/q \ge 1/p - \alpha/n$; if $1/q = 1/p - \alpha/n$, then q is called the Sobolev exponent.

If (1.9) holds, then $U_\alpha f$ is said to be L^q-differentiable of order ℓ at x_0 (cf. Meyers [9], Stein [19] and Ziemer [21]), where ℓ is a positive integer such that $\ell \le \alpha$. We discuss quasi every L^q-differentiability in case $\ell < \alpha$ and in fact show that $U_\alpha f$ is L^q-differentiable of order ℓ $C_{\alpha-\ell,\Phi_p}$-quasi everywhere. In view of the behavior at the origin of Bessel kernels, our results can be considered as generalizations of the results by Meyers [9], [10] concerning Bessel potentials of functions in $L^p(R^n)$. In case $\alpha = \ell$, $U_\ell f$ is shown to be L^q-differentiable of order ℓ almost everywhere. If (1.3) holds, then $U_\ell f$ is known to be ℓ times differentiable almost everywhere (see [11, Theorem 2]).

In the final section, we consider the Riesz potential of order α for a nonnegative measurable function f on a bounded open set $G \subset \mathbf{R}^n$ satisfying the Orlicz condition

$$\int_G f(y)^p[\log(e + f(y))]^a[\log(e + \log(e + f(y)))]^b dy < \infty, \qquad p = n/\alpha,$$

for some numbers p, a and b. We aim to show the exponential integrability such as

$$\int_G \exp[A\,(U_\alpha f(x))^\beta\,(\log(e+U_\alpha f(x)))^\gamma]dx < \infty \qquad \text{for any } A>0.$$

See [1], [3], [4], [5], [6], [15], [20], [21]. Moreover, we show double exponential integrability such as

$$\int_G \exp[A\exp(B(U_\alpha f(x))^\beta)]dx < \infty.$$

See [4], [5].

2 Fundamental facts

Throughout this paper, let M denote various constants independent of the variables in question.

First we collect properties which follow from conditions ($\varphi 1$) and ($\varphi 2$) ([7] and [18, Section 2]).

($\varphi 3$) φ satisfies the doubling condition, that is, there exists $A>1$ such that

$$A^{-1}\varphi(r) \leqq \varphi(2r) \leqq A\varphi(r) \qquad \text{whenever } r>0.$$

($\varphi 4$) For any $\gamma>0$, there exists $A(\gamma)>1$ such that

$$A(\gamma)^{-1}\varphi(r) \leqq \varphi(r^\gamma) \leqq A(\gamma)\varphi(r) \qquad \text{whenever } r>0.$$

($\varphi 5$) If $\gamma>0$, then

$$s^\gamma\varphi(s^{-1}) \leqq At^\gamma\varphi(t^{-1}) \qquad \text{whenever } 0<s<t.$$

Let $R_\alpha(x)=|x|^{\alpha-n}$ and consider the remainder term of Taylor's expansion:

$$R_{\alpha,\ell}(x,y) = R_\alpha(x-y) - \sum_{|\mu|\le\ell}\frac{x^\mu}{\mu!}[(D^\mu R_\alpha)(-y)].$$

In our discussions, the following estimates are fundamental (see [7] and [18, Section 3]).

Lemma 2.1. *If $y\in B(0,|x|/2)$, then*

$$|R_{\alpha,\ell}(x,y)| \le M|x|^\ell|y|^{\alpha-n-\ell}.$$

Lemma 2.2. *If $y\in B(0,2|x|)-B(0,|x|/2)$, then*

$$|R_{\alpha,\ell}(x,y)| \le M|x-y|^{\alpha-n}.$$

Lemma 2.3. *If $|y|\ge 2|x|$, then*

$$|R_{\alpha,\ell}(x,y)| \le M|x|^{\ell+1}|y|^{\alpha-n-\ell-1}.$$

3 Continuity

Throughout this section, let φ be a positive nondecreasing function on $(0,\infty)$ satisfying ($\varphi1$) and ($\varphi2$).

We have the following result by Hölder's inequality.

LEMMA 3.1 (cf. [12, Lemma 2.1]). *Let $p > 1$ and f be a nonnegative measurable function on R^n. If $0 \le 2r < a < 1$ and $0 < \delta < \beta$, then*

$$\int_{R^n - B(0,r)} |y|^{\beta-n} f(y)dy \le \int_{R^n - B(0,a)} |y|^{\beta-n} f(y)dy + Ma^{\beta-\delta}$$
$$+M\left(\int_r^a [t^{n-\beta p}\eta(t)]^{-p'/p} t^{-1} dt\right)^{1/p'} \left(\int_{B(0,a)} \Phi_p(f(y))\omega(|y|)dy\right)^{1/p},$$

and if $0 \le 2r < a < 1$ and $\delta > 0 \ge \beta$, then

$$\int_{R^n - B(0,r)} |y|^{\beta-n} f(y)dy \le \int_{R^n - B(0,a)} |y|^{\beta-n} f(y)dy + Mr^{\beta-\delta}$$
$$+M\left(\int_r^a [t^{n-\beta p}\eta(t)]^{-p'/p} t^{-1} dt\right)^{1/p'} \left(\int_{B(0,a)} \Phi_p(f(y))\omega(|y|)dy\right)^{1/p},$$

where $\eta(r) = \varphi(r^{-1})\omega(r)$ and $1/p + 1/p' = 1$.

For an integer ℓ, we consider the potential

$$U_{\alpha,\ell} f(x) = \int_{R^n} R_{\alpha,\ell}(x,y) f(y)dy;$$

in case $\ell \le -1$, $U_{\alpha,\ell}f(x)$ is nothing but $U_\alpha f(x)$, so that, in this paper, we assume that $\ell \ge 0$.

Write $U_{\alpha,\ell}f(x) = U_1(x) + U_2(x) + U_3(x)$ for $x \in R^n - \{0\}$, where

$$\begin{aligned} U_1(x) &= \int_{R^n - B(0,2|x|)} R_{\alpha,\ell}(x,y) f(y)dy, \\ U_2(x) &= \int_{B(0,|x|/2)} R_{\alpha,\ell}(x,y) f(y)dy, \\ U_3(x) &= \int_{B(0,2|x|) - B(0,|x|/2)} R_{\alpha,\ell}(x,y) f(y)dy. \end{aligned}$$

Setting $\eta(r) = \varphi(r^{-1})\omega(r)$ as above, we define

$$\kappa_1(r) = \begin{cases} \left(\int_r^1 [t^{n-\alpha p+(\ell+1)p}\eta(t)]^{-p'/p} t^{-1} dt\right)^{1/p'}, & \text{in case } p > 1, \\ \sup_{r \le t < 1} t^{\alpha-\ell-1-n}[\eta(t)]^{-1}, & \text{in case } p = 1, \end{cases}$$

for $0 < r \le 1/2$; further, set $\kappa_1(r) = \kappa_1(1/2)$ when $r > 1/2$.

REMARK 3.1. In view of the doubling conditions on φ and ω, we see that

$$\kappa_1(r) \ge M\left[r^{n-\alpha p+(\ell+1)p}\eta(r)\right]^{-1/p} \quad \text{whenever } 0 < r \le 1/2.$$

LEMMA 3.2. *Let f be a nonnegative measurable function on R^n. If $0 < 2|x| < a < 1$ and $0 < \delta < \alpha - \ell - 1$, then*

$$\begin{aligned} |U_1(x)| &\le M|x|^{\ell+1}\left\{\int_{R^n - B(0,a)} |y|^{\alpha-\ell-1-n} f(y)dy + Ma^{\alpha-\ell-1-\delta}\right\} \\ &\quad + M|x|^{\ell+1}\kappa_1(|x|)\left(\int_{B(0,a)} \Phi_p(f(y))\omega(|y|)dy\right)^{1/p}, \end{aligned}$$

and if $0 < 2|x| < a < 1$ and $\delta > 0 \ge \alpha - \ell - 1$, then

$$\begin{aligned} |U_1(x)| &\le M|x|^{\ell+1}\int_{R^n - B(0,a)} |y|^{\alpha-\ell-1-n} f(y)dy + M|x|^{\alpha-\delta} \\ &\quad + M|x|^{\ell+1}\kappa_1(|x|)\left(\int_{B(0,a)} \Phi_p(f(y))\omega(|y|)dy\right)^{1/p}, \end{aligned}$$

where M is a positive constant independent of x and a.

The case $p > 1$ follows readily from Lemma 2.3 and Lemma 3.1 with $r = |x|$, and the case $p = 1$ is trivial.

In view of Lemma 3.2, we have the following results.

COROLLARY 3.1. *Let f be a nonnegative measurable function on R^n satisfying (1.2) and (1.7). If $\alpha - \ell - 1 > 0$ and $\kappa_1(0) = \infty$, then*

$$\lim_{x \to 0}\left[|x|^{\ell+1}\kappa_1(|x|)\right]^{-1} U_1(x) = 0.$$

COROLLARY 3.2. *Let f be a nonnegative measurable function on R^n satisfying conditions (1.2) and (1.7). If $\alpha - \ell - 1 \le 0$ and*

$$\lim_{r \to 0} r^{\alpha-\delta}\left[r^{\ell+1}\kappa_1(r)\right]^{-1} = 0 \qquad \textit{for some } \delta > 0,$$

then

$$\lim_{x \to 0}\left[|x|^{\ell+1}\kappa_1(|x|)\right]^{-1} U_1(x) = 0.$$

In view of Lemmas 2.1 and 3.1, we can establish the following result.

LEMMA 3.3. *If $0 < \delta < \alpha - \ell$, then there exists a positive constant M such that*

$$|U_2(x)| \le M|x|^{\ell}\kappa_2(|x|)\left(\int_{B(0,|x|/2)} \Phi_p(f(y))\omega(|y|)dy\right)^{1/p} + M|x|^{\alpha-\delta}$$

for any $x \in B(0,1/2) - \{0\}$, where

$$\kappa_2(r) = \begin{cases} \left(\int_0^r [t^{n-\alpha p+\ell p}\eta(t)]^{-p'/p}t^{-1}dt\right)^{1/p'}, & \textit{in case } p > 1, \\ \sup_{0<t\le r} t^{\alpha-\ell-n}[\eta(t)]^{-1}, & \textit{in case } p = 1. \end{cases}$$

REMARK 3.2. As in Remark 3.1, we see that

$$\kappa_2(r) \ge M\left[r^{n-\alpha p+\ell p}\eta(r)\right]^{-1/p}.$$

With the aid of Lemma 3.3, we have the following result.

COROLLARY 3.3. *Let f be a nonnegative measurable function on R^n satisfying* (1.2). *If $0 < \delta < \alpha - \ell$, $\kappa_2(1) < \infty$ and*

$$\lim_{r\to 0} r^{\alpha-\delta}\left[r^{\ell}\kappa_2(r)\right]^{-1} = 0,$$

then

$$\lim_{x\to 0}\left[|x|^{\ell}\kappa_2(|x|)\right]^{-1}U_2(x) = 0.$$

REMARK 3.3. Let $\omega(r) = r^{\beta}$. If $\alpha - (n+\beta)/p < \ell + 1$, then

$$\kappa_1(r) \sim \left[r^{n-\alpha p+(\ell+1)p+\beta}\varphi(r^{-1})\right]^{-1/p} \qquad \text{as } r \to 0$$

and thus

$$\kappa_1(0) = \infty.$$

If in addition $n + \beta > 0$, then we see by ($\varphi5$) that

$$\limsup_{r\to 0} r^{\alpha-\delta}\left[r^{\ell+1}\kappa_1(r)\right]^{-1} \le M\limsup_{r\to 0} r^{(n+\beta)/p-\delta}\left[\varphi(r^{-1})\right]^{1/p} = 0$$

for $0 < \delta < (n+\beta)/p$.

REMARK 3.4. Let $\omega(r) = r^{\beta}$. If $\ell < \alpha - (n+\beta)/p$, then

$$\kappa_2(r) \sim \left[r^{n-\alpha p+\ell p+\beta}\varphi(r^{-1})\right]^{-1/p} \qquad \text{as } r \to 0.$$

If in addition $n+\beta>0$, then we see by (φ5) that

$$\limsup_{r\to 0} r^{\alpha-\delta}\left[r^{\ell}\kappa_2(r)\right]^{-1} \le M \limsup_{r\to 0} r^{(n+\beta)/p-\delta}\left[\varphi(r^{-1})\right]^{1/p} = 0$$

for $0<\delta<(n+\beta)/p$. If $p>1$ and $\ell=\alpha-(n+\beta)/p$, then $\kappa_2(1)<\infty$ is equivalent to

$$\int_0^1 [\varphi(r^{-1})]^{-p'/p} r^{-1} dr < \infty.$$

For $p>1$, set

$$\varphi^*(r) = \left(\int_0^r [t^{n-\alpha p}\varphi(t^{-1})]^{-p'/p} t^{-1} dt\right)^{1/p'}$$

and

$$\kappa_3(r) = [\omega(r)]^{-1/p}\varphi^*(r).$$

If $\varphi^*(1)<\infty$, then $U_\alpha f$ is continuous everywhere on R^n possibly except at the origin when f satisfies (1.1) and (1.2) (see [11, Theorem 1]).

LEMMA 3.4. *If $0<\delta<\alpha$, then there exists a positive constant M such that*

$$|U_3(x)| \le M\kappa_3(|x|)\left(\int_{B(0,2|x|)-B(0,|x|/2)} \Phi_p(f(y))\omega(|y|)dy\right)^{1/p} + M|x|^{\alpha-\delta}$$

for any $x\in B(0,1/2)-\{0\}$.

PROOF. Let $0<\delta<\alpha$, and consider the function

$$\tilde f(y) = \begin{cases} f(y), & \text{for } y\in B(0,2|x|)-B(0,|x|/2),\\ 0, & \text{otherwise.}\end{cases}$$

Note by Lemma 2.2 that

$$\begin{aligned} |U_3(x)| &\le M\int_{B(0,2|x|)-B(0,|x|/2)} |x-y|^{\alpha-n} f(y)dy \\ &= M\int_{B(0,3|x|)} |z|^{\alpha-n}\tilde f(x+z)dz. \end{aligned}$$

Now Lemma 3.4 can be proved by Lemma 3.1.

We consider the function

$$K(r) = r^{\ell+1}\kappa_1(r) + r^{\ell}\kappa_2(r) + \kappa_3(r).$$

Here note that

$$K(r) \ge M[r^{n-\alpha p}\eta(r)]^{-1/p}$$

for $r>0$.

THEOREM 3.1 ([18, Corollary 4.1]). *Assume that $\ell < \alpha$, $\lim_{r\to 0} K(r) = 0$ and*

$$\begin{array}{ll} \kappa_1(0) = \infty & \text{in case } \alpha - \ell - 1 > 0, \\ \lim_{r\to 0} r^{\alpha-\delta}\left[r^{\ell+1}\kappa_1(r)\right]^{-1} = 0 & \text{for some } \delta > 0 \text{ in case } \alpha - \ell - 1 \le 0, \\ \lim_{r\to 0} r^{\alpha-\delta}\left[r^{\ell}\kappa_2(r)\right]^{-1} = 0 & \text{for some } \delta \text{ such that } 0 < \delta < \alpha - \ell, \\ \lim_{r\to 0} r^{\alpha-\delta}\left[\kappa_3(r)\right]^{-1} = 0 & \text{for some } \delta > 0. \end{array}$$

If f is a nonnegative measurable function on R^n satisfying conditions (1.2) and (1.7), then

$$\lim_{x\to 0} [K(|x|)]^{-1} U_{\alpha,\ell} f(x) = 0.$$

PROOF. We may assume that $0 < \delta < \alpha$. Since $\lim_{r\to 0} r^{\alpha-\delta}[\kappa_3(r)]^{-1} = 0$, we see by Lemma 3.4 that

$$\lim_{x\to 0} [\kappa_3(|x|)]^{-1} U_3(x) = 0.$$

In view of Corollaries 3.1, 3.2 and 3.3, we have

$$\lim_{x\to 0} [K(|x|)]^{-1}\{U_1(x) + U_2(x)\} = 0,$$

and hence

$$\lim_{x\to 0} [K(|x|)]^{-1} U_{\alpha,\ell} f(x) = 0.$$

Thus we complete the proof of Theorem 3.1.

REMARK 3.5. Let $\omega(r) = r^\beta$. If $n + \beta > 0$, then we see by (φ5) that

$$\limsup_{r\to 0} r^{\alpha-\delta}[\kappa_3(r)]^{-1} = 0$$

for $0 < \delta < (n+\beta)/p$.

REMARK 3.6. Let $\omega(r) = r^\beta$, where $-n < \beta \le \alpha p - n$. Let ℓ be the integer such that

$$\ell \le \alpha - (n+\beta)/p < \ell + 1.$$

Then we see with the aid of Remarks 3.3, 3.4 and 3.5 that

$$\begin{array}{ll} K(r) \sim [r^{n-\alpha p+\beta}\varphi(r^{-1})]^{-1/p} & \text{when } \ \ell < \alpha - (n+\beta)/p < \ell + 1, n - \alpha p < 0, \\ K(r) \sim r^{-\beta/p}\left(\int_0^r [\varphi(t^{-1})]^{-p'/p} t^{-1} dt\right)^{1/p'} & \text{when } \ \ell < \alpha - (n+\beta)/p < \ell + 1, n - \alpha p = 0, \\ K(r) \sim r^{\ell}\left(\int_0^r [\varphi(t^{-1})]^{-p'/p} t^{-1} dt\right)^{1/p'} & \text{when } \ \ell = \alpha - (n+\beta)/p. \end{array}$$

In all cases, if $K(1) < \infty$, then

$$\lim_{r\to 0} K(r) = 0.$$

REMARK 3.7. Let $\omega(r) = r^\beta$, where $-n < \beta \le \alpha p - n$. If $\alpha - (n+\beta)/p < \ell + 1$ and f satisfies (1.2), then the proof of Lemma 3.1 shows that (1.7) is fulfilled.

COROLLARY 3.4 ([18, Corollary 4.1]). *Let $\omega(r) = r^\beta$ with $-n < \beta \le \alpha p - n$. Let f be a nonnegative measurable function on R^n satisfying conditions (1.1) and (1.2). If $\ell \le \alpha - (n+\beta)/p < \ell + 1$ and $K(1) < \infty$, then there exists a polynomial P_ℓ of degree at most ℓ such that*

$$\lim_{x \to 0} [K(|x|)]^{-1}[U_\alpha f(x) - P_\ell(x)] = 0$$

with K as in Remark 3.6.

In fact, since $\kappa_2(1) < \infty$, (1.6) holds, and further (1.7) holds by Remark 3.7. Hence

$$U_{\alpha,\ell} f(x) = U_\alpha f(x) - \sum_{|\mu| \le \ell} \frac{x^\mu}{\mu!} \int_{R^n} [(D^\mu R_\alpha)(-y)] f(y) dy.$$

With the aid of Remarks 3.3, 3.4, 3.5 and 3.6, Theorem 3.1 gives the present corollary.

Since $\lim_{r \to 0} r^{-\ell} K(r) = 0$, Corollary 3.4 implies that $U_\alpha f$ is ℓ times differentiable at the origin.

Here we discuss the best possibility of Corollary 3.4 as to the order of infinity in case $\alpha p = n$ and $\omega(r) = 1$.

PROPOSITION 3.1 ([18, Proposition 4.1]). *Assume $\varphi^*(1) < \infty$. Then, for any $\varepsilon > 0$, there exists a nonnegative measurable function f on R^n satisfying (4.2) with $p = n/\alpha$ such that $U_\alpha f(0) < \infty$ and*

$$\lim_{x \to 0} [K(|x|)]^{-\varepsilon - 1} \{U_\alpha f(x) - U_\alpha f(0)\} = -\infty.$$

4 Differentiability

In the section, we are concerned with differentiability properties for Riesz potentials of functions f satisfying

$$\int_{R^n} f(y)^p [\log(e + f(y))]^{-\sigma} dy < \infty. \tag{4.1}$$

THEOREM 4.1 ([14, Corollary 4.1]). *Let f be a nonnegative measurable function on R^n satisfying (1.1) and (4.1). If μ is a multi-index with length ℓ and x is in a fixed compact set in R^n, then*

(*i*) *in case $\alpha = \ell + (n/p)$ and $p - 1 + \sigma < 0$,*

$$D^\mu U_\alpha f(x+h) - D^\mu U_\alpha f(x) = o([\log(1/|h|)]^{(p-1+\sigma)/p}) \qquad \text{as } h \to 0;$$

(*ii*) *in case* $\ell < \alpha - (n/p) < \ell + 1$,

$$D^\mu U_\alpha f(x+h) - D^\mu U_\alpha f(x) = o(|h|^{\alpha - n/p - \ell}[\log(1/|h|)]^{\sigma/p}) \qquad \text{as } h \to 0;$$

(*iii*) *in case* $\alpha = \ell + 1 + (n/p)$ *and* $p - 1 + \sigma > 0$,

$$D^\mu U_\alpha f(x+h) - D^\mu U_\alpha f(x) = o(|h|[\log(1/|h|)]^{(p-1+\sigma)/p}) \qquad \text{as } h \to 0.$$

In case $\alpha = \ell + 1 + (n/p)$ and $p - 1 + \sigma < 0$, $D^\mu U_\alpha f$ is differentiable, and all partial derivatives of order $\ell + 1$ satisfy Hölder condition as in (*i*) of Theorem 4.1.

If we consider the second difference, then we can establish the following result.

THEOREM 4.2 ([14, Corollary 5.1]). *Let* f *be a nonnegative measurable function on* R^n *satisfying* (1.1) *and* (4.1). *If* x *is in a fixed compact set in* R^n, *then*

(*i*) *in case* $\alpha = n/p$ *and* $p - 1 + \sigma < 0$,

$$U_\alpha f(x+2h) - 2U_\alpha f(x+h) + U_\alpha f(x) = o([\log(1/|h|)]^{(p-1+\sigma)/p}) \quad \text{as } h \to 0;$$

(*ii*) *in case* $0 < \alpha - (n/p) < 2$,

$$U_\alpha f(x+2h) - 2U_\alpha f(x+h) + U_\alpha f(x) = o(|h|^{\alpha - n/p}[\log(1/|h|)]^{\sigma/p}) \quad \text{as } h \to 0;$$

(*iii*) *in case* $\alpha = 2 + (n/p)$ *and* $p - 1 + \sigma > 0$,

$$U_\alpha f(x+2h) - 2U_\alpha f(x+h) + U_\alpha f(x) = o(|h|^2[\log(1/|h|)]^{(p-1+\sigma)/p}) \quad \text{as } h \to 0.$$

Compare this result with Theorem 4.1 and (1.8).

5 Fine limits

To evaluate the size of exceptional sets, for a set $E \subset R^n$ and an open set $G \subset R^n$, we consider the relative Orlicz capacity

$$C_{\alpha,\Phi_p}(E;G) = \inf_g \int_G \Phi_p(g(y))dy, \qquad E \subset G,$$

where the infimum is taken over all nonnegative measurable functions g on G such that $U_\alpha g(x) \geq 1$ for every $x \in E$ (cf. Meyers [8] and Mizuta [12]). For simplicity, we write $C_{\alpha,\Phi_p}(E) = 0$ if $C_{\alpha,\Phi_p}(E \cap G; G) = 0$ for every bounded open set G. If a property holds except for a set E with $C_{\alpha,\Phi_p}(E) = 0$, then we say that the property holds C_{α,Φ_p}-quasi everywhere.

THEOREM 5.1 ([18, Corollary 5.1]). *Let* f *be a nonnegative measurable function on* R^n *satisfying* (1.1) *and* (1.4). *If* ℓ *is the nonnegative integer such that* $\ell \leq \alpha - (n+\beta)/p <$

$\ell+1$ *and* $\kappa(1)<\infty$, *then there exist a set* $E \subset R^n$ *and a polynomial* P_ℓ *of degree at most* ℓ *such that*

(5.1)
$$\lim_{x\to 0, x\in R^n-E} [\kappa(|x|)]^{-1}[U_\alpha f(x) - P_\ell(x)] = 0$$

and

(5.2)
$$\sum_{j=1}^{\infty} 2^{j(n-\alpha p)}[\varphi(2^j)]^{-1} C_{\alpha,\Phi_p}(E_j; B_j) < \infty,$$

where $E_j = \{x \in E : 2^{-j} \le |x| < 2^{-j+1}\}$, $B_j = \{x : 2^{-j-1} < |x| < 2^{-j+2}\}$ *and*

$$\kappa(r) = r^\ell \left(\int_0^r [t^{n-\alpha p+\beta+\ell p}\varphi(t^{-1})]^{-1/(p-1)} t^{-1} dt \right)^{1-1/p}.$$

REMARK 5.1. In view of [12, Lemma 7.3], we see that

$$C_{\alpha,\Phi_p}(A_j; B_j) \sim 2^{-j(n-\alpha p)}\varphi(2^j), \qquad A_j = B(0, 2^{-j+1}) - B(0, 2^{-j}).$$

6 Radial limits

We are concerned with the existence of radial limits. For this purpose, we have to modify the fine limit result as follows: there exist a set $E \subset R^n$ and a polynomial P_ℓ such that

(6.1)
$$\lim_{x\to 0, x\in R^n-E} |x|^{(n-\alpha p+\beta)/p}[U_\alpha f(x) - P_\ell(x)] = 0$$

and

(6.2)
$$\sum_{j=1}^{\infty} C_{\alpha,\Phi_p}(2^j E_j; B_0) < \infty;$$

note here that $r^{(n-\alpha p+\beta)/p} \le M[\kappa(r)]^{-1}$, and hence (6.1) is weaker than (5.1). It will be seen that (6.2) is more convenient than (5.2) to our aim of deriving the radial limit result.

THEOREM 6.1 ([18, Corollary 6.1]). *Let* f *be a nonnegative measurable function on* R^n *satisfying* (1.1) *and* (1.4) *for* $-n < \beta \le \alpha p - n$. *If* ℓ *is the nonnegative integer such that* $\ell \le \alpha - (n+\beta)/p < \ell + 1$ *and* $\kappa(1) < \infty$, *then there exist a set* $E^* \subset \partial B(0,1)$ *and a polynomial* P_ℓ *of degree at most* ℓ *such that*

(6.3)
$$\lim_{r\to 0} r^{(n-\alpha p+\beta)/p}[U_\alpha f(r\xi) - P_\ell(r\xi)] = 0 \qquad \text{for any } \xi \in \partial B(0,1) - E^*$$

and

(6.4)
$$C_{\alpha,\Phi_p}(E^*) = 0.$$

7 L^q-differentiability

Throughout this section, let φ be a positive nondecreasing function on $(0, \infty)$ satisfying $(\varphi 1)$ and $(\varphi 2)$.

For $q > 0$, $x_0 \in R^n$ and $r > 0$, we define the L^q-mean of a measurable function u over $B(x_0, r)$ by

$$V_q(u, x_0, r) = \left(\frac{1}{\sigma_n r^n} \int_{B(x_0,r)} |u(x)|^q dx \right)^{1/q},$$

where σ_n denotes the volume of the unit ball $B(0, 1)$.

We say that u is L^q-differentiable of order ℓ at x_0 if

$$\lim_{r \to 0} r^{-\ell} V_q \left(u(x) - P(x), x_0, r \right) = 0$$

for some polynomial P (see Meyers [9], Stein [19] and Ziemer [21]).

In this section, we discuss L^q-differentiability for Riesz potentials of functions f satisfying

(7.1)
$$\int_{R^n} \Phi_p(f(y))dy < \infty.$$

THEOREM 7.1 ([17, Theorem 5.1]). Let $\alpha p \leq n$. Let f be a nonnegative measurable function on R^n satisfying conditions (1.1) and (7.1). If ℓ is a nonnegative integer smaller than α, then $U_\alpha f$ is L^q-differentiable of order ℓ $C_{\alpha-\ell,\Phi_p}$-quasi everywhere for $q > 0$ with $1/q \geq 1/p - \alpha/n$.

For similar results for Bessel potentials of L^p-functions, see Meyers [9].

In case $\ell = \alpha$, we show the following result.

THEOREM 7.2 ([17, Theorem 5.2]). Let ℓ be a positive integer with $\ell p \leq n$. Let f be a nonnegative function in $L^p_{loc}(R^n)$ satisfying condition (1.1) with $\alpha = \ell$. Then $U_\ell f$ is L^q-differentiable of order ℓ almost everywhere for $q > 0$ with $1/q \geq 1/p - \ell/n$.

REMARK 7.1. For L^p-differentiability of Bessel potentials, we refer the reader to Ziemer [21, Theorem 3.4.2]. In case $\ell = \alpha = 1$ and $p < n$, Theorem 7.2 implies the result by Stein [19, Theorem 1, Chapter 8].

8 Exponential integrability

We give the following theorem, which deal with the limiting cases of Sobolev's imbeddings.

THEOREM 8.1 ([13, Theorem A]). *Let f be a nonnegative measurable function on a bounded open set $G \subset \mathbf{R}^n$ satisfying the Orlicz condition*

$$\int_G f(y)^p [\log(e + f(y))]^a [\log(e + \log(e + f(y)))]^b dy < \infty$$

for some numbers p, a and b. If $\alpha p = n$, $a < p-1$, $\beta = p/(p-1-a)$ *and* $\gamma = b/(p-1-a)$, *then*

$$\int_G \exp[A\,(U_\alpha f(x))^\beta\,(\log(e + U_\alpha f(x)))^\gamma]dx < \infty \qquad \text{for any } A > 0. \tag{8.1}$$

In case $a = b = 0$, inequality (8.1) is well known to hold (see [1], [15], [20], [21]). The case $a < p-1$ and $b = 0$ was proved by Edmunds-Krbec [6] and Edmunds-Gurka-Opic [4], [5] ; see also Brézis-Wainger [3].

In view of Theorem 8.1, we see that (8.1) is true for every $\beta > 0$ (and $\gamma > 0$) when $a \geqq p-1$. In particular, in case $a > p-1$, we know that $U_\alpha f$ is continuous on R^n (see Corollary 3.4 and Theorem 4.1).

In case $a = p-1$, we are also concerned with double exponential integrability given by Edmunds-Gurka-Opic [4], [5].

THEOREM 8.2 ([13, Theorem B]). *Let* f *be a nonnegative measurable function on a bounded open set* $G \subset \mathbf{R}^n$ *satisfying the Orlicz condition*

$$\int_G f(y)^p[\log(e + f(y))]^{p-1}[\log(e + \log(e + f(y)))]^b dy < \infty$$

for some numbers p and b. If $\alpha p = n$, $b < p-1$ *and* $\beta = p/(p-1-b)$, *then*

$$\int_G \exp[A\exp(B(U_\alpha f(x))^\beta)]dx < \infty \qquad \text{for any } A > 0 \text{ and } B > 0. \tag{8.2}$$

In case $b > p-1$, $U_\alpha f$ is continuous on R^n (see Corollary 3.4 and Theorem 4.1), so that (8.2) holds for every $\beta > 0$.

REMARK 8.1. Here we discuss the sharpness of β in case $p = n$. For $\delta > 0$, consider the function

$$u(x) = \int_{B(0,1)} |x-y|^{1-n} f(y)dy$$

with

$$f(y) = |y|^{-1}[\log(e/|y|)]^{\delta-1} \qquad \text{for } y \in B(0,1).$$

Then f satisfies

$$\int_{B(0,1)} f(y)^n[\log(e + f(y))]^a dx < \infty \tag{8.3}$$

if and only if $n(\delta - 1) + a < -1$. We see that

$$u(x) \geqq C\int_{\{y\in B(0,1):|y|>2|x|\}} |y|^{1-n} f(y)dy \geqq C[\log(e/|x|)]^\delta$$

for $|x| < 1/4$. Hence, if $\beta\delta > 1$, then

$$\int_{B(0,1)} \exp[u(x)^{\beta}]dx = \infty. \tag{8.4}$$

If $\beta > n/(n-1-a)$, then we can choose δ such that

$$1/\beta < \delta < (n-1-a)/n.$$

In this case, both (8.3) and (8.4) hold. This implies that the exponent β in Theorem 8.1 is sharp.

REMARK 8.2. For $\delta > 0$, consider the function

$$u(x) = \int_{B(0,1)} |x-y|^{1-n} f(y)dy$$

with

$$f(y) = |y|^{-1}[\log(e/|y|)]^{-1}[\log(e\log(e/|y|))]^{\delta-1} \qquad \text{for } y \in B(0,1).$$

Then f satisfies

$$\int_{B(0,1)} f(y)^n[\log(e+f(y))]^{n-1}[\log(e+\log(e+f(y)))]^b dx < \infty \tag{8.5}$$

if and only if $n(\delta-1)+b < -1$. We see that

$$u(x) \geqq C\int_{\{y\in B(0,1):|y|>2|x|\}} |y|^{1-n}f(y)dy \geqq C[\log(e\log(e/|x|))]^{\delta}$$

for $|x| < 1/4$. Hence, if $\beta\delta > 1$, then

$$\int_{B(0,1)} \exp\exp(u(x)^{\beta})dx = \infty. \tag{8.6}$$

If $\beta > n/(n-1-b)$, then we can choose δ such that

$$1/\beta < \delta < (n-1-b)/n.$$

In this case, both (8.5) and (8.6) hold. This implies that the exponent β in Theorem 8.2 is sharp.

REMARK 8.3. Here we also discuss the sharpness of γ in case $p = n$. For $a < n-1$ and $\delta > 0$, consider the function

$$u(x) = \int_{B(0,1)} |x-y|^{1-n} f(y)dy$$

with

$$f(y) = |y|^{-1}[\log(e/|y|)]^{-(a+1)/n}[\log(e\log(e/|y|))]^{\delta-1} \qquad \text{for } y \in B(0,1).$$

Then f satisfies

(8.7) $$\int_{B(0,1)} f(y)^n[\log(e+f(y))]^a[\log(e+\log(e+f(y)))]^b dx < \infty$$

if and only if $n(\delta-1)+b<-1$. We see that

$$u(x) \geqq C\int_{\{y\in B(0,1):|y|>2|x|\}} |y|^{1-n}f(y)dy \geqq C[\log(e/|x|)]^{1-(a+1)/n}[\log(e\log(e/|x|))]^{\delta-1}$$

for $|x|<1/4$. Hence, if $\beta = n/(n-1-a)$ and $\beta(\delta-1)+\gamma>0$, then

(8.8) $$\int_{B(0,1)} \exp[u(x)^\beta (\log(e+u(x)))^\gamma]dx = \infty.$$

If $\gamma > (b+1)/(n-1-a)$, then we can choose δ such that

$$(n-b-1)/n > \delta > (\beta-\gamma)/\beta = (n-(n-a-1)\gamma)/n.$$

In this case, both (8.7) and (8.8) hold.

Thus we do not know whether the exponent γ in Theorem 8.1 is sharp or not.

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