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On Group Topologies on the Group of Diffeomorphisms

HIROAKI SHIMOMURA and TAKESHI HIRAI
(Fukui University) (Kyoto University)
下村 宏彰 (福井大) 平井 武 (京都大)

Introduction.

Let M be a connected, non-compact, σ-compact C^r-manifold with $1 \leq r \leq \infty$. Denote by $\mathrm{Diff}(M)$ the group of all diffeomorphisms and by $\mathrm{Diff}_0(M)$ its subgroup consisting of diffeomorphisms with compact supports. Here we study group topologies on the group $G = \mathrm{Diff}_0(M)$. Usually, as seen in the beginning of [Ki], we have been considering on G the topology τ given by the following way of convergence: *a sequence $g_k, k = 1, 2, \cdots$, converges to g if supports of g and of all g_k are contained in a compact subset K and $g_k \to g$ on K uniformly together with all derivatives.*

This topology τ is normally understood as an inductive limit of topologies of canonical subgroups $G_n \nearrow G, n \to \infty$, as follows. First take an increasing sequence $M_0 \subset M_1 \subset M_2 \subset \cdots$ of relatively compact open subsets so that $\cup_{n=0}^{\infty} M_n = M$ and that each $K_n := \bar{M}_n$, the closure of M_n, is a manifold with boundary. Put

$$G_n = \mathrm{Diff}(K_n) := \{g \in G; \mathrm{supp}(g) \subset K_n\}.$$

Then we have an increasing sequence of subgroups as

$$G_0 \subset G_1 \subset G_2 \subset \cdots, \quad \bigcup_{n=0}^{\infty} G_n = G.$$

The topology τ_n on G_n is given by considering G_n as a topological subgroup of the Fréchet Lie group $\mathrm{Diff}(M_n'')$, where M_n'' is the compact manifold obtained by patching M_n and its mirror image M_n' through the boundary. For the Lie group structure of the group $\mathrm{Diff}(M)$ of a compact manifold M, we refer [Le] or [Om]. When $M = \mathbf{R}^d$ and $M_n = \{x \in \mathbf{R}^d; \| x \| < n\}$, the topology τ_n is nothing but the uniform convergence of $g_k \in G_n$ and also of all derivatives as $k \to \infty$.

In an algebraic sense, $G = \lim_{n\to\infty} G_n$, and as a topology on G, we have $\tau = \lim_{n\to\infty} \tau_n$. Since we will consider other topologies on G later, we denote this inductive limit topology as τ_{ind}.

On the other hand, as Tatsuuma[Ta] proved, when a consistent increasing sequence of topological groups (G_n, τ_n), with a group topology τ_n on G_n, is given, the inductive limit τ_{ind} of topologies τ_n is not necessarily a group topology, that is, it does not necessarily make the inductive limit group $G = \lim_{n\to\infty} G_n$ a topological group. This negative result is contrary to the affirmative statement in [Iw, Article 75] or in [Enc, Article 210]. In fact, he gave a counter example even in a case of simple abelian groups (Example 1.1).

It seems for us that this phenomenon is rather general for the case of non-locally-compact topological groups.

In this paper, we prove that this is the case for diffeomorphism group $\mathrm{Diff}_0(M)$ for any non-compact M. Thus our main theorem here is the following.

Theorem A. *Let M be a connected, non-compact, σ-compact C^r-manifold, $1 \le r \le \infty$. For the group* $\mathrm{Diff}_0(M)$, *the product map $G \times G \ni (g_1, g_2) \mapsto g_1 g_2 \in G$, is not continuous with respect to the inductive limit topology τ_{ind} on G.*

This fact does not affect so much the theory of unitary representations of the group G, because we can take, as our background, the topology $\tau_{p.d.}$ on G which is defined by means of the set of τ_{ind}-continuous positive definite functions (cf. §1). However it has certainly some effects, for instance, for determining *continuous* 1-cocycles $\alpha(g,p)$, $(g,p) \in G \times M$, depending on which continuity we choose (cf. [HS]).

Note that if a *sequence* $g_k \in G, k = 1, 2, \cdots$, is τ_{ind}-convergent to $g \in G$, then there exists a compact subset K of M such that $\mathrm{supp}(g_k)$ and $\mathrm{supp}(g)$ are contained in K, and the convergence is as in [Ki]. To see this last assertion, we remark that the restriction on $G_n = \mathrm{Diff}(K_n)$ of the inductive limit τ_{ind} on G is exactly the original τ_n. In fact, let O_n be a τ_n-open subset of G_n, then, for $k > n$, we can choose inductively a τ_k-open subset O_k of G_k such that $O_k \cap G_{k-1} = O_{k-1}$, since the restriction of τ_k onto G_{k-1} is equal to τ_{k-1}. Put $O = \cup_{k=n}^{\infty} O_k$, then O is τ_{ind}-open in G and $O \cap G_n = O_n$.

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§1. Some generalities on inductive limits.

Let us consider an inductive system $G_\alpha\,(\alpha \in A), \psi_{\alpha\beta}\,(\alpha, \beta \in A, \alpha \preceq \beta)$, of topological groups, where A is a directed set and $\psi_{\alpha\beta}: G_\alpha \to G_\beta$, are injective continuous homomorphisms. Put $G = \varinjlim G_\alpha$ and we identify each G_α with its image in G through $\psi_{\alpha\beta}$'s. Denote by τ_α the group topology on G_α and by $\tau_{ind} = \varinjlim \tau_\alpha$ their inductive limit. Note that, by definition, a subset U of G is open with respect to τ_{ind} (or τ_{ind}-open in short) if and only if $U \cap G_\alpha$ is τ_α-open in G_α for any $\alpha \in A$.

We see easily the following fact on τ_{ind}.

Lemma B. *On the inductive limit group $G = \varinjlim G_\alpha$, the following maps are continuous with respect to $\tau_{ind} = \varinjlim \tau_\alpha$:*

(i) *the inverse: $G \ni g \mapsto g^{-1} \in G$;*

(ii) *the left and right translations: for a fixed $h \in G$,*

$$G \ni g \mapsto gh \in G, \quad G \ni g \mapsto hg \in G.$$

However the product map $G \times G \ni (g_1, g_2) \mapsto g_1 g_2 \in G$ is not necessarily τ_{ind}-continuous as the following example of Tatsuuma shows.

Example 1.1([Ta]). Let $G_n = \mathbf{Q} \times F^n$, $F = \mathbf{R}$ or $\mathbf{Q}$ with the usual non-discrete topology, and imbed G_n into G_{n+1} as $x \mapsto (x, 0)$. Then, for $G = \lim_{n\to\infty} G_n = \mathbf{Q} \times \prod' \mathbf{R}$, the product map is not τ_{ind}-continuous. Or, there exists an open neighbourhood U of the identity element e of G such that V^2 is not contained in U for any open neighbourhood V of e.

Note that, if a *sequence* $g_k \in G, k = 1, 2, \cdots$, converges to e, then there exists a G_n such that $g_k \in G_n$ for all k, and they converge in G_n.

He also proved the following affirmative fact.

Proposition C([Ta]). *For an inductive sequence $(G_n, \tau_n), n = 1, 2, \cdots$, of topological groups, assume that all G_n's are locally compact. Then the inductive limit topology $\tau_{ind} = \lim_{n\to\infty} \tau_n$ gives a group topology on $G = \lim_{n\to\infty} G_n$.*

Example 1.2([Ya]). Let $GL(\infty, F)$ with $F = \mathbf{R}$ or $\mathbf{C}$ be the inductive limit group of $G_n = GL(n, F), n = 1, 2, \cdots$, where G_n is imbedded into G_{n+1} as

$$g \longmapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, by the above proposition, τ_{ind} is a group topology on $GL(\infty, F)$. A basis for τ_{ind}-neighbourhoods of e is given by A.Yamasaki. Rewriting it in a different form, we get another basis as follows. For $g \in GL(\infty, F)$, put $g = 1 + x, x =$

$(x_{ij})_{i,j=1}^{\infty}$. Take $\kappa = (\kappa_{ij})_{i,j=1}^{\infty}$, with $\kappa_{ij} > 0$, and put

$$V(\kappa) = \{g = 1 + x\,;\ |x_{ij}| < \kappa_{ij}\,(\forall i, j)\,\}.$$

Note 1.3. Generally speaking, why τ_{ind} does not give a group topology is that τ_{ind} has too many open neighbourhoods of e. So we should have some criterion to decrease the number of these nighbourhoods. In this context, we can refer the case of locally convex topological vector spaces. In that case the criterion is the convexity of neighbourhoods.

As a group topology on G weaker than τ_{ind}, one can propose the topology $\tau_{p.d.}$ defined by means of the set $\mathcal{P}(\tau_{ind})$ of all positive definite functions on G continuous with respect to τ_{ind}. Note that a positive definite function f is τ_{ind}-continuous on G if it is τ_{ind}-continuous at e, because the topology τ_{ind} is translation-invariant (by Lemma B(ii)), and the positive definiteness of f gives $f(e) \geq |f(g)|$, $f(g^{-1}) = \mathrm{Conj}\{f(g)\}$, and Krein's inequality [Kr]

$$|f(g) - f(h)|^2 \leq 2f(e)\,\{\,f(e) - \Re(f(gh^{-1}))\} \qquad (g, h \in G).$$

By definition, an open neighbourhood of e with respect to $\tau_{p.d.}$ is given as follows. Take a finite number of $f_j \in \mathcal{P}(\tau_{ind}), 1 \leq j \leq N$, and an $\epsilon > 0$, then

$$U(f_1, f_2, \cdots, f_N; \epsilon) = \{\, g \in G\,;\ |f_j(g) - f_j(e)| < \epsilon\,(\forall j)\,\}.$$

The topology $\tau_{p.d.}$ is also defined as a weakest topology on G which makes all τ_{ind}-continuous unitary representations continuous.

Finally we note that $\mathcal{P}(\tau_{ind}) = \mathcal{P}(\tau_{p.d.})$.

§2. Preparation for the proof of Theorem A.

Let $d = \dim M$. To express $G = \mathrm{Diff}_0(M)$ as an inductive limit, we choose $M_0 \subset M_1 \subset \cdots \subset M_n \subset \cdots$ under the following additional condition.

(Condition 1) There exists a coordinate neighbourhood $(V_M, \imath_M)$ containing the closure $\bar{M}_1$ and such that, with respect to a C^r-class Riemannian structure on M, M_0 and M_1 are open balls with the common center, and further that, under the coordinate map $\imath_M$, the Riemannian structure is of the canonical form on M_1:

$$ds^2 = dp_1^2 + dp_2^2 + \cdots + dp_d^2 \quad \text{for} \;\; p = (p_i)_{i=1}^{d} \in M_1 \overset{\imath_M}{\hookrightarrow} \mathbf{R}^d.$$

Denote by $\rho(p,q)$ the distance of two points $p,q\in M$. We fix the origin $\mathbf{O}$ of the coordinates on the boundary $\partial(M_0)$ of M_0, and put $\rho(p)=\rho(p,\mathbf{O})$.

Let $C^r(\bar{M}_0,M_1)$ denotes the set of all maps from $\bar{M}_0$ into M_1 which are restrictions on $\bar{M}_0$ of C^r-maps from some open sets containing $\bar{M}_0$ into M_1. Take $\phi\in C^r(\bar{M}_0,M_1)$. For $1\le k\le r$, finite, and $p\in\bar{M}_0$, put alike a jet at p

$$j_p^k\phi = (\partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_d^{\alpha_d}\phi(p))_{|\alpha|\le k},$$
$$\text{with}\quad \partial_i=\frac{\partial}{\partial p_i},\quad \alpha = (\alpha_1,\alpha_2,\cdots,\alpha_d),\ |\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_d.$$

Considering this value as an element of a Euclidean space $(\mathbf{R}^d)^{N_k}$ for an appropriate N_k, we take its norm:

$$\| j_p^k\phi \| := \left(\sum_{|\alpha|\le k} \| \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_d^{\alpha_d}\phi(p)\|^2\right)^{1/2},$$

and put for $\phi,\psi\in C^r(\bar{M}_0,M_1)\subset C^r(\bar{M}_0,\mathbf{R}^d)$,

$$d^k(\phi,\psi):=\sup_{p\in\bar{M}_0}\| j_p^k(\phi-\psi)\|.$$

We put also, taking the k-th homogeneous part,

$$j_p^{(k)}\phi:=(\partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_d^{\alpha_d}\phi(p))_{|\alpha|=k},\quad d^{(k)}(\phi,\psi):=\sup_{p\in\bar{M}_0}\| j_p^{(k)}(\phi-\psi)\|.$$

The next lemma is a key of our proof of Theorem A. Let $D_1,D_2\subset\mathbf{R}^d$ be connected open sets, and $C^r(D_1,D_2)$ be the set of all C^r-class maps ϕ from D_1 to D_2. For $\phi=(\phi_i)_{i=1}^d\in C^r(D_1,D_2)$, we have $j_p^{(1)}\phi=(\partial_j\phi_i)_{1\le i,j\le d}$. Considering it as a linear map on $\mathbf{R}^d$ canonically, we denote its operator norm by $\| j_p^{(1)}\phi\|_{op}$, where we take $\| x\|=(x_1^2+x_2^2+\cdots+x_d^2)^{1/2}$ as the norm of $x=(x_i)_{i=1}^d\in\mathbf{R}^d$.

Lemma 2.1. *Let $D\subset\mathbf{R}^d$ be an open ball and denote by* id *the identity map on D. Assume for $\phi\in C^r(D,D)$, the support* $\mathrm{supp}(\phi):=\mathrm{Cl}\{p\in D_1\,;\phi(p)\ne p=\mathrm{id}(p)\}$ *is compact, and*

$$\| j_p^{(1)}(\phi-\mathrm{id})\|_{op}=\| j_p^{(1)}\phi-1_d\|_{op}<1\ \ (\forall p\in D),$$

where 1_d denotes the $d\times d$ identity matrix. Then ϕ is a diffeomorphism on D.

Proof. Since $\det(j_p^{(1)}\phi)\ne 0$ $(\forall p\in D)$, by the theorem of implicit functions, we see that ϕ is an open map and locally diffeomorphic.

On the other hand, ϕ is globally 1-1. In fact, for $p,q\in D\subset\mathbf{R}^d$, take $p-q\in\mathbf{R}^d$ and put $p_t=q+t(p-q)$ $(0\le t\le 1)$, then

$$\phi(p)-\phi(q)=\int_0^1\frac{d}{dt}\phi(p_t)\,dt=\int_0^1\left(j_{p_t}^{(1)}\phi\right)(p-q)\,dt.$$

From the similar formula for $\psi = \phi - \mathrm{id}$, we have

$$\| \psi(p) - \psi(q) \| \leq \int_0^1 \| j_{p_t}\psi \|_{op} \| p - q \| \ dt < \| p - q \| .$$

Hence $\| \phi(p) - \phi(q) \| \geq \| p - q \| - \| \psi(p) - \psi(q) \| > 0$.

Now let us prove that ϕ is onto. To do so, it is enough to prove that $\phi(D)$ is relatively closed, i.e., $D \cap \mathrm{Cl}(\phi(D)) = \phi(D)$, because we know already that $\phi(D)$ is open. Here $\mathrm{Cl}(\phi(D))$ denotes the closure of $\phi(D)$ in $\mathbf{R}^d$. Take a $p \in D \cap \mathrm{Cl}(\phi(D))$. Then there exists a sequence $q_n \in D$ such that $\phi(q_n) \to p$ as $n \to \infty$. Since ϕ is 1-1 and = id near the boundary $\partial(D)$, q_n has an accumulation point q inside D. Thus we get $p = \phi(q)$. Q.E.D.

§3. Behavior of a diffeomorphism on M_0 and $\bar{M}_0$.

3.1. A basis of neighbouhoods of $e \in G_0$. We denote the identity map id on M also by e, since it is the identity element of G. Put

$$\Omega = \{ g \in G ; g\bar{M}_0 \subset M_1 \} \subset G.$$

Then Ω is τ_{ind}-open in G, as is easily seen. Note that, for $g \in \Omega$, its restriction $g|_{\bar{M}_0}$ on $\bar{M}_0$ belongs to $C^r(\bar{M}_0, M_1)$.

We define subsets W_k of Ω as follows depending on the class C^r:

$$\begin{aligned} W_k &:= \{ g \in \Omega ; d^k(g, e) \leq 1/k \} \quad \text{in Case } r = \infty, \\ W_k &:= \{ g \in \Omega ; d^r(g, e) \leq 1/k \} \quad \text{in Case } r < \infty. \end{aligned}$$

Then we have the following lemma.

Lemma 3.1. *Put $W_{k,0} := W_k \cap G_0$ for $k = 1, 2, \cdots$. Then they form a basis of neighbourhoods of the identity element $e \in G_0$ with respect to the topology τ_0.*

3.2. Convex combination of maps. Take $g \in \Omega$. For $0 \leq s \leq 1$, we can put

$$g_s := s \cdot \mathrm{id}_{\bar{M}_0} + (1 - s) \cdot g|_{\bar{M}_0} \ \in C^r(\bar{M}_0, M_1). \tag{3.1}$$

More generally we put, for $\phi \in C^r(\bar{M}_0, M_1)$,

$$\phi_s := s \cdot \mathrm{id}_{\bar{M}_0} + (1 - s) \cdot \phi \in C^r(\bar{M}_0, M_1).$$

Further put

$$\begin{aligned} \alpha_k(\phi) &:= \inf\{ s ; 0 \leq s \leq 1, d^k(\phi_s, \mathrm{id}) \leq 1/k \} \quad \text{in Case } r = \infty, \\ \alpha_k(\phi) &:= \inf\{ s ; 0 \leq s \leq 1, d^r(\phi_s, \mathrm{id}) \leq 1/k \} \quad \text{in Case } r < \infty. \end{aligned}$$

Since $d^k(\phi_s, \mathrm{id}) = \sup_{p \in \bar{M_0}} \| j_p^k(\phi_s - \mathrm{id}) \| = (1-s) \cdot d^k(\phi, \mathrm{id})$, we have according as $r = \infty$ or $r < \infty$,

$$(3.2) \qquad \alpha_k(\phi) = 0 \vee \left(1 - \frac{1}{k \cdot d^k(\phi, \mathrm{id})}\right) \quad \text{in Case } r = \infty,$$

$$(3.2') \qquad \alpha_k(\phi) = 0 \vee \left(1 - \frac{1}{k \cdot d^r(\phi, \mathrm{id})}\right) \quad \text{in Case } r < \infty.$$

Define further, for $\phi \in C^r(\bar{M_0}, M_1)$,

$$P_k\phi = \phi_{\alpha_k(\phi)} = \alpha_k(\phi) \cdot \mathrm{id}_{\bar{M_0}} + (1 - \alpha_k(\phi)) \cdot \phi \in C^r(\bar{M_0}, M_1).$$

Then we have the following facts.

(イ) Let $g \in W_k \subset \Omega$. Then $\alpha_k(g) = 0$, whence $P_k g = g|_{\bar{M_0}}$.

(ロ) Let $g \in G_0 \subset \Omega$. Assume $g \in W_{k,0} = W_k \cap G_0$ with $k \geq 2$. Then, for any $s, 0 \leq s \leq 1$, we can extend g_s outside of M_0 as $g_s = \mathrm{id}$, and get $g_s \in G_0 \subset G$.

Proof. Since M_0 is an open ball, we have $g_s \in C_0^r(M_0, M_0)$. Moreover, for any $p \in M_0$,

$$\| j_p^{(1)}(g_s - \mathrm{id}) \|_{op} \leq d^{(1)}(g_s, \mathrm{id}) \leq d^1(g_s, \mathrm{id}) \leq 1/k < 1.$$

By Lemma 2.1 applied to $D = M_0$, we see $g_s \in \mathrm{Diff}_0(M_0) \subset G_0 \subset G$.

3.3. A crutial inequlity on M_0. Now put for $g \in \Omega$

$$(3.3) \qquad \beta_k := \inf_{g \in W_{k,0}} \int_{\bar{M_0}} \rho(g(p))\, dp = \inf_{g \in W_{k,0}} \int_{\bar{M_0}} \| g(p) \| \; dp_1 dp_2 \cdots dp_d,$$

where $p = (p_i)_{i=1}^d$, $dp = dp_1 dp_2 \cdots dp_d$, and $\| g(p) \| = (\Sigma_{i=1}^d g_i(p)^2)^{1/2}$ with $g(p) = (g_i(p))_{i=1}^d$.

The inequlity in the following lemma reflects the fact that G_0 is not locally compact and is crutial for our proof of Theorem A.

Lemma 3.2. *Let $k \geq 2$. Then, for any $g \in W_{k,0} = W_k \cap G_0$, we have*

$$\int_{\bar{M_0}} \rho(g(p))\, dp \; > \; \beta_k.$$

Proof. Step 1. Since $g \in G_0$, $\mathrm{supp}(g) \subset \bar{M_0}$ and so g and the identity map id have, at the origin $\mathbf{O}$, C^r-class contact. Hence

$$j_{\mathbf{O}}^{k'}(g) = j_{\mathbf{O}}^{k'}(\mathrm{id}) \quad (\forall k' \leq r, \text{finite}).$$

We can consider $g - \mathrm{id}$ as an element of $C^r(M_1, \mathbf{R}^d)$, then

$$j^{k'}_{\mathbf{O}}(g - \mathrm{id}) = 0 \qquad (\forall k' \leq r, \text{finite}).$$

We fix $k \geq 2$, and take $k' = k$ in Case $r = \infty$, and $k' = r$ in Case $r < \infty$. Then there exists an open neighbourhood U_M of $\mathbf{O}$ in M such that

$$\begin{aligned} \| j^{k'}_p(g - \mathrm{id}) \| &< \frac{1}{2k} \qquad (\forall p \in U_M \cap M_0), \\ j^{k'}_p(g - \mathrm{id}) &= \mathbf{0} \qquad (\forall p \notin M_0). \end{aligned}$$

Now take an $\eta = (\eta_i)_{i=1}^d \in C^r_0(U_M \cap M_0, \mathbf{R}^d)$ satisfying

$$\text{(3.4)} \qquad \| j^{k'}_p \eta \| < \frac{1}{2k} \quad \text{and} \quad \| j^0_p \eta \| = \| \eta \| < \mathrm{diam}(M_1) - \mathrm{diam}(M_0),$$

where $\mathrm{diam}(M_1)$ denotes the diameter of M_1. Put $\phi = g - \eta$. Then,

$$\begin{aligned} \phi(\bar{M}_0) \subset M_1 \qquad &\text{and} \qquad \phi = \mathrm{id} \quad \text{on} \quad M_1 \setminus M_0, \\ \| j^{k'}_p(\phi - \mathrm{id}) \| &< \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k} \qquad (\forall p \in U_M \cap M_0). \end{aligned}$$

Hence $\phi \in C^r(M_1, M_1)$ and, for any $p \in M_1$,

$$\| j^{(1)}_p(\phi - \mathrm{id}) \|_{op} \leq \| j^{k'}_p(\phi - \mathrm{id}) \| < \frac{1}{k} < 1.$$

Therefore we can apply Lemma 2.1 to ϕ and $D = M_1$, and see that $\phi \in \mathrm{Diff}_0(M_1)$. Since $\mathrm{supp}(\phi) \subset \bar{M}_0$, we get $\phi \in G_0 = \mathrm{Diff}(\bar{M}_0)$ and so $\phi \in W_{k,0} = W_k \cap G_0$.

STEP 2. Let us compair the following two values:

$$\begin{aligned} A &:= \int_{\bar{M}_0} \rho(g(p))\, dp = \int_{\bar{M}_0} \left(\sum_{i=1}^d g_i(p)^2 \right)^{1/2} dp, \\ B &:= \int_{\bar{M}_0} \rho(\phi(p))\, dp = \int_{\bar{M}_0} \left(\sum_{i=1}^d (g_i(p) - \eta_i(p))^2 \right)^{1/2} dp. \end{aligned}$$

To get $A > B (\geq \beta_k)$, it is sufficient to have the following:

$$\begin{aligned} |g_i(p)| &\geq |g_i(p) - \eta_i(p)| \qquad (\forall i, \forall p \in \bar{M}_0), \\ |g_{i_0}(p_0)| &> |g_{i_0}(p_0) - \eta_{i_0}(p_0)| \qquad (\exists i_0, \exists p_0 \in \bar{M}_0), \end{aligned}$$

On the other hand, since the maps g and id are sufficiently near to each other on $U_M \cap M_0$, there certainly exist i_0 and $p_0 \in U_M \cap M_0$ such that $g_{i_0}(p_0) \neq 0$.

Then there exists a small neighbourhood $U(p_0)$ of p_0 such that, for $\epsilon = 1$ or -1 and some $\kappa > 0$,

$$\epsilon \cdot g_{i_0}(p) > \kappa \quad (\forall p \in U(p_0)).$$

We can choose $\eta = (\eta_i)_{i=1}^d$ in such a way that $\eta_i = 0$ for $i \neq i_0$, and $\eta_{i_0} \in C_0^r(U(p_0) \cap U_M \cap M_0, \mathbf{R}^d)$ satisfies the condition (3.4) and

$$\epsilon \cdot \eta_{i_0}(p_0) > 0, \qquad \kappa \geq \epsilon \cdot \eta_{i_0}(p) \geq 0 \quad (\forall p).$$

Under this choice of η the above sufficient condition for $A > B$ holds.

This gives that $A > \beta_k$, which is to be proved. Q.E.D.

§4. A τ_{ind} -neighbourhood of $e \in G$.

4.1. Neighbourhood U. We define a τ_{ind}-neighbourhood U of $e \in G$, for which it will be proved that $V^2 \not\subset U$ for any τ_{ind}-neighbourhood V of $e \in G$.

Let $M_0^c = M \setminus M_0$, and put, for $g \in \Omega \subset G$,

$$F_k(g) := \left| \int_{\bar{M_0}} \rho((P_k g)(p))\, dp - \beta_k \right| + \int_{M_0^c} \rho(g(p), \mathrm{id}(p))\, dp. \tag{4.1}$$

where $\mathrm{id}(p) = p$. Then the following fact is a consequence of Lemma 3.2.

Lemma 4.1. *Let $k \geq 2$. Then,* $F_k(g) > 0 \ (\forall g \in \Omega)$.

Proof. Assume that the 2nd term in $F_k(g)$ is equal to zero. Then, $g = \mathrm{id}$ on M_0^c, and so $\mathrm{supp}(g) \subset \bar{M_0}$ whence $g \in G_0 \subset C^r(\bar{M_0}, M_1)$. Then,

$$P_k g \in C^r(\bar{M_0}, M_1) \quad \subset \quad C^r(M_1, M_1),$$
$$\mathrm{supp}(P_k g) \subset \mathrm{supp}(g) \subset \bar{M_0} \quad \text{and} \quad d^{k'}(P_k g, \mathrm{id}) \leq 1/k < 1,$$

where $k' = k$ or $= r$ according as $r = \infty$ or $r < \infty$. Therefore we can apply Lemma 2.1 to $\phi = P_k g$ and $D = M_1$, and see that $P_k g \in \mathrm{Diff}(\bar{M_0}) = G_0$. Then by Lemma 3.2 we get

$$\int_{\bar{M_0}} \rho((P_k g)(p))\, dp > \beta_k.$$

This means that the 1st term in (3.4) of $F_k(g)$ is positive, and so $F_k(g) > 0$.

4.2. Proof of Theorem A. Choose non-empty open sets O_k in such a way that $O_k \subset M_k \setminus \bar{M_{k-1}}$ for $k \geq 2$. Fix $\gamma > 1$, and for $k \geq 2$, put

$$U_k := \left\{ g \in \Omega;\ F_k(g) > \gamma \cdot \int_{O_k} \rho(g(p), \mathrm{id}(p))\, dp \right\}.$$

Since $G_n = \mathrm{Diff}(\bar{M}_n) = \{ g \in G ; \mathrm{supp}(g) \subset \bar{M}_n \}$, we see that, if $n < k$, then $g = \mathrm{id}$ on O_k. Then, by Lemma 4.1, $U_k \cap G_n = \Omega \cap G_n$, and this is τ_n-open in G_n. In particular, $G_0 = \Omega \cap G_0 \subset U_k$. Put

$$U = \bigcap_{k=2}^{\infty} U_k \subset \Omega.$$

Lemma 4.2. *The subset U is τ_{ind}-open in G.*

Proof. For any $n \geq 2$, the intersection $U \cap G_n$ is τ_{ind}-open in G_n, because

$$U \cap G_n = \cap_{k=2}^{n}(U_k \cap G_n) \cap (\Omega \cap G_n).$$

Now we come to the final stage of the proof of Theorem A, and it is enough for us to prove the following lemma.

Lemma 4.3. *There does not exist any τ_{ind}-neighbourhood V of $e \in G$ such that $V^2 \subset U$.*

Proof. Suppose the contrary and let V be such that $V^2 \subset U$. Since $V \cap G_0$ is τ_0-open and $W_{k,0}$'s form a basis of τ_0-neighbourhoods of $e \in G_0$, there exsits a $W_{k,0}$ such that $V \cap G_0 \supset W_{k,0}$. Put $V_k = V \cap \mathrm{Diff}_0(O_k)$. Then

$$W_{k,0} V_k \subset V^2 \subset U \subset U_k \subset \Omega.$$

Hence, for any $g \in W_{k,0}, h \in V_k$,

$$F_k(g \circ h) > \gamma \cdot \int_{O_k} \rho((g \circ h)(p), \mathrm{id}(p))\, dp.$$

Note that $\mathrm{supp}(g) \subset \bar{M}_0$, $\mathrm{supp}(h) \subset M_k \setminus M_{k-1}$, and that

$$g \circ h = g \ \text{ on } \bar{M}_0, \ \ g \circ h = h \ \text{ on } O_k, \ \ g \circ h = \mathrm{id} \ \ \text{anywhere else.}$$

Hence

$$\left| \int_{\bar{M}_0} \rho((P_k g)(p))\, dp - \beta_k \right| > (\gamma - 1) \cdot \int_{O_k} \rho(h(p), \mathrm{id}(p))\, dp.$$

Further, since $g \in W_{k,0} = W_k \cap G_0$, we have $P_k g = g$, and the above inequality turns out to be

$$\int_{\bar{M}_0} \rho(g(p))\, dp - \beta_k > (\gamma - 1) \cdot \int_{O_k} \rho(h(p), \mathrm{id}(p))\, dp.$$

Taking the infimum over $g \in W_{k,0}$, we get 0 on the left hand side and so

$$0 = \int_{O_k} \rho(h(p), \mathrm{id}(p))\, dp.$$

Hence $h = \mathrm{id}$. This means that $V \cap \mathrm{Diff}_0(O_k) = \{ \mathrm{id} \}$. A contradiction.

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