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Box and Ball System with a Carrier and Ultra-Discrete Modified KdV Equation

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Abstract

A new soliton cellular automaton is proposed. It is defined by an array of an infinite number of boxes, a finite number of balls and a carrier of balls. Moreover, it reduces to a discrete equation obtained from discrete modified Korteweg–de Vries equation through a limit. Algebraic expression of soliton solutions is also proposed.

In 1990, Takahashi and Satsuma proposed a soliton cellular automaton (SCA) [1]. Its state is defined by using an infinite array of boxes and a finite number of balls. Therefore, the SCA is now called 'box and ball system'(BBS). Time evolution rule is defined by the following equation;

$$T_j^{t+1} = \min(L - T_j^t, \sum_{i=-\infty}^{j-1} T_i^t - \sum_{i=-\infty}^{j-1} T_i^{t+1}) \ , \tag{1}$$

where T_j^t is a number of balls in j-th box at time t and L means every box holds L balls at most. (Though the box capacity L is restricted to 1 in the original version of BBS in ref. [1], we can extend the system to the one with boxes of capacity more than 1 [2,3].) The remarkable feature of the system is the existence of N-soliton solutions and an infinite number of conserved quantities [4].

Recently, Tokihiro *et. al.* including two of authors, have revealed the algebraic properties of BBS by finding the direct relation to discrete soliton equations [5]. The key is the following identity;

$$\lim_{\varepsilon\to+0} \varepsilon \log(e^{\frac{A}{\varepsilon}} + e^{\frac{B}{\varepsilon}} + \cdots) = \max(A, B, \cdots) \ . \tag{2}$$

Using this identity, the discrete Lotka-Volterra (d-LV) equation [6];

$$\frac{w_j^{t+1}}{w_j^t} = \frac{1+\delta w_{j-1}^t}{1+\delta w_{j+1}^{t+1}} \ , \tag{3}$$

reduces to the ultra-discrete Lotka-Volterra (u-LV) equation;

$$W_j^{t+1} - W_j^t = \max(0, W_{j-1}^t - L) - \max(0, W_{j+1}^{t+1} - L) \ , \tag{4}$$

if we take $w_j^t = \exp(W_j^t/\varepsilon)$, $\delta = \exp(-L/\varepsilon)$ and a limit $\varepsilon \to +0$. Note that if the parameter L and initial W are all integer, W_j^t for any j and t is always integer.

If we define $\widetilde{W}_j^t$ by

$$\widetilde{W}_j^t = \sum_{i=-\infty}^{j} (T_{i+1}^t - T_i^{t+1}) \ , \tag{5}$$

and introduce a transformation of coordinates $\widetilde{W}_j^t = W_{j-t}^j$, then we can derive eq. (4) from eq. (1). Thus we can see solutions of BBS can be expressed by those of u-LV equation. Indeed, N-soliton solutions of BBS can be derived from those of u-LV equation [5]. The discretization procedure described above is called 'ultra-discretization' and several ultra-discrete equations are successfully derived from difference equations preserving their algebraic properties. [7–9]

Tsujimoto and Hirota proposed a discrete version of modified Korteweg–de Vries (d-mKdV) equation [10];

$$\frac{v_j^{t+1}(1+\delta v_{j+1}^{t+1})}{1+av_j^{t+1}} = \frac{v_j^t(1+\delta v_{j-1}^t)}{1+av_j^t} \ , \tag{6}$$

where δ and a are parameter constants, and, j and t are integer variables. If we define a new variable $r_n(t)$ by $v_n^t = r_n(-\delta t)$ and take a limit $\delta \to 0$, eq. (6) reduces to the following modified version of Lotka–Volterra equation;

$$\dot{r_j} = r_j(1+ar_j)(r_{j+1} - r_{j-1}) \ . \tag{7}$$

Moreover, if we define $s(x,t)$ by $r_j(t) = -\frac{1}{2a} + \sqrt{-1}\varepsilon s((j - \frac{1}{2a}t)\varepsilon, \frac{\varepsilon^3}{3}t)$ and take a limit $\varepsilon \to 0$, then eq. (7) reduces to the following modified Korteweg–de Vries (mKdV) equation;

$$s_t + 6as^2 s_x + \frac{1}{4a} s_{xxx} = 0 \ . \tag{8}$$

Maruno *et. al.* showed eq. (6) can be bilinearized and has N-soliton solution with Casorati determinant [11]. Thus, eq. (6) is a fully-discrete soliton equation analogous to the continuous mKdV equation (8).

In this letter, we show that d-mKdV eq. (6) can reduce to a ultra-discrete modified KdV (u-mKdV) equation under appropriate transformations of variables and the limit (2). Then, we show that u-mKdV equation is related to an extended version of BBS introducing a carrier of balls. Finally, we discuss a structure of N-soliton solutions of the system.

First, we derive u-mKdV eq. from d-mKdV eq. Introducing a variable $\tilde{v}_j^t \equiv v_j^t/(1+av_j^t)$, eq. (6) is rewritten to

$$\tilde{v}_j^{t+1} \frac{1 + (\delta - a)\tilde{v}_{j+1}^{t+1}}{1 - a\tilde{v}_{j+1}^{t+1}} = \tilde{v}_j^t \frac{1 + (\delta - a)\tilde{v}_{j-1}^t}{1 - a\tilde{v}_{j-1}^t} \ . \tag{9}$$

Then, introducing another variable V_j^t by $\tilde{v}_j^t = \exp(V_j^t/\varepsilon)$ and taking $\delta = \exp(-L/\varepsilon)$ and $a = -\exp(-M/\varepsilon)$, eq. (9) reduces to

$$V_j^{t+1} + \varepsilon \log \frac{1 + (e^{-L/\varepsilon} + e^{-M/\varepsilon}) e^{V_{j+1}^{t+1}/\varepsilon}}{1 + e^{(V_{j+1}^{t+1} - M)/\varepsilon}} = V_j^t + \varepsilon \log \frac{1 + (e^{-L/\varepsilon} + e^{-M/\varepsilon}) e^{V_{j-1}^t/\varepsilon}}{1 + e^{(V_{j-1}^t - M)/\varepsilon}} \ . \tag{10}$$

If $L \geq M$, we obtain a trivial equation from eq. (10) under a limit $\varepsilon \to +0$. Therefore, we consider $L < M$ case. Taking $\varepsilon \to +0$, then

$$\begin{aligned} &V_j^{t+1} + \max(0, V_{j+1}^{t+1} - L) - \max(0, V_{j+1}^{t+1} - M) \\ &= V_j^t + \max(0, V_{j-1}^t - L) - \max(0, V_{j-1}^t - M) \ , \end{aligned} \tag{11}$$

through the limit (2).

If we take $M \to \infty$, the last terms of both sides of eq. (11) disappear and eq. (11) becomes u-LV eq. (4). This corresponds to the relation between d-mKdV eq. (6) and d-LV eq. (3) when we take $a = 0$. If L, M and initial V are all integer, V_j^t for any j and t is always integer. We call eq. (11) 'ultra-discrete modified KdV' (u-mKdV) equation.

Next, we define 'box and ball system with a carrier' (BBSC) and show its evolution rule is derived from u-mKdV eq. Prepare an array of an infinite number of boxes and a finite number of balls. Assume that all balls are the same, that is, they can not be distinguished from each other. All boxes are also the same and each box holds L balls at most. A 'state' is defined by putting balls into boxes appropriately. Therefore, any state can be distinguished by the number of balls and the distribution of balls in the array of boxes. Figure 1 shows an example of a state for $L = 3$. Note that the array of boxes is fixed in space and we can identify every box by integer site number j increasing from left to right.

We assume any state can evolve into another state from integer time t to $t+1$. In order to define the evolution rule, prepare a 'carrier' of balls. Assume that the carrier can carry M balls at most. From t to $t+1$, the carrier moves from $-\infty$ site to ∞ site and passes each box from left to right. While the carrier passes the j-th box, the following action occurs. Assume that the carrier carries m $(0 \le m \le M)$ balls before it passes the j-th box. Also assume that there are ℓ $(0 \le \ell \le L)$ balls in the j-th box. There are vacant spaces of $M - m$ balls in the carrier and those of $L - \ell$ balls in the box. Then, when the carrier passes the box, the carrier puts $\min(m, L-\ell)$ balls into the box and gets $\min(\ell, M-m)$ balls from the box. In other words, the carrier puts its balls into the box as many as possible, and simultaneously, gets balls from the box as many as possible. The action of carrier is illustrated in Fig. 2.

According to the above rule, the number of balls in the j-th box changes from ℓ to $\ell + \min(m, L-\ell) - \min(\ell, M-m) = \min(m, L-\ell) + \max(0, \ell+m-M)$. (Note the identity $-\min(A,B) = \max(-A,-B)$.) Finally, if U_j^t denotes the number of balls in the j-th box at time t, an evolution equation to U_j^t is

$$U_j^{t+1} = \min(L - U_j^t, \sum_{i=-\infty}^{j-1} U_i^t - \sum_{i=-\infty}^{j-1} U_i^{t+1}) + \max(0, \sum_{i=-\infty}^{j} U_i^t - \sum_{i=-\infty}^{j-1} U_i^{t+1} - M) \ . \qquad (12)$$

Note that $U_j^t \to 0$ $(j \to \pm\infty)$ because the total number of balls is finite and that the carrier carries $\sum_{i=-\infty}^{j-1} U_i^t - \sum_{i=-\infty}^{j-1} U_i^{t+1}$ balls just before passing the j-th box. All dependent and independent variables of eq. (12) are integer and the dependent variable U always satisfies $0 \leq U \leq L$. Therefore, BBSC can be considered to be a cellular automaton.

Figure 3 shows an evolution of the state of Fig. 1 for $L = 3$ and $M = 5$. In this figure, each number denotes the number of balls in a box and '.' denotes an empty box. Let us define S_j^t by

$$S_j^t = \sum_{i=-\infty}^{j} U_i^t \ . \tag{13}$$

Using $U_j^t = S_j^t - S_{j-1}^t$ and eq. (12), we can derive

$$S_{j+1}^{t+1} - S_j^t = -\max(0, S_{j+1}^t - S_j^{t+1} - L) + \max(0, S_{j+1}^t - S_j^{t+1} - M) \ . \tag{14}$$

In the derivation, we use the identity $\max(A, B) = A + \max(0, B - A)$. Moreover, introduce $\widetilde{V}_j^t = S_{j+1}^t - S_j^{t+1}$, then $\widetilde{V}_j^t$ satisfies

$$\begin{aligned} &\widetilde{V}_{j+1}^{t+1} + \max(0, \widetilde{V}_{j+1}^t - L) - \max(0, \widetilde{V}_{j+1}^t - M) \\ &= \widetilde{V}_j^t + \max(0, \widetilde{V}_j^{t+1} - L) - \max(0, \widetilde{V}_j^{t+1} - M) \ . \end{aligned} \tag{15}$$

If we introduce a coordinates transformation, $\widetilde{V}_j^t = V_{j-t}^j$, then V_j^t satisfies eq. (11). Therefore, we can conclude that BBSC (eq. (12)) reduces to u-mKdV eq. (11) through transformation of variables and coordinates.

Next, we discuss structure of basic solutions to BBSC. Figure 4 (a) and (b) shows examples of evolution of a state of BBSC. We can observe groups of neighboring balls separated by empty boxes at every time. Let us call each group 'ball group'. Moreover, let us define 'size' of a group by the number of balls included.

Figure 4 (a) shows the following: For $t \leq 3$, there are 3 ball groups of which size are 5, 2 and 1, respectively. For $t = 4 \sim 6$, they interact each other. For $t \geq 7$, again 3 ball groups of the same size appear and they never interact. After the interaction, shift of orbit occurs for each ball group. We can observe similar phenomena in Fig. 4 (b). In the figure, there are 4 ball groups of which size are 12, 5, 4 and 2, respectively. Note that we identify a ball group only by its size, not by its shape. For example, the ball group '23' at $t = 0$, '32' at

$t = 1$ and '131' at $t = 2$ in Fig. 4 (b) are an identical group. After the interaction, 4 ball groups reappear and their sizes are the same as those before interaction. Figure 4 (a) and (b) imply BBSC is a soliton system. In spite of the interaction, every ball group preserves its own size and speed. Therefore, we can consider each ball group is a soliton.

The most simple solution is 1-soliton solution. A general expression of 1-soliton solution is

$$U_j^t = f_{j+1}^t - f_j^t - g_{j+1}^t + g_j^t \ , \tag{16}$$

with

$$f_j^t = \max(0, kj - \omega t - \xi^0) \ ,$$
$$g_j^t = \max(0, kj - \omega t - \xi^0 - n) \ ,$$

where n is a size of ball group, ξ^0 is an initial phase, $k = \min(n, L)$, and $\omega = \min(n, M)$. In Fig. 5, we show an example of 1-soliton solution in the case of $L = 4$, $M = 7$, $n = 19$, $\xi^0 = 3$. Since site number j is not specified explicitly in this figure, ξ^0 has a freedom of additional constant.

Let us define a speed of a soliton (ball group) by an average number of boxes which the soliton passes per unit time. Then, a speed of a soliton of size n is $\min(n, M)/\min(n, L)$. Therefore, the maximum speed is M/L. This is a remarkable feature of BBSC distinguishable from BBS, because the speed of a soliton of BBS is unbounded.

General expression of N-soliton solution is

$$U_j^t = f_{j+1}^t - f_j^t - g_{j+1}^t + g_j^t \ , \tag{17}$$

with

$$f_j^t = \max_{\mu_i=0,1} \left[\sum_{i=1}^{N} \mu_i \xi_i - \sum_{i<i'}^{(N)} \mu_i \mu_{i'} a_{ii'} \right] \ ,$$
$$g_j^t = \max_{\mu_i=0,1} \left[\sum_{i=1}^{N} \mu_i (\xi_i - n_i) - \sum_{i<i'}^{(N)} \mu_i \mu_{i'} a_{ii'} \right] \ ,$$

where

$$\xi_i = k_i j - \omega_i t - \xi_i^0 \ ,$$
$$a_{ii'} = 2\min(\omega_i, \omega_{i'}) \ ,$$
$$k_i = \min(n_i, L) \ ,$$
$$\omega_i = \min(n_i, M) \ .$$

Here n_i and ξ_i^0 are a size and an initial phase of each soliton, respectively. $\max\limits_{\mu_i=0,1}[X(\mu_i)]$ denotes the maximum value in 2^N possible values of $X(\mu_i)$ obtained by replacing each μ_i by 0 or 1. $\sum\limits_{i<i'}^{(N)}$ denotes the summation over all possible pairs chosen from N elements.

Note that we derived the above expression of solution empirically and can not yet prove it is truly a general expression. However, we confirmed the expression numerically for wide range of initial data. About the solutions in Fig. 4, we obtain them by setting $N = 3$, $n_1 = 5$, $n_2 = 2$, $n_3 = 1$, $\xi_1^0 = 0$, $\xi_2^0 = 6$, $\xi_3^0 = 13$ (Fig. 4(a)) and $N = 4$, $n_1 = 12$, $n_2 = 5$, $n_3 = 4$, $n_4 = 2$, $\xi_1^0 = 2$, $\xi_2^0 = 12$, $\xi_3^0 = 22$, $\xi_4^0 = 22$ (Fig. 4(b)) in the above expression.

Finally, we give concluding remarks. We proposed a new soliton system, BBSC. This system is an extended system to BBS in ref. [1] and can reduce to u-mKdV eq. (11) newly obtained from d-mKdV eq. (6). Moreover, we proposed a general expression of soliton solutions to BBSC. However, this expression is derived empirically. In ref. [11], algebraic expression of N-soliton solution to d-mKdV eq. is shown. Therefore, it may be possible to derive solutions to u-mKdV eq. from those to d-mKdV eq. using the limit (2). Such a derivation is not automatic and we have not yet succeeded. This is a future problem to be solved.

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FIGURES

FIG. 1. An example of a state for $L = 3$.

FIG. 2. Action of carrier while passing a box.

FIG. 3. Evolution of the state of Fig. 1 for $L = 3$ and $M = 5$. Each number denotes the number of balls in a box and '.' denotes an empty box.

FIG. 4. Examples of evolution. (a) $L = 1$ and $M = 3$, (b) $L = 3$ and $M = 6$.

FIG. 5. Example of 1-soliton for $L = 4$ and $M = 7$.

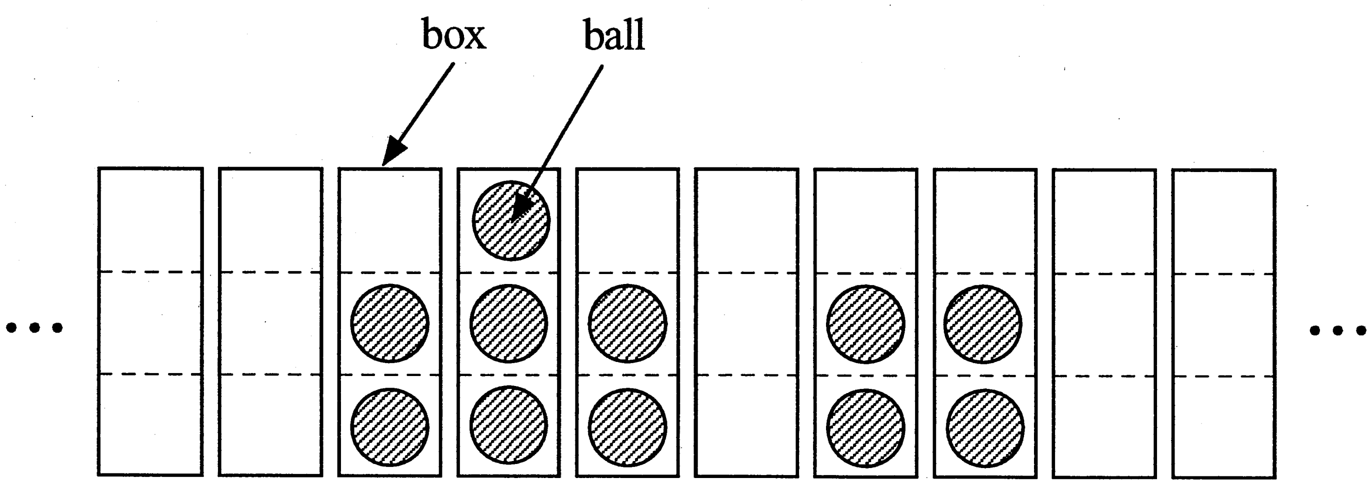

Figure 1: An example of a state for L = 3.

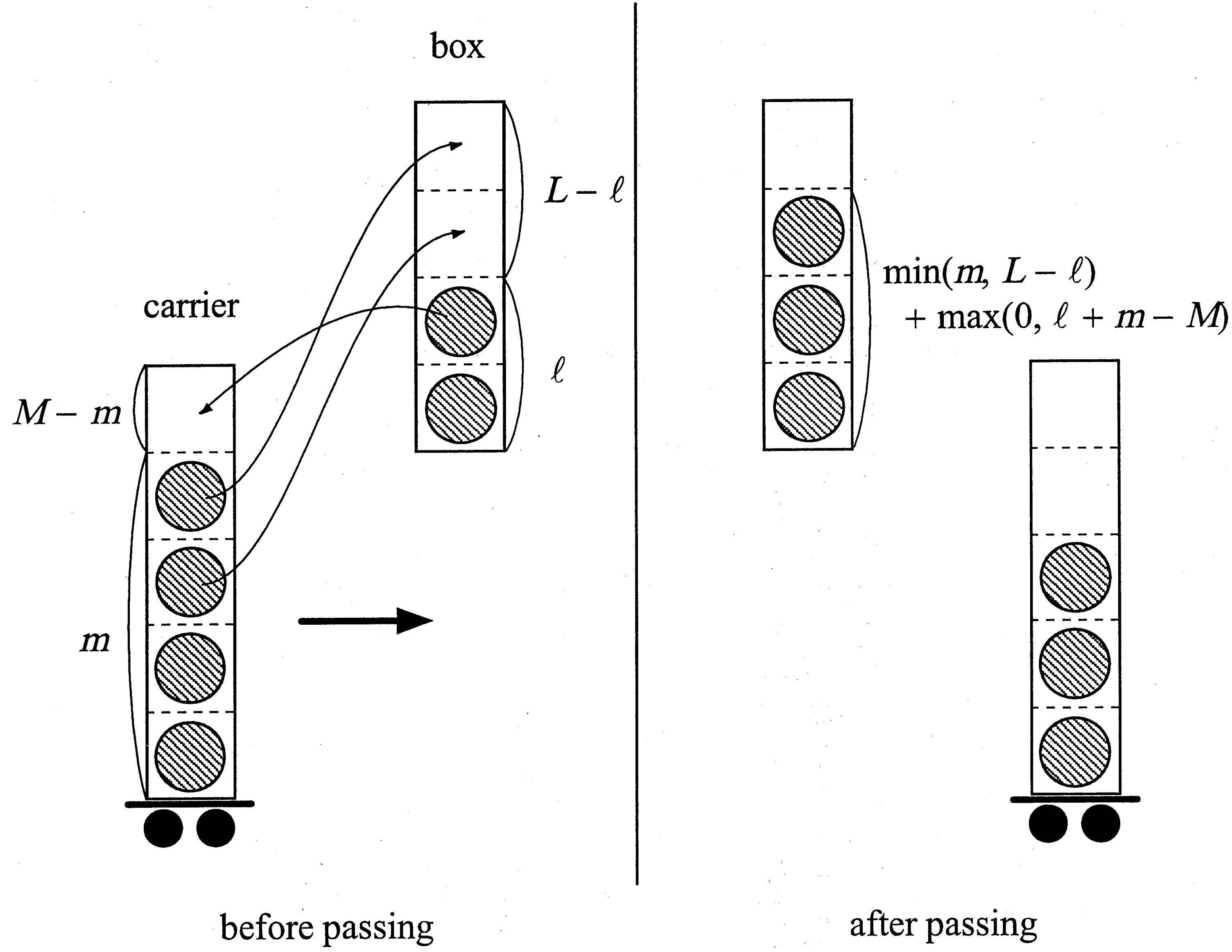

Figure 2: Action of carrier while passing a box.

```
t = 0 : .....232.22...............
    1 : .......33113..............
    2 : ........132.32............
    3 : ..........23.231..........
    4 : ............31133.........
    5 : .............22.232.......
    6 : ..............13..331.....
```

Figure 3 : Evolution of the state of Fig. 1 for L = 3 and M = 5. Each number denotes the number of balls in a box and '.' denotes an empty box.

```
t = 0 : ...11111.....11.....1.............................
    1 : ......11111....11....1............................
    2 : .........11111...11...1...........................
    3 : ............11111..11..1..........................
    4 : ...............1111..11.11........................
    5 : ..................111..1.1111.....................
    6 : .....................11.1..11111..................
    7 : .......................1.11...11111...............
    8 : ........................1..11....11111............
    9 : .........................1...11.....11111.........
   10 : ..........................1....11......11111......
   11 : ...........................1.....11.......11111...
```

(a)

```
t =  0 : ...13332..23...13...2...........................
     1 : .....13332..32...31..2..........................
     2 : .......13332.131..22..2.........................
     3 : .........13332.23..13..2........................
     4 : ...........13331.33..31.2.......................
     5 : .............1333.133.22.2......................
     6 : ...............1332.2311313.....................
     7 : .................1331.32.3133...................
     8 : ...................133.13.21333.................
     9 : .....................132.312.2333...............
    10 : .......................13.2131.3333.............
    11 : .........................312.23..3333...........
    12 : ..........................213..32..3333.........
    13 : ...........................2.31.131..3333.......
    14 : ............................2.22..23...3333.....
    15 : .............................2.13...32...3333...
```

(b)

Figure 4 : Examples of evolution. (a) L = 1 and M = 3, (b) L = 3 and M = 6.

```
t = 0 : ...144442.................
    1 : .....244441...............
    2 : .......34444..............
    3 : .........44443............
    4 : ..........144442..........
    5 : ............244441........
    6 : ..............34444.......
    7 : ................44443.....
    8 : .................144442...
```

Figure 5 : Example of 1-soliton for L = 4 and M = 7.