数理解析研究所講究録
1020 巻 1997 年 143-159

Discretization of Coupled Modified KdV Equations

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ABSTRACT: The coupled modified KdV equation

$$\frac{\partial v_i}{\partial t}+3[\sum_{j,k=1}^{N}c_{j,k}v_jv_k]\frac{\partial v_i}{\partial x}+\frac{\partial^3 v_i}{\partial x^3}=0,\quad i=1,2,\cdots,N,$$

is discretized in the form

$$v_{i,n}^{m+1}-v_{i,n}^{m}+\delta[1+\sum_{j,k=1}^{N}c_{j,k}v_{j,n}^{m}v_{k,n}^{m}]\Gamma_n^m[v_{i,n+1}^{m}-v_{i,n-1}^{m+1}]=0,\quad i=1,2,\cdots,N,$$

$$\Gamma_{n+1}^{m}=[1+\sum_{j,k=1}^{N}c_{j,k}v_{j,n}^{m}v_{k,n}^{m}]\Gamma_n^m/[1+\sum_{j,k=1}^{N}c_{j,k}v_{j,n}^{m+1}v_{k,n}^{m+1}],$$

where Γ_n^m is an auxiliary variable. We integrate the difference equation numerically and compare the results with exact solutions.

KEYWORDS: solitons,coupled system,exact solutions,difference scheme

1 Introduction

We have proposed [1]a method of constructing nonlinear partial differential equations without destroying integrability. The method uses the bilinear formalism and follows 3 steps. First, a given nonlinear partial differential equation is transformed into the bilinear form by the dependent variable transformation. Secondly the bilinear differential equation is discretized. Thirdly the bilinear difference equation is transformed back into the nonlinear difference equation by the associated dependent variable transformation.

In this paper we discuss how to discretize the bilinear differential equation taking the modified KdV equation as an example. Then we construct a system of coupled modified KdV difference-difference equations.

The difference equation obtained is found to be of importance in integrating the coupled modified KdV equations numerically. We compare the results with exact solutions.

2 Discretization of the Modified KdV Equation

We have the modified KdV equation

$$\frac{\partial v}{\partial t} + 6v^2 \frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} = 0, \tag{1}$$

which is transformed into the bilinear form [2]

$$(D_t + D_x^3) g \cdot f = 0, \tag{2}$$

$$D_x^2 f \cdot f = 2g^2, \tag{3}$$

through the dependent variable transformation $v = g/f$. The bilinear differential equation has N-soliton solution. We write 2-soliton solution among them explicitly

$$f = 1 + a(1,1)e^{2\eta_1} + a(1,2)e^{\eta_1+\eta_2} + a(2,2)e^{2\eta_2} + a(1,1,2,2)e^{2\eta_1+2\eta_2}, \tag{4}$$

$$g = e^{\eta_1} + e^{\eta_2} + a(1,1,2)e^{2\eta_1+\eta_2} + a(1,2,2)e^{\eta_1+2\eta_2}, \tag{5}$$

where

$$\eta_i = p_i(x - x_0) - \omega t, \tag{6}$$

$$\omega_i = p_i^3, \quad i = 1,2, \tag{7}$$

$$a(1,1) = \frac{1}{(2p_1)^2}, \tag{8}$$

$$a(1,2) = \frac{2}{(p_1+p_2)^2}, \tag{9}$$

$$a(2,2) = \frac{1}{(2p_2)^2}, \tag{10}$$

$$c(1,2) = [\frac{(p_1-p_2)}{(p_1+p_2)}]^2, \tag{11}$$

$$a(1,1,2) = a(1,1) * c(1,2), \tag{12}$$

$$a(1,2,2) = a(2,2) * c(1,2), \tag{13}$$

$$a(1,1,2,2) = a(1,1) * a(2,2) * c(1,2)^2, \tag{14}$$

where p_i and x_0 are parametes related to an amplitude and a position of a soliton respectively.

A discrete analogue of the modified KdV equation is obtained [3)]

$$\frac{\partial v_n}{\partial t} + (1 + v_n^2)(v_{n+1} - v_{n-1}) = 0, \qquad (15)$$

which is reduced to eq.(1) in the limit of small ϵ ($x = n\epsilon$).

Equation(15) is transformed into the bilinear form

$$D_t g_n \cdot f_n + g_{n+1} f_{n-1} - g_{n-1} f_{n+1} = 0, \qquad (16)$$

$$f_{n+1} f_{n-1} - f_n^2 = g_n^2, \qquad (17)$$

through the same dependent variable transformation $v_n = g_n/f_n$. The bilinear differential-difference equation has N-soliton solution. We write 2-soliton solution among them explicitly

$$f = 1 + a(1,1)e^{2\eta_1} + a(1,2)e^{\eta_1+\eta_2} + a(2,2)e^{2\eta_2} + a(1,1,2,2)e^{2\eta_1+2\eta_2}, \qquad (18)$$

$$g = e^{\eta_1} + e^{\eta_2} + a(1,1,2)e^{2\eta_1+\eta_2} + a(1,2,2)e^{\eta_1+2\eta_2}, \qquad (19)$$

where

$$e^{\eta_i} = P_i^{(n-n_0)} e^{-\omega_i t}, \qquad (20)$$

$$\omega_i = P_i - 1/P_i, \quad i = 1, 2, \qquad (21)$$

$$a(1,1) = \frac{P_1^2}{(P_1^2 - 1)^2}, \qquad (22)$$

$$a(1,2) = \frac{P_1 P_2}{(P_1 P_2 - 1)^2}, \qquad (23)$$

$$a(2,2) = \frac{P_2^2}{(P_2^2 - 1)^2}, \qquad (24)$$

$$c(1,2) = [\frac{(P_1 - P_2)}{(P_1 * P_2 - 1)}]^2, \qquad (25)$$

and $a(1,1,2), a(1,2,2)$ and $a(1,1,2,2)$ have the same forms as eqs.(12) (13) and (14). P_i and n_0 are parametes related to an amplitude and a position of a soliton respectively.

We note that the bilinear equations (2) and (3) are invariant under the gauge transformation

$$f \rightarrow fe^{\alpha x+\beta t}, \tag{26}$$

$$g \rightarrow ge^{\alpha x+\beta t}, \tag{27}$$

where α and β are constants. Also invariant are the bilinear equations (16) and (17) under the transformation

$$f_n \rightarrow f_n e^{\alpha n+\beta t}, \tag{28}$$

$$g_n \rightarrow g_n e^{\alpha n+\beta t}. \tag{29}$$

Based on these facts we postulate that the discrete-time bilinear equations are invariant under the gauge transformation

$$f_n^m \rightarrow f_n^m e^{\alpha n+\beta m}, \tag{30}$$

$$g_n^m \rightarrow g_n^m e^{\alpha n+\beta m}, \tag{31}$$

where $t = m\delta$,m being integers.

We have by definition

$$D_t g_n \cdot f_n = \frac{dg}{dt} f - g\frac{df}{dt}. \tag{32}$$

Replacing the differential operator by the forward difference operator

$$\frac{df}{dt} \rightarrow \delta^{-1}[f_n^{m+1} - f_n^m], \tag{33}$$

$$\frac{dg}{dt} \rightarrow \delta^{-1}[g_n^{m+1} - g_n^m], \tag{34}$$

we obtain a difference analogue of the bilinear operator

$$D_t g \cdot f \rightarrow \delta^{-1}[g_n^{m+1} f_n^m - g_n^m f_n^{m+1}]. \tag{35}$$

Accordingly we have a discrete-time bilinear equation

$$g_n^{m+1} f_n^m - g_n^m f_n^{m+1} + \delta[g_{n+1}^m f_{n-1}^m - g_{n-1}^m f_{n+1}^m] = 0. \tag{36}$$

However this equation is not invariant under the gauge transformation (30) and (31). We find that following two forms are gauge invariant

$$g_n^{m+1} f_n^m - g_n^m f_n^{m+1} + \delta[g_{n+1}^m f_{n-1}^{m+1} - g_{n-1}^{m+1} f_{n+1}^m] = 0, \tag{37}$$

$$g_n^{m+1} f_n^m - g_n^m f_n^{m+1} + \delta[g_{n+1}^{m+1} f_{n-1}^m - g_{n-1}^m f_{n+1}^{m+1}] = 0. \tag{38}$$

Tsujimoto [4] has pointed out that eq.(37) is transformed into eq.(38) by the coordinates trasformation:

$$m \rightarrow m - n, \tag{39}$$

$$n \rightarrow n. \tag{40}$$

Hereafter we use eq.(37) as a difference analogue of eq.(16). As a difference analogue of eq.(17) we use the bilinear eqaution (17) as it is because it does not concerned with the time-development. Then we have the time-discretization of the bilinear forms (16) and (17)

$$g_n^{m+1} f_n^m - g_n^m f_n^{m+1} + \delta[g_{n+1}^m f_{n-1}^{m+1} - g_{n-1}^{m+1} f_{n+1}^m] = 0, \tag{41}$$

$$f_{n+1}^m f_{n-1}^m - (f_n^m)^2 = (g_n^m)^2. \tag{42}$$

We have found that eqs.(41) and (42) exhibit 2-soliton solution of the following

$$f = 1 + a(1,1)e^{2\eta_1} + a(1,2)e^{\eta_1+\eta_2} + a(2,2)e^{2\eta_2} + a(1,1,2,2)e^{2\eta_1+2\eta_2}, \tag{43}$$

$$g = e^{\eta_1} + e^{\eta_2} + a(1,1,2)e^{2\eta_1+\eta_2} + a(1,2,2)e^{\eta_1+2\eta_2}, \tag{44}$$

where

$$e^{\eta_i} = P_i^{(n-n_0)}/\Omega_i^m, \tag{45}$$

$$\Omega_i = \frac{1-\delta/P_i}{1-\delta P_i}, \quad i = 1,2, \tag{46}$$

and $a(1,1), a(1,2), a(2,2), c(1,2), a(1,1,2), a(1,2,2)$ and $a(1,1,2,2)$ have the same forms as before.

Now we transform the bilinear eqs.(41) and (42) back into the nonlinear difference-difference equations. Let

$$g_n^m = v_n^m f_n^m. \tag{47}$$

Substituting this into eqs.(41) and (42) we find

$$v_n^{m+1} - v_n^m + \delta\frac{f_{n+1}^m f_{n-1}^{m+1}}{f_n^{m+1} f_n^m}[v_{n+1}^m - v_{n-1}^{m+1}] = 0, \tag{48}$$

$$\frac{f_{n+1}^m f_{n-1}^m}{(f_n^m)^2} = 1 + (v_n^m)^2. \tag{49}$$

The term $f_{n+1}^m f_{n-1}^{m-1}/f_n^{m+1} f_n^m$ in eq.(48) is very similar to the term $f_{n+1}^m f_{n-1}^m/(f_n^m)^2$ which is $l.h.s$ of eq.(49). Let us define the ratio of the two terms by Γ_n^m which becomes an auxiliary variable

$$\Gamma_n^m = \frac{f_{n+1}^m f_{n-1}^{m-1}}{f_n^{m+1} f_n^m} \div \frac{f_{n+1}^m f_{n-1}^m}{(f_n^m)^2}$$

$$= \frac{f_{n-1}^{m+1} f_n^m}{f_n^{m+1} f_{n-1}^m}. \tag{50}$$

Substituting eq.(49) and Γ_n^m into eq.(48) we obtain

$$v_n^{m+1} - v_n^m + \delta[1 + (v_n^m)^2]\Gamma_n^m [v_{n+1}^m - v_{n-1}^{m+1}] = 0. \tag{51}$$

On the other hand taking the ratio of Γ_{n+1}^m and Γ_n^m we obtain

$$\begin{aligned} \frac{\Gamma_{n+1}^m}{\Gamma_n^m} &= \frac{f_n^{m+1} f_{n+1}^m}{f_{n+1}^{m+1} f_n^m} \div \frac{f_{n-1}^{m+1} f_n^m}{f_n^{m+1} f_{n-1}^m} \\ &= \frac{f_{n+1}^m f_{n-1}^m}{(f_n^m)^2} \div \frac{f_{n+1}^{m+1} f_{n-1}^{m+1}}{(f_n^{m+1})^2} \\ &= \frac{1 + (v_n^m)^2}{1 + (v_n^{m+1})^2}. \end{aligned} \tag{52}$$

Hence the following two equations

$$v_n^{m+1} - v_n^m + \delta[1 + (v_n^m)^2]\Gamma_n^m [v_{n+1}^m - v_{n-1}^{m+1}] = 0, \tag{53}$$

$$\frac{\Gamma_{n+1}^m}{\Gamma_n^m} = \frac{1 + (v_n^m)^2}{1 + (v_n^{m+1})^2}. \tag{54}$$

constitute of the modified KdV difference-difference equation.

Here we remark that the nonlinear Schrödinger equation

$$i\frac{\partial}{\partial t}\psi + \frac{\partial^2}{\partial x^2}\psi + 2|\psi|^2\psi = 0 \tag{55}$$

has been discretized [5] by the same fashion in the form

$$i\delta^{-1}[\psi_n^{m+1} - \psi_n^m] - [\psi_n^{m+1} + \psi_n^m] + [1 + |\psi_n^m|^2]\Gamma_n^m [\psi_{n+1}^m + \psi_{n-1}^{m+1}] = 0, \tag{56}$$

$$\Gamma_{n+1}^m = \frac{1 + |\psi_n^m|^2}{1 + |\psi_n^{m+1}|^2}\Gamma_n^m. \tag{57}$$

Multi-soliton solutions to the nonlinear Schrödinger difference-difference equation are obtained by Tsujimoto [6].

In the limit of samll δ eq.(56) is reduced to a differential-difference equation obtained by Ablowitz and Ladik [7].

3 System of Coupled Modified KdV Equation

Svinolupov [8] has shown that a system of coupled modified KdV equations

$$\frac{\partial v_i}{\partial t}+3[\sum_{j,k=1}^{N} c_{j,k}v_jv_k]\frac{\partial v_i}{\partial x}+\frac{\partial^3 v_i}{\partial x^3}=0, \quad i=1,2,\cdots,N, \tag{58}$$

where the coefficients $c_{j,k}$ are arbitrary constants, possesses an infinite series of generalized symmetries and local conservation laws. Iwao and the author [9] have shown that it posseses multi-soliton solutions expressed by pfaffians under the conditions on the coefficients,namely $c_{jj}=0$ and $c_{jk}=c_{kj}$.

Habibullin and Svinolupov [10] have found that the boundary conditions imposed at the point x=0 are compatible with the integrability of eq.(58). The author has shown [11] that a system of the coupled modified KdV differential-difference equations under the boundary conditions $v_n^{(i)}=0$ at $n=0,\ i=1,2,\cdots,N$

$$\frac{\partial v_n^{(i)}}{\partial t}-[\sum_{j,k=1}^{N} c_{j,k}v_n^j v_n^k][v_{n+1}^{(i)}-v_{n-1}^{(i)}]=0, \quad i=1,2,\cdots,N \tag{59}$$

exhibits exact solutions expressed by pfaffians of the following form

$$v_n^{(i)}=g_n^{(i)}/f_n, \quad i=1,2,\cdots,N, \tag{60}$$

where

$$f_{2n} = \mathrm{pf}(0, 1, 2, \cdots, 2n-1), \tag{61}$$

$$g_{2n}^{(i)} = \mathrm{pf}(p_i, 1, 2, \cdots, 2n-1), \tag{62}$$

$$f_{2n+1} = \mathrm{pf}(1, 2, 3, \cdots, 2n), \tag{63}$$

$$g_{2n+1}^{(i)} = \mathrm{pf}(p_i, 0, 1, \cdots, 2n), \tag{64}$$

where the elements of the pfaffians are defined as follows

$$\mathrm{pf}(p_i, m) = h^{(i)}(t, m), \tag{65}$$

$$\frac{1}{2}[\mathrm{pf}(l, m+1) - \mathrm{pf}(l+1, m)] = \sum_{j,k=1}^{N} c_{j,k} h^{(j)}(t, l) h^{(k)}(t, m), \tag{66}$$

$$\frac{d}{dt}\mathrm{pf}(p_i, m) = \mathrm{pf}(p_i, m+1), \tag{67}$$

$$\frac{d}{dt}\mathrm{pf}(l, m) = \mathrm{pf}(l+1, m) + \mathrm{pf}(l, m+1), \tag{68}$$

for $i = 1, 2, \cdots, N$ and for non-negative integers l and m, and $h^{(i)}(t, m)$ are arbitrary functions of t and satisfy the relations

$$\frac{d}{dt}h^{(i)}(t, m) = h^{(i)}(t, m+1). \tag{69}$$

Following the same procedure developed in the previous section we transform the system of coupled modified KdV equations

$$\frac{\partial v_i}{\partial t} + 3[\sum_{j,k=1}^{N} c_{j,k} v_j v_k]\frac{\partial v_i}{\partial x} + \frac{\partial^3 v_i}{\partial x^3} = 0, \quad i = 1, 2, \cdots, N, \tag{70}$$

into the bilinear forms

$$(D_t - D_x^3)g_i \cdot f = 0,, \quad i = 1, 2, \cdots, N, \tag{71}$$

$$D_x^2 f \cdot f = \sum_{j,k=1}^{N} c_{j,k} g_j g_k, \tag{72}$$

through the dependent variable transformation

$$v_i = g_i/f, \quad i = 1, 2, \cdots, N. \tag{73}$$

We have a difference analogue of eqs.(71) and (72)

$$D_t g_n^{(i)} \cdot f_n - g_{n+1}^{(i)} f_{n-1} + g_{n-1}^{(i)} f_{n+1} = 0, \tag{74}$$

$$f_{n+1} f_{n-1} - f_n^2 = \sum_{j,k=1}^{N} c_{j,k} g_n^{(j)} g_n^{(k)}. \tag{75}$$

Following the procedure described in the previous section we obtain a time-discretization of the bilinear forms

$$g_{i,n}^{m+1} f_n^m - g_{i,n}^m f_n^{m+1} + \delta[g_{i,n+1}^m f_{n-1}^{m+1} - g_{i,n-1}^{m+1} f_{n+1}^m] = 0, \tag{76}$$

$$f_{n+1}^m f_{n-1}^m - (f_n^m)^2 = \sum_{j,k=1}^{N} c_{j,k} g_{j,n}^m g_{k,n}^m, \tag{77}$$

which are transformed back into the coupled nonlinear difference-difference equations

$$v_{i,n}^{m+1} - v_{i,n}^m + \delta[1 + \sum_{j,k=1}^{N} c_{j,k} v_{j,n}^m v_{k,n}^m]\Gamma_n^m [v_{i,n+1}^m - v_{i,n-1}^{m+1}] = 0, \quad i = 1, 2, \cdots, N, \tag{78}$$

$$\Gamma_{n+1}^m = [1 + \sum_{j,k=1}^{N} c_{j,k} v_{j,n}^m v_{k,n}^m]\Gamma_n^m / [1 + \sum_{j,k=1}^{N} c_{j,k} v_{j,n}^{m+1} v_{k,n}^{m+1}], \tag{79}$$

through the dependent variable transformation

$$g_{i,n}^m = v_{i,n}^m f_{i,n}^m, \quad i = 1, 2, \cdots, N, \tag{80}$$

$$\Gamma_n^m = \frac{f_{n-1}^{m+1} f_n^m}{f_n^{m+1} f_{n-1}^m}. \tag{81}$$

We have found[12)] that multi-soliton solutions to the coupled modified KdV difference-difference equations are expressed by pfaffians,which have the same structure as those [9)] of the coupled modified KdV differential equations.

Let $N = 2$ and $c_{11} = c_{22} = 0, c_{12} = c_{21} = 1/2.$ in eqs.(78) and (79). Then we have the simplest form of the coupled modified KdV difference-difference equations

$$v_{1,n}^{m+1} - v_{1,n}^m + \delta[1 + v_{1,n}^m v_{2,n}^m]\Gamma_n^m[v_{1,n+1}^m - v_{1,n-1}^{m+1}] = 0, \tag{82}$$

$$v_{2,n}^{m+1} - v_{2,n}^m + \delta[1 + v_{1,n}^m v_{2,n}^m]\Gamma_n^m[v_{2,n+1}^m - v_{2,n-1}^{m+1}] = 0, \tag{83}$$

$$\Gamma_{n+1}^m = [1 + v_{1,n}^m v_{2,n}^m]\Gamma_n^m/[1 + v_{1,n}^{m+1} v_{2,n}^{m+1}], \tag{84}$$

which exhibit the following 1+1 soliton solution

$$f = 1 + a_{12}\exp[\eta_1 + \eta_2], \tag{85}$$

$$g_1 = \exp[\eta_1], \tag{86}$$

$$g_2 = \exp[\eta_2], \tag{87}$$

where

$$\exp[\eta_i] = P_i^{n-n_i}/\Omega_i^m, \tag{88}$$

$$\Omega_i = \frac{1-\delta/P_i}{1-\delta P_i}, \quad i=1,2, \tag{89}$$

$$a_{12} = \frac{P_1P_2}{(P_1P_2-1)^2}. \tag{90}$$

This expression of the 1+1 soliton solution shows that a soliton $v_1 = g_1/f$ is damping and another soliton $v_2 = g_2/f$ is growing as time m increases.

The difference-difference equations (82),(83) and (84) are integrated numerically using the initial conditions of the 1+1 soliton solution with the following parameters: $\delta = 1/5, \quad P_1 = 2, \quad P_2 = 3, \quad n_1 = 12, \quad n_2 = 7$. The numerical results agree exactly with the theoretical ones except round-off errors (see Fig.1).

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Figure Captions

Fig.1 Numerical integration of eqs.(82),(83) and (84) with the initial condition of the 1+1 soliton solultion expressed by eqs.(85)-(90). The soliton $v_1 = g_1/f$ is damping and the other soliton g_2/f is growing as time m increases from 1 to 50. Numerical results agree exactly with the theoretical ones except round-off errors.

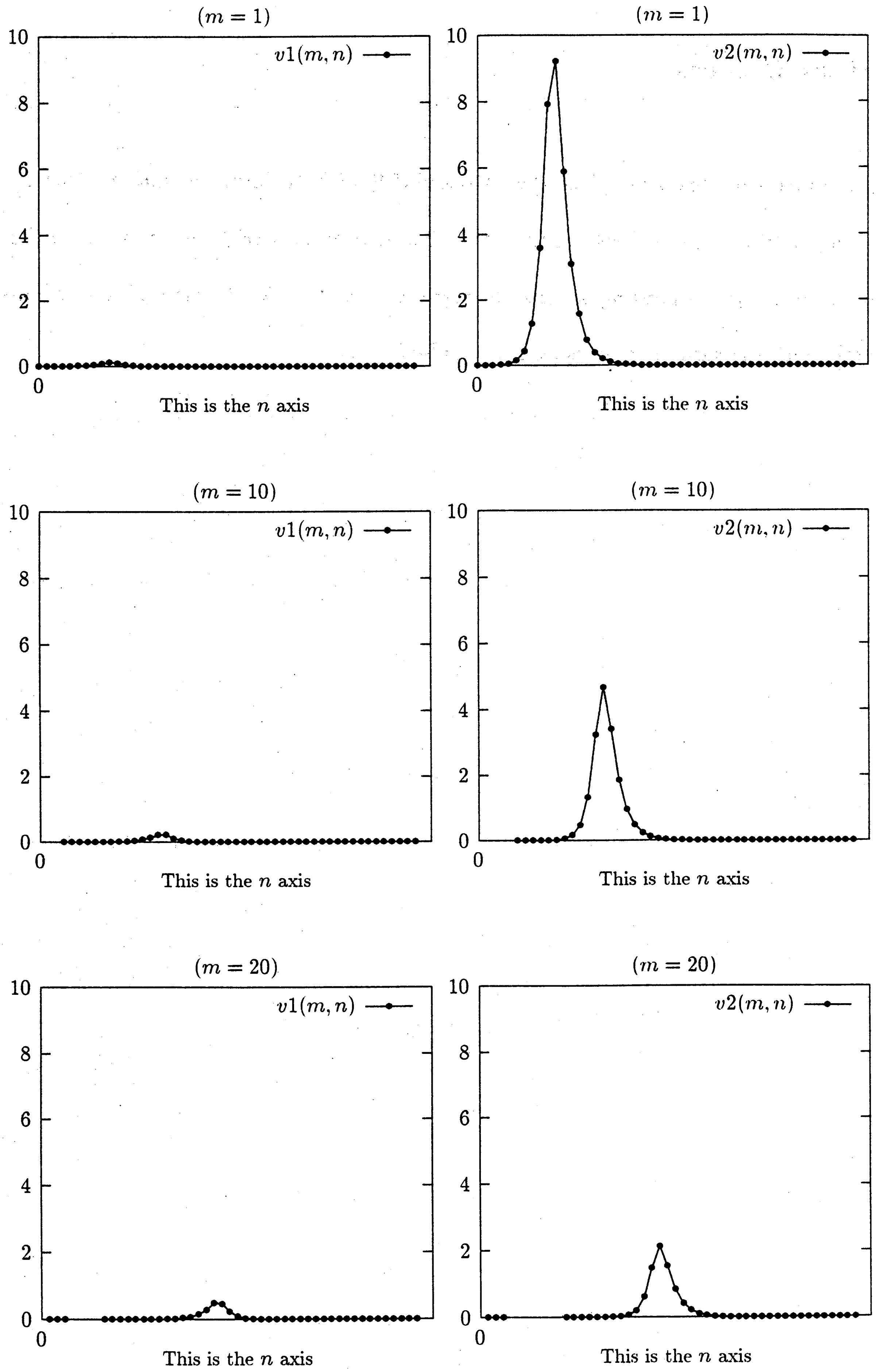
(m = 1)
v1(m, n)
(m = 1)
v2(m, n)
(m = 10)
v1(m, n)
(m = 10)
v2(m, n)
(m = 20)
v1(m, n)
(m = 20)
v2(m, n)
10
8
6
4
2
0
0
This is the n axis

Fig. 1

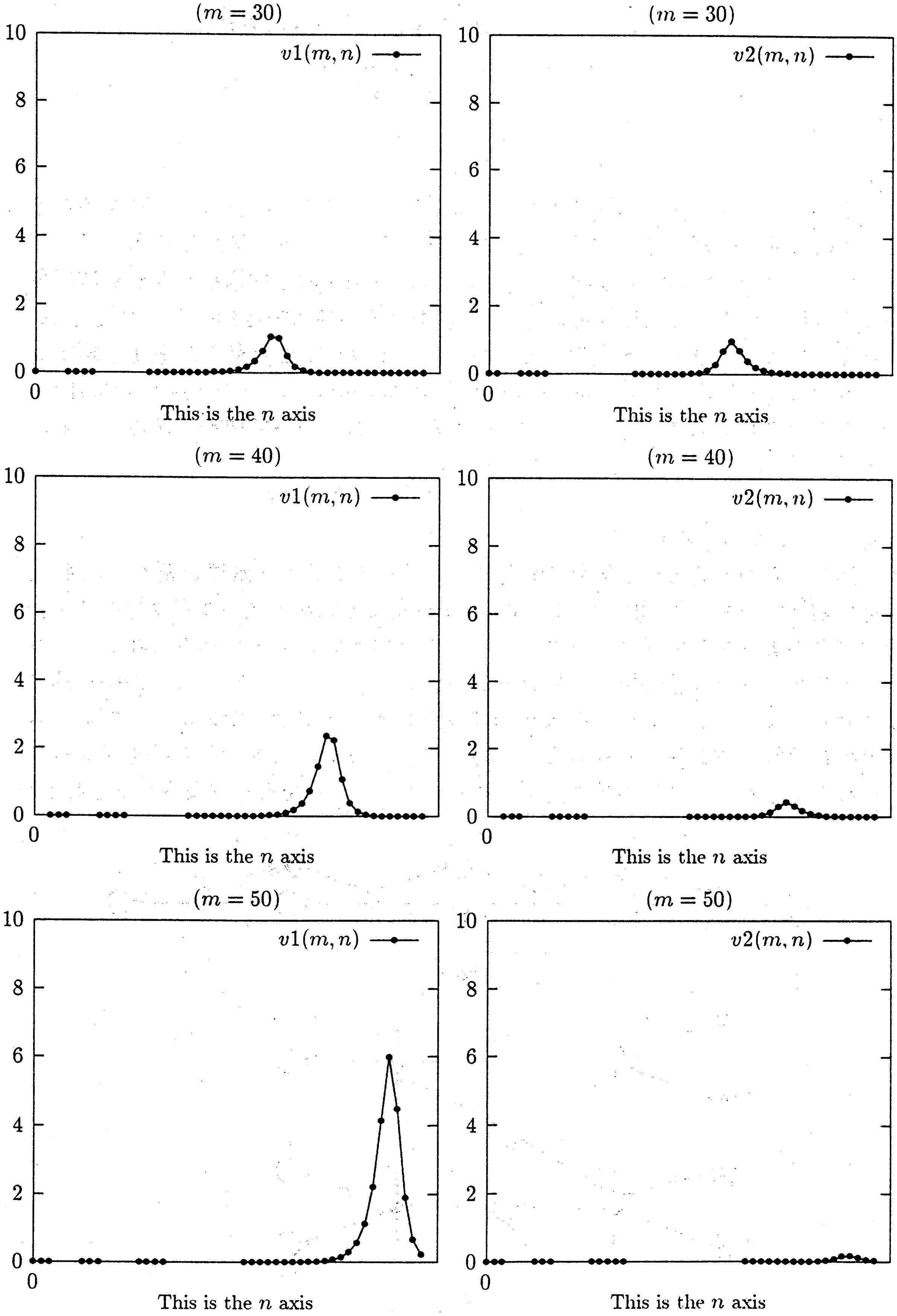

Fig. 1 continued