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Strong Normalization of Pure GL_w-$\lambda\mu$-Terms

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Abstract

The notion of GL_w-$\lambda\mu$-terms is introduced, in terms of Parigot's $\lambda\mu$-terms with proper restrictions, as proof terms of classical substructural logics without contraction rules. GL_w-$\lambda\mu$-terms are obtained as a natural extension of BCK-λ-terms. The main theorem says that any pure GL_w-$\lambda\mu$-term is strongly normalizable. We observe that some GL_w-$\lambda\mu$-terms are not stratified, namely, a certain GL_w-$\lambda\mu$-term is not well-typed. There exist some variants of $\lambda\mu$-calculus, for instance, Parigot[Pari93-1], Ong[Ong96] and Fujita[Fuji95-1]. The stratification property of pure GL_w-$\lambda\mu$-terms is lost for any formulation of them. However, our strong normalization proof of pure GL_w-$\lambda\mu$-terms is available for all of them.

1 Introduction

The $\lambda\mu$-calculus is originally introduced by M.Parigot[Pari92] for giving a computational meaning to classical proofs via the Curry-Howard isomorphism [How80]. In terms of $\lambda\mu$-calculus with proper restrictions, we introduce a notion of GL_w-$\lambda\mu$-terms as a proof of classical substructural logics without contraction rules. In other words, well-typed GL_w-$\lambda\mu$-terms denote proofs of classical substructural logics without contraction rules, and vice versa. The notion of GL_w-$\lambda\mu$-terms is obtained by a natural extension of BCK-λ-terms.

One can prove by a simple induction on the length of terms that all BCK-λ-terms are stratified and strongly normalizable. However, in the presence of μ-reductions, the length of terms may not decrease under μ-reductions, for instance, $(\mu\alpha.\alpha x)N \triangleright \mu\alpha.\alpha(xN)$. Moreover, the application of μ-reductions can make the number of redexes increase, for example, $(\mu\alpha.\alpha(\mu\beta.M))N_1N_2 \triangleright (\mu\alpha.\alpha((\mu\beta.M)N_1))N_2$.

A strong normalization property is usually proved by using the reducibility candidate, which is defined by induction on the structure of types. There are some variants of $\lambda\mu$-calculus, for instance, Parigot[Pari93-1], Ong[Ong96], Fujita[Fuji95-1]. We observe that in all the formulations of them a certain GL_w-$\lambda\mu$-term is not stratified, namely, some term does not have a type. However, in this paper we prove that every pure GL_w-$\lambda\mu$-term is strongly normalizable in the three formulations.

2 GL_w-$\lambda\mu$-Terms for Proofs of Classical Substructural Logics without Contractions

From the view point that the right structural rules in classical logics can be simulated by the μ-operator in the $\lambda\mu$-calculus, we adopt the $\lambda\mu$-calculus. However, minor modifications of the system and proper restrictions on proof terms are used in this paper. In order to manage multiple-consequence, the concept of names was introduced by Parigot, for which μ-variables (greek letters) were used. Here, giving names is dealt with as a special form of application, which might make no distinction between λ-variables and μ-variables. This kind of treatment also appears in λ_Δ-calculus [RS94]. However, in classical systems every occurrence of the name α is to be moved to the left side position by the substitution $[\alpha := \lambda k.\alpha k]$. Hence, we can harmlessly make the restriction that names take only the left side position of applications. A second modification is that in appearance, inference rules have one consequence, as in NJ. This avoids the possibility that closed terms in our usual sense might contain free μ-variables which are names of $\perp$. Quite similar modifications also appear in [Ong96].

The syntax of the $\lambda\mu$-term M is defined by λ-variables x and μ-variables α:
$M ::= x \mid MM \mid \lambda x.M \mid \alpha M \mid \mu\alpha.M$.
The term of the form αM is called a named term whose name is α. The set of λ-free variables and λ-bound variables in M are usually defined, and respectively denoted by $\lambda FV(M)$ and $\lambda BV(M)$. The set of μ-free variables and μ-bound variables in M are also naturally defined, and are denoted by $\mu FV(M)$ and $\mu BV(M)$, respectively. If $\lambda FV(M) = \phi$, then we call M λ-closed. If $\mu FV(M) = \phi$, then we call M μ-closed. When M is λ-closed and μ-closed, we call M closed.

We consider the following reduction rules. We implicitly use α-conversion.
β-reduction rules: contract $(\lambda x.M)M_1$ to $M[x := M_1]$.
μ-reduction rules (structural reduction rules): contract $(\mu\alpha.M)M_1$ to $(\mu\alpha.M)[\alpha \Leftarrow M_1]$ where

$$\begin{aligned}
x[\alpha \Leftarrow M_1] &= x;\\
(\lambda x.M)[\alpha \Leftarrow M_1] &= \lambda x.M[\alpha \Leftarrow M_1];\\
(MM')[\alpha \Leftarrow M_1] &= M[\alpha \Leftarrow M_1]M'[\alpha \Leftarrow M_1];\\
(\mu\beta.M)[\alpha \Leftarrow M_1] &= \mu\beta.M[\alpha \Leftarrow M_1];\\
(\alpha M)[\alpha \Leftarrow M_1] &= \alpha(M[\alpha \Leftarrow M_1]M_1);\\
(\beta M)[\alpha \Leftarrow M_1] &= \beta M[\alpha \Leftarrow M_1] \text{ if } \beta \not\equiv \alpha.
\end{aligned}$$

The one step reduction relation $\triangleright$ is inductively defined as follows:

$$(\lambda x.M)N \triangleright M[x := N] \qquad (\mu\alpha.M)N \triangleright \mu\alpha.M[\alpha \Leftarrow N]$$

$$\frac{M \triangleright N}{\lambda x.M \triangleright \lambda x.N} \qquad \frac{M \triangleright N}{\mu\alpha.M \triangleright \mu\alpha.N}$$

$$\frac{M \triangleright N}{LM \triangleright LN} \qquad \frac{M \triangleright N}{\alpha M \triangleright \alpha N} \qquad \frac{M \triangleright N}{MR \triangleright NR}$$

We have two kinds of types, types indexed with λ-variables and negated types indexed with μ-variables. In the following, Γ, called a context, is a set of indexed types with λ-variables and $\neg\Delta$ is a set of negation types indexed with μ-variables where distinct types never have the same index. The set of type assignment rules $(TA_{\lambda\mu})$ is defined as follows together with the rule that infers $\Gamma, \neg\Delta \vdash x : A$ from $x{:}A \in \Gamma$.

$$\frac{\Gamma, x{:}A_1, \neg\Delta \vdash M : A_2}{\Gamma, \neg\Delta \vdash \lambda x.M : A_1 \to A_2} (\to I) \qquad \frac{\Gamma_1, \neg\Delta_1 \vdash M_1 : A_1 \to A_2 \quad \Gamma_2, \neg\Delta_2 \vdash M_2 : A_1}{\Gamma_1, \Gamma_2, \neg\Delta_1, \neg\Delta_2 \vdash M_1 M_2 : A_2} (\to E)$$

$$\frac{\Gamma, \neg\Delta, \alpha{:}\neg A \vdash M : \bot}{\Gamma, \neg\Delta \vdash \mu\alpha.M : A} (\bot E) \qquad \frac{\alpha{:}\neg A \in \neg\Delta \quad \Gamma, \neg\Delta \vdash M : A}{\Gamma, \neg\Delta \vdash \alpha M : \bot} (\bot I)$$

The first two rules are called logical rules and the latter two are called naming rules. When there is a $TA_{\lambda\mu}$ deduction of a statement $\Gamma, \neg\Delta \vdash M : A$, we say M is stratified.

Let Γ be $\{x_1 : A_1, \cdots, x_m : A_m\}$ and $\neg\Delta$ be $\{\alpha_1 : \neg A_1, \cdots, \alpha_n : \neg A_n\}$, then a set of λ-variables $\lambda Subjects(\Gamma)$ is defined by $\{x_1, \cdots, x_m\}$ and a set of μ-variables $\mu Subjects(\neg\Delta)$ is $\{\alpha_1, \cdots, \alpha_n\}$. For a term M and a context Γ, a restricted context $\Gamma \uparrow \lambda FV(M)$ is defined such that
$\{\ \} \uparrow \lambda FV(M) = \{\ \}$;
$(\{x{:}A\} \cup \Gamma) \uparrow \lambda FV(M) = \{x{:}A\} \cup (\Gamma \uparrow \lambda FV(M))$ if $x \in \lambda FV(M)$;
$(\{x{:}A\} \cup \Gamma) \uparrow \lambda FV(M) = \Gamma \uparrow \lambda FV(M)$ if $x \notin \lambda FV(M)$.
For a term M and $\neg\Delta$, $\neg\Delta \uparrow \mu FV(M)$ is similarly defined.

Putting proper restrictions on $\lambda\mu$-terms makes it possible to define the notions of GL_w-$\lambda\mu$-terms which would correspond to proofs of the respective classical substructural logics. We give the definitions below, which is a natural extension of BCK-λ-terms.

Definition 1 (GL_w-$\lambda\mu$-**Terms**)
1. Every λ-variable is a GL_w-$\lambda\mu$-term.
2. If M_1 and M_2 are GL_w-$\lambda\mu$-terms where $\lambda FV(M_1) \cap \lambda FV(M_2) = \phi$ and $\mu FV(M_1) \cap \mu FV(M_2) = \phi$, then so is $M_1 M_2$.
3. If M and N are GL_w-$\lambda\mu$-terms, respectively, then so are $\lambda x.M$ and $\mu\alpha.N$, respectively.
4. If M is a GL_w-$\lambda\mu$-term where $\alpha \notin \mu FV(M)$, then so is αM.

Clause 2 forbids the left and right contraction rules on applications, and right contractions are not allowed by clause 4 in the other cases. By the definition, each λ-free variable and μ-free variable in a GL_w-$\lambda\mu$-term, respectively, appear at most once. Every λ-abstraction and μ-abstraction in a GL_w-$\lambda\mu$-term bind at most one λ-variable and μ-variable, respectively.

We show a closed example with a type, $\lambda y.\mu\alpha.y(\lambda x.\mu\beta.\alpha x) : \neg(A \to B) \to A$.

Similarly to the definition of GL_w-$\lambda\mu$-terms, GL-$\lambda\mu$-terms and GL_c-$\lambda\mu$-terms can be given. Well-typed GL-$\lambda\mu$-terms and GL_c-$\lambda\mu$-terms, respectively are proofs of classical substructural logics without weakening and contractions; and without weakening, respectively (see [Fuji95-2][Fuji]). When no conditions are applied on terms, the terms are exactly $\lambda\mu$-terms.

3 Well-Typed GL_w-$\lambda\mu$-Terms Are Proofs of GL_w

Following [Ono90][Ono93], we define GL by the implicational and negational fragment of Gentzen's LK without the contraction rules or the weakening rules. It is shown that well-typed GL_w-$\lambda\mu$-terms correspond to proofs of GL_w. In other words, following the notion of formulae-as-types [How80], types inhabited by GL_w-$\lambda\mu$-terms are provable in GL_w. We define GL_w as the following sequent calculus system, namely, GL together with the right and left weakening rules.

$$A \Rightarrow A$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}(\neg \Rightarrow) \qquad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}(\Rightarrow \neg)$$

$$\frac{\Gamma_1 \Rightarrow \Delta_1, A_1 \quad A_2, \Gamma_2 \Rightarrow \Delta_2}{A_1 \supset A_2, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}(\supset\Rightarrow) \qquad \frac{A_1, \Gamma \Rightarrow \Delta, A_2}{\Gamma \Rightarrow \Delta, A_1 \supset A_2}(\Rightarrow\supset)$$

$$\frac{\Gamma_1 \Rightarrow \Delta_1, A \quad A, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}(cut)$$

$$\frac{\Gamma_1, B, A, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, A, B, \Gamma_2 \Rightarrow \Delta}(e \Rightarrow) \qquad \frac{\Gamma \Rightarrow \Delta_1, B, A, \Delta_2}{\Gamma \Rightarrow \Delta_1, A, B, \Delta_2}(\Rightarrow e)$$

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}(w \Rightarrow) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}(\Rightarrow w)$$

It is known that the cut elimination theorem (Grišin, Wroński-Krzystek) holds for GL_w [Ono90].

For a sequence Γ, $\tilde{\Gamma}$ is defined as a set of types with distinct λ-variables. For a sequence Δ, $\neg\tilde{\Delta}$ is a set of negated types with distinct μ-variables. They are inductively defined as follows:
$\tilde{nil} = \{ \}$; $\quad (A,\tilde{\ }\Gamma) = \{x\!:\!A\} \cup \tilde{\Gamma}$, and $\neg\tilde{nil} = \{ \}$; $\quad \neg(A,\tilde{\ }\Delta) = \{\alpha\!:\!\neg A\} \cup \neg\tilde{\Delta}$.
Now we prove that GL_w proofs are represented as GL_w-$\lambda\mu$-terms.

Theorem 1 (GL_w Proofs as GL_w-$\lambda\mu$-Terms)
If $\Gamma \Rightarrow \Delta$ is provable in GL_w, then there exists a GL_w-$\lambda\mu$-term M such that $\tilde{\Gamma}, \neg\tilde{\Delta} \vdash M : \bot$ is derivable in $TA_{\lambda\mu}$.

Proof. We prove this by induction on the number of sequents contained in the derivation of GL_w. We show some of the cases in a step-by-step case analysis on the last rule.

Case 1-1. $(\supset\Rightarrow)$:
By the induction hypotheses, for GL_w-$\lambda\mu$-terms M_1 and M_2, we have $\tilde{\Gamma}_1, \neg\tilde{\Delta}_1, \alpha : \neg A_1 \vdash M_1 : \bot$, and we have $x : A_2, \tilde{\Gamma}_2, \neg\tilde{\Delta}_2 \vdash M_2 : \bot$. Moreover, we can use λ-variables and μ-variables such that $\lambda FV(M_1) \cap \lambda FV(M_2) = \phi$ and $\mu FV(M_1) \cap \mu FV(M_2) = \phi$. Using the first deduction, we obtain a GL_w-$\lambda\mu$-term such that $z : A_1 \to A_2, \tilde{\Gamma}_1, \neg\tilde{\Delta}_1 \vdash z\mu\alpha.M_1 : A_2$ for a fresh variable z. Then we have $z : A_1 \to A_2, \tilde{\Gamma}_1, \tilde{\Gamma}_2, \neg\tilde{\Delta}_1, \neg\tilde{\Delta}_2 \vdash (\lambda x.M_2)(z\mu\alpha.M_1) : \bot$. Since it is satisfied that $\lambda FV(z\mu\alpha.M_1) \cap \lambda FV(\lambda x.M_2) = \phi$ and $\mu FV(z\mu\alpha.M_1) \cap \mu FV(\lambda x.M_2) = \phi$, the term is also a GL_w-$\lambda\mu$-term.

Case 1-2. $(\Rightarrow\supset)$:
By the induction hypothesis, $x\!:\!A_1, \tilde{\Gamma}, \neg\tilde{\Delta}, \alpha\!:\!\neg A_2 \vdash M : \bot$ for some GL_w-$\lambda\mu$-term M. Then using a new variable β, we obtain $\tilde{\Gamma}, \neg\tilde{\Delta}, \beta\!:\!\neg(A_1 \to A_2) \vdash \beta(\lambda x.\mu\alpha.M) : \bot$ where the term is also a GL_w-$\lambda\mu$-term.

Case 2. (cut):
By the induction hypotheses, for some GL_w-$\lambda\mu$-terms M_1 and M_2, we have $\tilde{\Gamma}_1, \neg\tilde{\Delta}_1, \alpha : \neg A \vdash M_1 : \bot$, and we have $x : A, \tilde{\Gamma}_2, \neg\tilde{\Delta}_2 \vdash M_2 : \bot$ Here, we can use λ-variables and μ-variables such that $\lambda FV(M_1) \cap \lambda FV(M_2) = \phi$ and $\mu FV(M_1) \cap \mu FV(M_2) = \phi$.

By the two deductions, $\tilde{\Gamma}_1, \tilde{\Gamma}_2, \neg\tilde{\Delta}_1, \neg\tilde{\Delta}_2 \vdash (\lambda x.M_2)(\mu\alpha.M_1) : \bot$ is obtained. Since $\lambda FV(\lambda x.M_2) \cap \lambda FV(\mu\alpha.M_1) = \phi$, and $\mu FV(\lambda x.M_2) \cap \mu FV(\mu\alpha.M_1) = \phi$, the term is a GL_w-$\lambda\mu$-term.

Case 3-1. $(w \Rightarrow)$:
By the induction hypothesis, for a GL_w-$\lambda\mu$-term M, we have $\tilde{\Gamma}, \neg\tilde{\Delta} \vdash M : \bot$. Here, $\lambda xy.x : \bot \to \neg A$ is a GL_w-$\lambda\mu$-term. Then for a fresh variable z, we obtain $z{:}A, \tilde{\Gamma}, \neg\tilde{\Delta} \vdash (\lambda xy.x)Mz : \bot$ where the term is a GL_w-$\lambda\mu$-term.

Case 3-2. $(\Rightarrow w)$:
By the induction hypothesis, for a GL_w-$\lambda\mu$-term M, there is $\tilde{\Gamma}, \neg\tilde{\Delta} \vdash M : \bot$. Then we have $\tilde{\Gamma}, \neg\tilde{\Delta}, \alpha{:}\neg A \vdash \alpha\mu\beta.M : \bot$ for a fresh μ-variables α and β. Since $\alpha \notin \mu FV(\mu\beta.M)$, the term is a GL_w-$\lambda\mu$-term. □

For a term M and a context Γ, a sequence of formulae $\Gamma \uparrow^* \lambda FV(M)$ is defined as follows:
$\{\,\} \uparrow^* \lambda FV(M) = nil$;
$(\{x{:}A\} \cup \Gamma) \uparrow^* \lambda FV(M) = A, \Gamma \uparrow^* \lambda FV(M)$ if $x \in \lambda FV(M)$;
$(\{x{:}A\} \cup \Gamma) \uparrow^* \lambda FV(M) = \Gamma \uparrow^* \lambda FV(M)$ if $x \notin \lambda FV(M)$.
For a term M and $\neg\Delta$, a sequence of formulae $(\neg\Delta) \uparrow^* \mu FV(M)$ is similarly defined. That is, $\Gamma \uparrow^* \lambda FV(M)$ and $(\neg\Delta) \uparrow^* \mu FV(M)$ are the sequences obtained by omitting λ-variables and μ-variables from the restricted $\Gamma \uparrow \lambda FV(M)$ and $\neg\Delta \uparrow \mu FV(M)$, respectively. For the inverse direction of the above theorem, we prove that GL_w-$\lambda\mu$-terms are represented as GL_w proofs.

Theorem 2 (GL_w-$\lambda\mu$-Terms as GL_w Proofs) Let M be a GL_w-$\lambda\mu$-term. If $\Gamma, \neg\Delta \vdash M : A$ is derivable in $TA_{\lambda\mu}$, then $\Gamma \uparrow^* \lambda FV(M) \Rightarrow (\neg\Delta) \uparrow^* \mu FV(M), A$ is provable in GL_w.

Proof. By induction on the number of types contained in the $TA_{\lambda\mu}$ deductions. We show some of the cases in a step-by-step case analysis on the last rule.

Case 1: $(\to I)$, i.e., M is $\lambda x.M_1$.

Case 1-1. $x \in \lambda FV(M_1)$:
By the induction hypothesis, we have $\Gamma \uparrow^* \lambda FV(M_1), A_1 \Rightarrow (\neg\Delta) \uparrow^* \mu FV(M_1), A_2$ under exchange rules in GL_w from $\Gamma, x : A_1, \neg\Delta \vdash M_1 : A_2$ for a GL_w-$\lambda\mu$-term M_1. Hence, $(\Rightarrow\supset)$ and exchange rules give $\Gamma \uparrow^* \lambda FV(M_1) \Rightarrow A_1 \supset A_2, (\neg\Delta) \uparrow^* \mu FV(M_1)$ in GL_w where $\Gamma \uparrow^* \lambda FV(M_1) = \Gamma \uparrow^* \lambda FV(\lambda x.M_1)$ and $(\neg\Delta) \uparrow^* \mu FV(M_1) = (\neg\Delta) \uparrow^* \mu FV(\lambda x.M_1)$ under exchange rules.

Case 1-2. $x \notin \lambda FV(M_1)$:
By the induction hypothesis, we have $\Gamma \uparrow^* \lambda FV(M_1) \Rightarrow A_2, (\neg\Delta) \uparrow^* \mu FV(M_1)$ in GL_w. Hence, by $(w \Rightarrow)$ and $(\Rightarrow\supset)$ we have $\Gamma \uparrow^* \lambda FV(\lambda x.M_1) \Rightarrow A_1 \supset A_2, (\neg\Delta) \uparrow^* \mu FV(\lambda x.M_1)$ in GL_w.

Case 2: $(\to E)$, i.e., M is M_1M_2 and $\lambda FV(M_1) \cap \lambda FV(M_2) = \phi$ and $\mu FV(M_1) \cap \mu FV(M_2) = \phi$:
By the induction hypotheses, there are proofs of $\Gamma_1 \uparrow^* \lambda FV(M_1) \Rightarrow A_1 \supset A_2, (\neg\Delta_1) \uparrow^* \mu FV(M_1)$ and $\Gamma_2 \uparrow^* \lambda FV(M_2) \Rightarrow A_1, (\neg\Delta_2) \uparrow^* \mu FV(M_2)$ in GL_w. By $(\supset\Rightarrow)$ and the second sequent, we have $A_1 \supset A_2, \Gamma_2 \uparrow^* \lambda FV(M_2) \Rightarrow A_2, (\neg\Delta_2) \uparrow^* \mu FV(M_2)$ and hence $\Gamma_1 \uparrow^* \lambda FV(M_1), \Gamma_2 \uparrow^* \lambda FV(M_2) \Rightarrow A_2, (\neg\Delta_1) \uparrow^* \mu FV(M_1), (\neg\Delta_2) \uparrow^* \mu FV(M_2)$ by (cut). Here $\Gamma_1 \uparrow^* \lambda FV(M_1), \Gamma_2 \uparrow^* \lambda FV(M_2) = (\Gamma_1, \Gamma_2) \uparrow^* \lambda FV(M_1M_2)$ and $(\neg\Delta_1) \uparrow^* \mu FV(M_1), (\neg\Delta_2) \uparrow^* \mu FV(M_2) = (\neg\Delta_1, \neg\Delta_2) \uparrow^* \mu FV(M_1M_2)$ under exchange rules.

Case 3: $(\bot I)$, i.e., M is αM_1 and $\alpha \notin \mu FV(M_1)$:
By $\alpha \notin \mu FV(M_1)$, $(\neg\Delta)\uparrow^* \mu FV(M_1)$ contains no A whose index is α. The induction hypothesis is, therefore, the required result.

Case 4: $(\bot E)$, i.e., M is $\mu\alpha.M_1$.
Case 4-1. $\alpha \in \mu FV(M_1)$:
By $\alpha \in \mu FV(M_1)$, the induction hypothesis is the result.
Case 4-2. $\alpha \notin \mu FV(M_1)$:
Since $\alpha \notin \mu FV(M_1)$, $(\neg\Delta)\uparrow^* \mu FV(M_1)$ contains no A whose index was α. Hence, the application of the right weakening rules leads to $\Gamma \uparrow^* \lambda FV(\mu\alpha.M_1) \Rightarrow (\neg\Delta)\uparrow^* \mu FV(\mu\alpha.M_1), A$ in GL_w from the induction hypothesis $\Gamma\uparrow^* \lambda FV(M_1) \Rightarrow (\neg\Delta)\uparrow^* \mu FV(M_1)$. □

From Theorems 1 and 2, we can identify stratified GL_w-$\lambda\mu$-terms as GL_w proofs. Moreover, with the following theorem [Ono90], the set of types inhabited by closed GL_w-$\lambda\mu$-terms corresponds to the set of theorems of $FL_{ew} + \neg\neg A \supset A$ with respect to the implicational fragment with the multiplicative constant 0.

Theorem 3 [Ono90] Let FL_e be Full Lambek Calculus with exchange rules, i.e., the intuitionistic fragment of GL. Then $GL_w = FL_{ew} + \neg\neg A \supset A$.

Let BCK be the Hilbert-type system (axioms-based logic) consisting of modus ponens and substitution rules together with axioms (B): $(B \supset C) \supset (A \supset B) \supset A \supset C$; and (C): $(A \supset B \supset C) \supset B \supset A \supset C$ and (K): $A \supset B \supset A$. Since the sequent system FL_{ew} contains the right weakening rules, FL_{ew} corresponds to BCK with $0 \supset A$. Here, BCK with $\neg\neg A \supset A$ can derive $0 \supset A$[1] (see also [Bunder93]). Then the statement of Corollary 1 follows.

Corollary 1
$\vdash_{TA_{\lambda\mu}} M : X$ for some closed GL_w-$\lambda\mu$-term M iff X is a theorem in $BCK + \neg\neg A \supset A$.

It is known that every linear λ-term (BCK-λ-term) is stratified by Theorem 4.1 in [Hind89]. However, the corresponding classical terms no longer have this property, i.e., some GL_w-$\lambda\mu$-terms are *not* stratified. For instance, $\mu\alpha.(\lambda xy.yx)$ is a GL_w-$\lambda\mu$-term, but it is not stratified. Strictly speaking, this example shows that the stratification property is lost even for the full intuitionistic fragment of GL_w-$\lambda\mu$-terms, which might correspond to terms as FL_{ew} proofs. However, every pure GL_w-$\lambda\mu$-term is strongly normalizable with respect to $\triangleright_{\beta s}$.

4 Every Pure GL_w-$\lambda\mu$-Term Is Strongly Normalizable

One can prove by induction on the length of terms that every pure BCK-λ-term is strongly normalizable. However, in the presence of μ-reductions, the length of terms may not decrease under $\triangleright$, for instance, $(\mu\alpha.\alpha x)N \triangleright \mu\alpha.\alpha(xN)$. The reduction of μ is logically a kind of permutative reduction rules. We observe some examples in the following. Let a context with a hole be $\mathcal{E}$ such that

[1]The proof of $0 \supset A$ can be given by "BcK" where $c : \neg\neg A \supset A$ and $\neg A \equiv A \supset 0$.

$\mathcal{E}[\,] ::= [\,] \mid \mathcal{E}M \mid M\mathcal{E}$.

Example 1. The number of μ-redexes increases:
$(\mu\alpha.\alpha(\mu\beta.M))N_1N_2 \triangleright (\mu\alpha.\alpha((\mu\beta.M)N_1))N_2$

Example 2. β-reductions introduce μ-redexes:
$(\lambda x.xN)\mu\alpha.M \triangleright (\mu\alpha.M)N$.

Example 3. μ-reductions introduce β-redexes:
$(\mu\alpha.\alpha(\lambda x.M))N \triangleright \mu\alpha.\alpha((\lambda x.M)N)$.

Example 4. The length of named terms increases under μ-reductions:
$(\mu\alpha.\mathcal{E}[\alpha M])N \triangleright \mu\alpha.\mathcal{E}[\alpha(MN)]$.

Example 5. When $\alpha \notin \mu FV(M)$, the length of $(\mu\alpha.M)N$ decreases under μ-reductions:
$(\mu\alpha.M)N \triangleright \mu\alpha.M$.

Example 6. β-reductions make the length of named terms increase:
$(\lambda x.(\mu\alpha.\alpha x))N \triangleright \mu\alpha.\alpha N$.

Example 7. β-reductions make the length of named terms decrease:
$\mu\alpha.\beta((\lambda x.x)M) \triangleright \mu\alpha.\beta M$.

Following the simple observations, for proving the strong normalization property we consider a pair of two induction measures, namely, the first one is the length of a whole term, and the second is the length of a whole term minus the length of a named term.

Definition 2 *The length of a term M denoted by $|M|$ is defined as follows:*
$|x| = 1$;
$|\lambda x.M| = |\mu\alpha.M| = |\alpha M| = 1 + |M|$;
$|MN| = |M| + |N|$.
The sum of length of named terms is defined such that
$\|x\| = 0$;
$\|\lambda x.M\| = \|M\|$;
$\|MN\| = \|M\| + \|N\|$;
$\|\alpha M\| = |M| + \|M\|$;
$\|\mu\alpha.M\| = \|M\|$.
The number of named terms is defined as follows:
$\#x = 0$;
$\#\lambda x.M = \#M$;
$\#MN = \#M + \#N$;
$\#\alpha M = 1 + \#M$;
$\#\mu\alpha.M = \#M$.
$\mathcal{L}(M) = \#M * |M| - \|M\|$.
The degree of a term M is defined by
$d(M) = \langle |M|, \mathcal{L}(M)\rangle$.
The degrees are compared by the lexicographical order.

Lemma 1 $|(\lambda x.M)N| > |M[x := N]|$.

Proof. By induction on the structure of M. □

Lemma 2 $|(\mu\alpha.M)N| \geq |\mu\alpha.M[\alpha \Leftarrow N]|$.

Proof. By induction on the structure of M. □

Lemma 3 *If* $M \rhd N$, *then* $\#M \geq \#N$.

Proof. By induction on the derivation of $M \rhd N$.
Case 1. $(\lambda x.M)N \rhd M[x := N]$:
By induction on M:
Case 1-1-1. $M \equiv x$:
$\#(\lambda x.x)N = \#N$.
Case 1-1-2. $M \equiv y \not\equiv x$:
$\#(\lambda x.y)N = \#N \geq 0 = \#y$.

Case 1-2. $M \equiv M_1M_2$:
Case 1-2-1. $x \in \lambda FV(M_1)$:
$\#(\lambda x.M_1M_2)N = \#M_1 + \#M_2 + \#N = \#((\lambda x.M_1)N) + \#M_3 \geq \#(M_1[x := N]) + \#M_3$.
Case 1-2-2. $x \in \lambda FV(M_2)$:
Same as the above.
Case 1-2-3. $x \notin \lambda FV(M_1M_2)$:
Similarly to the above.

Case 1-3. $M \equiv \lambda y.M_1$:
$\#((\lambda xy.M_1)N) = \#M_1 + \#N = \#((\lambda x.M_1)N) \geq \#(M_1[x := N]) = \#(\lambda y.M_1[x := N])$.

Case 1-4. $M \equiv \alpha M_1$:
$\#((\lambda x.\alpha M_1)N) = 1 + \#M_1 + \#N = 1 + \#((\lambda x.M_1)N) \geq 1 + \#(M_1[x := N]) = \#(\alpha M_1[x := N])$.

Case 1-5. $M \equiv \mu\alpha.M_1$:
$\#((\lambda x.\mu\alpha.M_1)N) = 1 + \#M_1 + \#N = \#((\lambda x.M_1)N) \geq \#(M_1[x := N]) = \#(\mu\alpha.M_1[x := N])$.

Case 2. $(\mu\alpha.M)N \rhd \mu\alpha.M[\alpha \Leftarrow N]$:
By induction on M:
Case 2-1. $M \equiv x$:
$\#((\mu\alpha.x)N) = \#N \geq 0 = \#(\mu\alpha.x)$.

Case 2-2. $M \equiv M_1M_2$:
Case 2-2-1. $\alpha \in \mu FV(M_1)$:
$\#((\mu\alpha.M_1M_2)N) = \#M_1 + \#M_2 + \#N = \#((\mu\alpha.M_1)N) + \#M_2 \geq \#(M_1[\alpha \Leftarrow N]) + \#M_2 = \#(\mu\alpha.M_1[\alpha \Leftarrow N]M_2)$.
Case 2-2-2. $\alpha \in \mu FV(M_2)$:
Same as the above.
Case 2-2-3. $\alpha \notin \mu FV(M_1M_2)$:
Similarly to the above.

Case 2-3. $M \equiv \lambda x.M_1$:

$\#((\mu\alpha.\lambda x.M_1)N) = \#M_1 + \#N = \#((\mu\alpha.M_1)N \geq \#(M_1[\alpha \Leftarrow N]) = \#(\mu\alpha.\lambda x.M_1[\alpha \Leftarrow N])$.

Case 2-4-1. $M \equiv \alpha M_1$ where $\alpha \notin \mu FV(M_1)$:
$\#((\mu\alpha.\alpha M_1)N) = 1 + \#M_1 + \#N1 + \#M_1\#N = \#(\mu\alpha.\alpha(M_1N))$.
Case 2-4-2. $M \equiv \beta M_1$ where $\beta \not\equiv \alpha$:
$\#((\mu\alpha.\beta M_1)N) = 1 + \#M_1 + \#N = 1 + \#((\mu\alpha.M_1)N) \geq 1 + \#(M_1[\alpha \Leftarrow N]) = \#(\mu\alpha.\beta(M_1[\alpha \Leftarrow N]))$.

Case 2-5. $M \equiv \mu\beta.M_1$:
$\#((\mu\alpha\beta.M_1)N) = \#M_1 + \#N = \#((\mu\alpha.M_1)N) \geq \#(M_1[\alpha \Leftarrow N]) = \#(\mu\alpha\beta.(M_1[\alpha \Leftarrow N]))$.

The rest of the cases is similarly confirmed. □

Proposition 1 *If $M \rhd N$, then $d(M) > d(N)$.*

Proof. By induction on the derivation of $M \rhd N$.
Case 1. $(\lambda x.M)N \rhd M[x := N]$:
By $|(\lambda x.M)N| > |M[x := N]|$.

Case 2. $(\mu\alpha.M)N \rhd \mu\alpha.M[\alpha \Leftarrow N]$:
Case 2-1. $\alpha \notin \mu FV(M)$:
$|(\mu\alpha.M)N| > |\mu\alpha.M[\alpha \Leftarrow N]| = |\mu\alpha.M|$.

Case 2-2. $\alpha \in \mu FV(M)$:
Let M be $\mathcal{E}[\alpha M']$ where $\alpha \notin \mu FV(\mathcal{E}[M'])$.

Case 2-2-1. $M' \not\equiv \beta M''$:
$|(\mu\alpha.\mathcal{E}[\alpha M'])N| = |\mu\alpha.\mathcal{E}[\alpha(M'N)]|$.
Let d_1 be
$\mathcal{L}((\mu\alpha.\mathcal{E}[\alpha M'])N) = (\#\mathcal{E}+1+\#M'+\#N)*|(\mu\alpha.\mathcal{E}[\alpha M'])N| - \|\mathcal{E}\| - |M'| - \|M'\| - \|N\|$.
Let d_2 be
$\mathcal{L}(\mu\alpha.\mathcal{E}[\alpha(M'N)]) = (\#\mathcal{E}+1+\#M'+\#N)*|\mu\alpha.\mathcal{E}[\alpha(M'N)]| - \|\mathcal{E}\| - |M'N| - \|M\| - \|N\|$.
Then $d_1 > d_2$.

Case 2-2-2. $M' \equiv \beta M''$ where $\beta \not\equiv \alpha$:
$|(\mu\alpha.\mathcal{E}[\alpha(\beta M'')])N| = |\mu\alpha.\mathcal{E}[\alpha((\beta M'')N)]|$.
Let d_1 be $\mathcal{L}((\mu\alpha.\mathcal{E}[\alpha(\beta M'')])N)$
$= (\#\mathcal{E}+2+\#M''+\#N)*|(\mu\alpha.\mathcal{E}[\alpha(\beta M'')])N| - \|\mathcal{E}\| - (1+|M''|) - |M''| - \|M''\| - \|N\|$.
Let d_2 be $\mathcal{L}(\mu\alpha.\mathcal{E}[\alpha(\beta M'')N])$
$= (\#\mathcal{E}+2+\#M''+\#N)*|\mu\alpha.\mathcal{E}[\alpha(\beta M'')N)]| - \|\mathcal{E}\| - (1+|M''|+|N|) - |M''| - \|M''\| - \|N\|$.
Then $d_1 > d_2$.

In the following we assume that $|M| = |N|$ and that $\mathcal{L}(M) = \#M * |M| - \|M\| > \mathcal{L}(N) = \#N * |N| - \|N\|$.
Case 3. $\lambda x.M \rhd \lambda x.N$ is derived from $M \rhd N$:
$\mathcal{L}(\lambda x.M) = \#M * (1 + |M|) - \|M\|$
$\mathcal{L}(\lambda x.N) = \#N * (1 + |N|) - \|N\|$
$\mathcal{L}(\lambda x.M) - \mathcal{L}(\lambda x.N) = (1 + |M|) * (\#M - \#N) + \|N\| - \|M\|$

$\geq |M| * (\#M - \#N) + \|N\| - \|M\| = \mathcal{L}(M) - \mathcal{L}(N) > 0.$

Case 4. $\mu\alpha.M \rhd \mu\alpha.N$ is derived from $M \rhd N$:
$\mathcal{L}(\mu\alpha.M) = \#M * (1 + |M|) - \|M\|$
$\mathcal{L}(\mu\alpha.N) = \#N * (1 + |N|) - \|N\|$
$\mathcal{L}(\mu\alpha.M) - \mathcal{L}(\mu\alpha.N) = (1 + |M|) * (\#M - \#N) + \|N\| - \|M\|$
$\geq |M| * (\#M - \#N) + \|N\| - \|M\| = \mathcal{L}(M) - \mathcal{L}(N) > 0.$

Case 5. $\alpha M \rhd \alpha N$ is derived from $M \rhd N$:
$\mathcal{L}(\alpha M) = (1 + \#M) * (1 + |M|) - |M| - \|M\|$
$\mathcal{L}(\alpha N) = (1 + \#N) * (1 + |N|) - |N| - \|N\|$
$\mathcal{L}(\alpha M) - \mathcal{L}(\alpha N) = (1 + |M|) * (\#M - \#N) + \|N\| - \|M\|$
$\geq |M| * (\#M - \#N) + \|N\| - \|M\| = \mathcal{L}(M) - \mathcal{L}(N) > 0.$

Case 6. $LM \rhd LN$ is derived from $M \rhd N$:
$\mathcal{L}(LM) = (\#L + \#M) * (|L| + |M|) - \|L\| - \|M\|$
$\mathcal{L}(LN) = (\#L + \#N) * (|L| + |N|) - \|L\| - \|N\|$
$\mathcal{L}(LM) - \mathcal{L}(LN) = (|L| + |M|) * (\#M - \#N) + \|N\| - \|M\|$
$\geq |M| * (\#M - \#N) + \|N\| - \|M\| = \mathcal{L}(M) - \mathcal{L}(N) > 0.$

Case 7. $MR \rhd NR$ is derived from $M \rhd N$:
$\mathcal{L}(MR) = (\#R + \#M) * (|R| + |M|) - \|R\| - \|M\|$
$\mathcal{L}(NR) = (\#R + \#N) * (|R| + |N|) - \|R\| - \|N\|$
$\mathcal{L}(MR) - \mathcal{L}(NR) = (|R| + |M|) * (\#M - \#N) + \|N\| - \|M\|$
$\geq |M| * (\#M - \#N) + \|N\| - \|M\| = \mathcal{L}(M) - \mathcal{L}(N) > 0.$ □

Corollary 2 *Every pure GL_w-$\lambda\mu$-term is strongly normalizable.*

Proof. By the above lemma. □

Remarks 1 *The above proof with a slight modification is also available for GL_w-$\lambda\mu$-terms in the other formulations of Parigot[Pari93-1] and Ong[Ong96].*

In Parigot[Pari92][Pari93-1][Pari93-2], more reduction rules are considered as the following $(S1)$ and $(S2)$. Since after the application of the reduction rules, the length of a term decreases, every pure GL_w-$\lambda\mu$-term is also strongly normalizable.
$(S1)$: contract $\alpha\mu\beta.M$ to $M[\beta := \alpha]$.
$(S2)$: contract $\mu\alpha.\alpha M$ to M if $\alpha \notin \mu FV(M)$.
η-reduction rules: contract $\lambda x.Mx$ to M if $x \notin \lambda FV(M)$.

Corollary 3 *Every pure GL_w-$\lambda\mu$-term is strongly normalizable with respect to β, η, μ, $(S1)$ and $(S2)$ as well.*

Remarks 2 *In contrast to BCK-λ-terms, a certain GL_w-$\lambda\mu$-term is not stratified.*

For instance, $\mu\alpha.\lambda x.x$ is given in our formulation and in Ong[Ong96] (see appendix B).

5 Remarks on Formulations

Depending on the treatment of $\perp$, there are some variants of $\lambda\mu$-calculus, for instance, Parigot[Pari93-1], Ong[Ong96] and Fujita[Fuji95-1]. In the style of Parigot (see appendix A), the proof term, $\lambda y.\mu\alpha.[\delta_1]y(\lambda x.\mu\delta_2.[\alpha]x)$, of type $\neg\neg A \to A$ becomes a GL_w-$\lambda\mu$-term in our sense, where we do not have to use the weakening rules to prove it. When we consider the term as a GL_w-$\lambda\mu$-term, the principal type of the term, $\neg(A \to B) \to A$, is provable in GL_w. Moreover, this proof term should be closed in our usual sense, but it does contain a free name.

If one takes the formulation in the style of Parigot to consider proofs of substructural logics, then one has to fix ones attention to the treatment of names of $\perp$.

We use a special name δ for the constant type $\perp$. Another definition of μ-free variables is given below. Here, the occurrence of δ is not counted as a free variable.

Definition 3 (μ-free variables)
$\mu FV'(x) = \phi$;
$\mu FV'(\lambda x.M) = \mu FV'(M)$;
$\mu FV'(M_1 M_2) = \mu FV'(M_1) \cup \mu FV'(M_2)$;
$\mu FV'(\mu\alpha.[\delta]M) = (\mu FV'(M))/\{\alpha\}$,
$\mu FV'(\mu\alpha.[\beta]M) = (\mu FV'(M) \cup \{\beta\})/\{\alpha\}$ *where* $\beta \not\equiv \delta$.

We redefine the notion of GL_w-$\lambda\mu$-terms in this formulation.

Definition 4 (GL_w-$\lambda\mu$-terms in the style of Parigot)
1. Every λ-variable is a GL_w-$\lambda\mu$-term.
2. If M_1 and M_2 are GL_w-$\lambda\mu$-terms where $\lambda FV(M_1) \cap \lambda FV(M_2) = \phi$ and $\mu FV'(M_1) \cap \mu FV'(M_2) = \phi$, then so is $M_1 M_2$.
3. If M is a GL_w-$\lambda\mu$-term, then so is $\lambda x.M$.
4. If M is a GL_w-$\lambda\mu$-term where $\beta \notin \mu FV'(M)$, then so is $\mu\alpha.[\beta]M$.

Proposition 2 (Well-typed GL_w-$\lambda\mu$-terms are proofs of GL_w)
(1) If we have $\Gamma \Rightarrow \Delta$ in GL_w, then there is a GL_w-$\lambda\mu$-term M such that $\Gamma \vdash M : \perp; \Delta$.
(2) For a GL_w-$\lambda\mu$-term M if we have $\Gamma \vdash M : A; \Delta$, then $\Gamma \Rightarrow A, \Delta$ in GL_w.

Proposition 3 *Every pure GL_w-$\lambda\mu$-term is strongly normalizable.*

We can assume that $(\lambda x.xN)\mu\delta.[\alpha]M$ is a GL_w-$\lambda\mu$-term. Even in this formulation, this term is not stratified.

To define classical proof terms corresponding to BCK-λ-terms, we may consider yet another formulation based on Felleisen's λ_c with $\mathcal{C} : \neg\neg A \to A$ [FFKD86][Grif90]. However, $\mathcal{C}(\lambda xy.zxy)$ is not stratified.

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6 Appendix

A $\lambda\mu$-calculus of Parigot

Proof Terms:
$M ::= x \mid \lambda x.M \mid MM \mid \mu\alpha.[\alpha]M$

Inference Rules:
If $\Gamma(x) = A$, then $\Gamma \vdash x : A; \Delta$.

$$\frac{\Gamma, x : A_1 \vdash M : A_2; \Delta}{\Gamma \vdash \lambda x.M : A_1 \to A_2; \Delta}\ (\to I) \qquad \frac{\Gamma_1 \vdash M_1 : A_1 \to A_2; \Delta_1 \quad \Gamma_2 \vdash M_2 : A_1; \Delta_2}{\Gamma_1, \Gamma_2 \vdash M_1 M_2 : A_2; \Delta_1, \Delta_2}\ (\to E)$$

$$\frac{\Gamma \vdash M : B; \Delta}{\Gamma \vdash \mu\alpha.[\beta]M : A; (\Delta, B^\beta)/A^\alpha}\ (\mu)$$

B $\lambda\mu$-calculus à la Ong

Proof Terms:
$M ::= x \mid \lambda x.M \mid MM \mid \mu\alpha.M \mid [\alpha]M$

Inference Rules:
If $\Gamma(x) = A$, then $\Gamma; \Delta \vdash x : A$.

$$\frac{\Gamma, x : A_1; \Delta \vdash M : A_2}{\Gamma; \Delta \vdash \lambda x.M : A_1 \to A_2}\ (\to I) \qquad \frac{\Gamma_1; \Delta_1 \vdash M_1 : A_1 \to A_2 \quad \Gamma_2; \Delta_2 \vdash M_2 : A_1}{\Gamma_1, \Gamma_2; \Delta_1, \Delta_2 \vdash M_1 M_2 : A_2}\ (\to E)$$

$$\frac{\Gamma; \Delta \vdash M : B}{\Gamma; \Delta, B^\beta \vdash [\beta]M : \bot}\ (\mu 1) \qquad \frac{\Gamma; \Delta, A^\alpha \vdash M : \bot}{\Gamma; \Delta \vdash \mu\alpha.M : A}\ (\mu 2)$$