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Arithmetization of another formulation of a subsystem of Kaneko-Nagashima's GL

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Abstract

We propose a system G of game logic related to Kaneko-Nagashima's GL_ω. Our aim is to make the system more constructive than GL_ω. Though G is an infinitary system, formulae and sequents are finitary. We define a Gödel numbering of formulae, sequents and derivations, and we consider some problems concerning undecidable sentences.

1 The Language and the Rules of the System G

Terms, formulae and sequents of the semiformal deductive system G are defined in this section. Derivations (proof figures) are defined in a later section. G has an infinitary inference rule ($\to$ C); all other elements of G are finitary.

Symbols.
Free variables: $a_0, a_1, \ldots$.
Bound variables: $x_0, x_1, \ldots$.
Logical symbols: $\neg$, $\supset$, $\wedge$, $\vee$, $\forall$, $\exists$.
Modal (epistemic) symbols: K_1, K_2, C.
Predicate symbols: $=$.
Function symbols: 0, S, $+$, $\times$.

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Auxiliary symbols: (,), $\rightarrow$.

Remark. Though other functions, predicates may be allowed without difficulty, we confine ourself to this language for the sake of notational simplicity.

Terms, formulae, cedents and sequents are defined as usual. $S(t)$, the successor of t, is abbreviated as t'.

$(A \supset B) \wedge (B \supset A)$ is abbreviated as $A \sim B$.

$\exists x\, F(x) \wedge \forall y \forall z\, [F(y) \wedge F(z) \supset y = z]$ is abbreviated as $\exists! x\, F(x)$.

Sequents are defined as usual.

For any formula A, we define $K_{i,k}A$ ($i = 1, 2$; $k \in \mathbb{N}$) inductively as follows:

$$\mathrm{K}_{i,0}A \text{ is } A, \qquad \mathrm{K}_{i,k+1}A \text{ is } \mathrm{K}_i\mathrm{K}_{j,k}A \text{ where } i \neq j.$$

For any formula A, we define N_kA ($k \in \mathbb{N}$) as follows:

$$N_0A \text{ is } A, \quad N_{2k+1}A \text{ is } \mathrm{K}_{1,k+1}A, \quad N_{2k+2}A \text{ is } \mathrm{K}_{2,k+1}A$$

Schemata for Initial Sequents:

$$\begin{array}{rcl}
A & \longrightarrow & A \\
\forall x\, \mathrm{K}_1(F(x)) & \longrightarrow & \mathrm{K}_1(\forall x\, F(x)) \\
\forall x\, \mathrm{K}_2(F(x)) & \longrightarrow & \mathrm{K}_2(\forall x\, F(x)) \\
\forall x\, \mathrm{C}(F(x)) & \longrightarrow & \mathrm{C}(\forall x\, F(x)) \\
 & \longrightarrow & t = t \\
s = t, F(s) & \longrightarrow & F(t) \\
t' = 0 & \longrightarrow & \\
s' = t' & \longrightarrow & s = t \\
 & \longrightarrow & t + 0 = t \\
 & \longrightarrow & (t + s)' = t + s' \\
 & \longrightarrow & t \times 0 = 0 \\
 & \longrightarrow & t \times s' = t \times s + t
\end{array}$$

Inference Rules:

$$\frac{\Gamma \longrightarrow \Theta}{A, \Gamma \longrightarrow \Theta}\ (\mathrm{t} \rightarrow) \qquad \frac{\Gamma \longrightarrow \Theta}{\Gamma \longrightarrow \Theta, A}\ (\rightarrow \mathrm{t})$$

$$\frac{A, A, \Gamma \longrightarrow \Theta}{A, \Gamma \longrightarrow \Theta}\ (\mathrm{c} \rightarrow) \qquad \frac{\Gamma \longrightarrow \Theta, A, A}{\Gamma \longrightarrow \Theta, A}\ (\rightarrow \mathrm{c})$$

$$\frac{\Gamma, A, B, \Delta \longrightarrow \Theta}{\Gamma, B, A, \Delta \longrightarrow \Theta}\ (\mathrm{i} \rightarrow) \qquad \frac{\Gamma \longrightarrow \Theta, A, B, \Lambda}{\Gamma \longrightarrow \Theta, B, A, \Lambda}\ (\rightarrow \mathrm{i})$$

$$\frac{\Gamma \longrightarrow \Theta, A \quad A, \Delta \longrightarrow \Lambda}{\Gamma, \Delta \longrightarrow \Theta, \Lambda}\ (\mathrm{cut})$$

$$\frac{\Gamma \longrightarrow \Theta, A}{\neg A, \Gamma \longrightarrow \Theta}\ (\neg \rightarrow) \qquad \frac{A, \Gamma \longrightarrow \Theta}{\Gamma \longrightarrow \Theta, \neg A}\ (\rightarrow \neg)$$

$$\frac{A, \Gamma \longrightarrow \Theta}{A \wedge B, \Gamma \longrightarrow \Theta}\ (\wedge \rightarrow 1) \qquad \frac{\Gamma \longrightarrow \Theta, A \quad \Gamma \longrightarrow \Theta, B}{\Gamma \longrightarrow \Theta, A \wedge B}\ (\rightarrow \wedge)$$

$$\frac{B, \Gamma \longrightarrow \Theta}{A \wedge B, \Gamma \longrightarrow \Theta}\ (\wedge \rightarrow 2)$$

$$\frac{A, \Gamma \longrightarrow \Theta \quad B, \Gamma \longrightarrow \Theta}{A \vee B, \Gamma \longrightarrow \Theta}\ (\vee \rightarrow) \qquad \frac{\Gamma \longrightarrow \Theta, A}{\Gamma \longrightarrow \Theta, A \vee B}\ (\rightarrow \vee 1)$$

$$\frac{\Gamma \longrightarrow \Theta, B}{\Gamma \longrightarrow \Theta, A \vee B}\ (\rightarrow \vee 2)$$

$$\frac{\Gamma \longrightarrow \Theta, A \quad B, \Delta \longrightarrow \Lambda}{A \supset B, \Gamma, \Delta \longrightarrow \Theta, \Lambda}\ (\supset \rightarrow) \qquad \frac{A, \Gamma \longrightarrow \Theta, B}{\Gamma \longrightarrow \Theta, A \supset B}\ (\rightarrow \supset)$$

$$\frac{F(t), \Gamma \longrightarrow \Theta}{\forall x\, F(x), \Gamma \longrightarrow \Theta}\ (\forall \rightarrow) \qquad \frac{\Gamma \longrightarrow \Theta, F(a)}{\Gamma \longrightarrow \Theta, \forall x\, F(x)}\ (\rightarrow \forall)^{(1)}$$

$$\frac{F(a), \Gamma \longrightarrow \Theta}{\exists x\, F(x), \Gamma \longrightarrow \Theta}\ (\exists \rightarrow)^{(1)} \qquad \frac{\Gamma \longrightarrow \Theta, F(t)}{\Gamma \longrightarrow \Theta, \exists x\, F(x)}\ (\rightarrow \exists)$$

$$\frac{\Gamma, \mathrm{K}(\Delta) \longrightarrow \Theta}{\mathrm{K}(\Gamma, \Delta) \longrightarrow \mathrm{K}\Theta}\ (\mathrm{K} \rightarrow \mathrm{K})^{(3)}$$

$$\frac{N_k A, \Gamma \longrightarrow \Theta}{\mathrm{C}(A), \Gamma \longrightarrow \Theta}\ (\mathrm{C} \rightarrow)\ (k \in \mathbb{N}) \qquad \frac{\{\Gamma \longrightarrow \Theta, N_k A \mid k \in \mathbb{N}\}}{\Gamma \longrightarrow \Theta, \mathrm{C}(A)}\ (\rightarrow \mathrm{C})^{(2)}$$

$$\frac{F(a), \Gamma \longrightarrow \Theta, F(a')}{F(0), \Gamma \longrightarrow \Theta, F(t)}\ (\mathrm{MI})^{(1)}$$

(1) *Restriction on variable*: The free variable designated by a, the *eigenvariable*, must not occur in the lower sequent.
(2) *Restriction will be stated later*.
(3) K is either K_1 or K_2. Θ consists of at most one formula.

2 Derivations and the Coding

In this section we define *derivations* and the *coding* of derivations simultaneously. Let $(F_0, F_1, F_2, \ldots)$ be an effective enumeration of all primitive recursive functions[1]. First we introduce some total recursive functions and total recursive predicates needed for Gödel numbering.

$\langle a_0, a_1, \ldots, a_k \rangle$ denotes the sequence number $p_0{}^{a_0{}'} \cdot p_1{}^{a_1{}'} \cdot \cdots \cdot p_k{}^{a_k{}'}$ where $p_0 = 2$, $p_1 = 3$, $p_2 = 5$, ... is the series of prime numbers. Let Seqnum(a) be the number

[1] Primitive recursiveness is not essential. For some other subrecursive classes, the argument almost parallels.

theoretic predicate denoting that a is a sequence number. Definition is

$$\mathrm{Seqnum}(a) \sim a > 0 \land \exists k{<}a\, \forall i{<}a\, [p_i | a \sim i < k].$$

We define $[a]_i = (\mu x{<}a\, \neg(p_i^{x'} | a)) \dot{-} 1$ and $\mathrm{lh}(a) = \sum_{i<a} \mathrm{sg}([a]_i)$. If $a = \langle a_0, a_1, \ldots, a_k \rangle$ then $\mathrm{lh}(a) = k'$ and $[a]_i = a_i$ for any $i < \mathrm{lh}(a)$.
Note. $[a]_i = (a)_i \dot{-} 1$.

For any two sequence numbers $a = \langle a_0, \ldots, a_k \rangle$ and $b = \langle b_0, \ldots, b_l \rangle$, let

$$a * b = \langle a_0, \ldots, a_k, b_0, \ldots, b_l \rangle.$$

We assign Gödel numbers to the symbols and the names of inference rules. The Gödel number of any symbol # is denoted as $\ulcorner \# \urcorner$ and similarly for the names of inference rules.

Successive odd numbers (≥ 3) are assigned to the symbols and the names of inference rules: 0, S, $+$, $\times$, $=$, $\neg$, $\supset$, $\land$, $\lor$, $\forall$, $\exists$, K_1, K_2, C, $\longrightarrow$, (t $\to$), ($\to$ t), (c $\to$), ($\to$ c), (i $\to$), ($\to$ i), (cut), ($\neg \to$), ($\to \neg$), ($\land \to$ 1), ($\land \to$ 2), ($\to \land$), ($\lor \to$), ($\to \lor$1), ($\to \lor$2), ($\supset \to$), ($\to \supset$), ($\forall \to$), ($\to \forall$), ($\exists \to$), ($\to \exists$), (K $\to$ K), (C $\to$), ($\to$ C), (MI), a_k ($k = 0, 1, \ldots$), x_k ($k = 0, 1, \ldots$). For example, $\ulcorner + \urcorner = 7$ and $\ulcorner(\mathrm{cut})\urcorner = 45$.

Gödel numbers of terms and formulae are defined as usual. The Gödel number of a formal expression E is denoted as $\ulcorner E \urcorner$. The Gödel number of a sequent

$$A_1, A_2, \ldots, A_k \longrightarrow B_1, B_2, \ldots, B_m$$

is

$$\langle \ulcorner \longrightarrow \urcorner, \langle \ulcorner A_1 \urcorner, \ulcorner A_2 \urcorner, \ldots, \ulcorner A_k \urcorner \rangle, \langle \ulcorner B_m \urcorner, \ldots, \ulcorner B_2 \urcorner, \ulcorner B_1 \urcorner \rangle \rangle.$$

We omit "the Gödel number of" if no confusions are likely to occur. For instance, we say "a is a formula" instead of "a is the Gödel number of a formula".

Let Formula(a) be a number theoretic predicate meaning that a is a formula. Now we define

$$\begin{aligned} N(\ulcorner A \urcorner, k) &= \ulcorner N_k(A) \urcorner, \\ \mathrm{Cedent}(a) &\sim \mathrm{Seqnum}(a) \land \forall i{<}\mathrm{lh}(a)\, \mathrm{Formula}([a]_i), \\ \mathrm{Sequent}(a) &\sim \mathrm{lh}(a) = 3 \land \\ &\quad \land [a]_0 = \ulcorner \longrightarrow \urcorner \land \mathrm{Cedent}([a]_1) \land \mathrm{Cedent}([a]_2). \end{aligned}$$

Let InitialSequent(a) be a number theoretic predicate meaning that a is an initial sequent.

Let $\mathrm{Infer}_1(j, b, a_1)$ or $\mathrm{Infer}_2(j, b, a_1, a_2)$ be the number theoretic predicate meaning that

$$\frac{a_1}{b}\ (j) \qquad \text{or} \qquad \frac{a_1 \quad a_2}{b}\ (j)$$

is an instance of a one-premise or two-premise inference rule respectively.

Definition 1 *We define derivation and its Gödel number simultaneously by induction.*
(1) *If $\mathcal{S}$ is an initial sequent, then $\mathcal{S}$ is a derivation of $\mathcal{S}$ with the Gödel number* $\langle 0, \ulcorner \mathcal{S} \urcorner \rangle$.
(2) *If $\mathcal{H}_1$ is a derivation of a sequent $\mathcal{S}_1$ and*

$$\frac{\mathcal{S}_1}{\mathcal{S}}\ (J)$$

is an instance of a one-premise inference rule, then

$$\frac{\mathcal{H}_1}{\mathcal{S}}\ (J)$$

is a derivation of $\mathcal{S}$ with the Gödel number $\langle \ulcorner (J) \urcorner, \ulcorner \mathcal{S} \urcorner, \ulcorner \mathcal{H}_1 \urcorner \rangle$.
(3) *If $\mathcal{H}_1$ is a derivation of a sequent $\mathcal{S}_1$, $\mathcal{H}_2$ is a derivation of a sequent $\mathcal{S}_2$ and*

$$\frac{\mathcal{S}_1 \quad \mathcal{S}_2}{\mathcal{S}}\ (J)$$

is an instance of a two-premise inference rule, then

$$\frac{\mathcal{H}_1 \quad \mathcal{H}_2}{\mathcal{S}}\ (J)$$

is a derivation of $\mathcal{S}$ with the Gödel number $\langle \ulcorner (J) \urcorner, \ulcorner \mathcal{S} \urcorner, \ulcorner \mathcal{H}_1 \urcorner, \ulcorner \mathcal{H}_2 \urcorner \rangle$.
(4) *If $\mathcal{H}_k$ is a derivation of a sequent $\mathcal{S}_k$ for each $k \in \mathbb{N}$ and if*

$$\frac{\mathcal{S}_0 \quad \mathcal{S}_1 \quad \cdots}{\mathcal{S}}\ (\rightarrow \mathrm{C})$$

is an instance of the rule $(\rightarrow \mathrm{C})$, and if $\ulcorner \mathcal{H}_k \urcorner$ is a primitive recursive function $F_e(k)$ of k, then

$$\frac{\mathcal{H}_0 \quad \mathcal{H}_1 \quad \cdots}{\mathcal{S}}\ (\rightarrow \mathrm{C})$$

is a derivation of $\mathcal{S}$ with the Gödel number $\langle \ulcorner (\rightarrow \mathrm{C}) \urcorner, \ulcorner \mathcal{S} \urcorner, e \rangle$. □

Lemma 1 *The nonmodal fragment* G_0 *of* G *is the first order arithmetic.* □

Theorem 2 G *is conservative over* G_0 □

Proof (outline). Let $\mathcal{H}$ be a derivation of a nonmodal sequent $\mathcal{S}$. Delete all modal symbols K_1, K_2, C from $\mathcal{H}$. For every occurrences of $(\rightarrow \mathrm{C})$ in $\mathcal{H}$, delete all premises but the leftmost one. □

Corollary 3 *Any undecidable sentence in* G_0 *is undecidable in* G. □

Problem 4 *What is the relation between* G *and Kaneko-Nagashima's* GL_ω*? Is primitive recursively restricted* GL_ω *conservative over* G*?*

Problem 5 *Construct a semantics for* G.

3 Undecidability

Let prov(a, b) be a number theoretic predicate denoting "a is a derivation of a sequent b". This predicate is inductively defined as follows:

$$\begin{aligned}
&\mathrm{prov}(a,b)\\
&\quad\sim\ [a=\langle 0,b\rangle \wedge \mathrm{InitialSequent}(b)] \vee\\
&\qquad \vee(\exists j,u,x<a)[a=\langle j,b,u\rangle \wedge \mathrm{Infer}_1(j,b,x)\wedge \mathrm{prov}(u,x)]\vee\\
&\qquad \vee(\exists j,u,v,x,y<a)[a=\langle j,b,u,v\rangle \wedge\\
&\qquad\qquad \wedge\ \mathrm{Infer}_2(j,b,x,y)\wedge \mathrm{prov}(u,x)\wedge \mathrm{prov}(v,y)]\vee\\
&\qquad \vee(\exists e,x,u,v<a)\big[a=\langle \ulcorner(\rightarrow \mathrm{C})\urcorner, b, e\rangle \wedge b=\langle \ulcorner\rightarrow\urcorner, u, v\rangle \wedge\\
&\qquad\quad \wedge\mathrm{Cedent}(u)\wedge \mathrm{Cedent}(v)\wedge \mathrm{Formula}(x)\wedge \mathrm{lh}(v)>0\wedge\\
&\qquad\quad \wedge[v]_0=\langle \ulcorner \mathrm{C}\urcorner, x\rangle\wedge\\
&\qquad\quad \wedge\forall k\,\big(\mathrm{prov}(F_e(k),[F_e(k)]_1)\wedge [[[F_e(k)]_1]_2]_0=N(x,k)\big)\big]
\end{aligned}$$

Conjecture 6 *The predicate* prov(a, b) *is* Π_1. □

Let $\mathrm{prov_F}(a, b)$ be a number theoretic predicate denoting "a is a derivation of a formula b":

$$\mathrm{prov_F}(a,b) \sim \mathrm{Formula}(b) \wedge \mathrm{prov}\,(a, \langle \ulcorner\rightarrow\urcorner, \langle\rangle, \langle b\rangle\rangle)\,.$$

Conjecture 7 *The predicate* $\mathrm{prov_F}(a, b)$ *is* Π_1. □

Problem 8 *Is the predicate* $\mathrm{prov_F}(a, b)$ *proper* Π_1*?* □

Problem 9 *Is the predicate* $\mathrm{prov_F}(a, b)$ *numeralwise expressible in* G*?* □

Theorem 10 *If* G *is* ω*-consistent and if* $\mathrm{prov_F}$ *is* Π_1*, an undecidable sentence can be constructed from* $\mathrm{prov_F}$. □

Proof. Case 1: The predicate $\mathrm{prov_F}$ is numeralwise expressible. The argument is similar to Gödel's. Let $\mathbf{P}(u, v)$ be a formula numeralwise expressing $\mathrm{prov_F}$. By diagonalization lemma, there exists a sentence A satisfying

$$\vdash A \sim \neg\exists x\, \mathbf{P}(x, \overline{\ulcorner A\urcorner}).$$

(i) Proof of $\nvdash A$. If there is a derivation $\mathcal{H}$ of A, then $\mathrm{prov_F}(\ulcorner\mathcal{H}\urcorner, \ulcorner A\urcorner)$, hence $\vdash \mathbf{P}(\overline{\ulcorner\mathcal{H}\urcorner}, \overline{\ulcorner A\urcorner})$, hence $\vdash \exists x\, \mathbf{P}(x, \overline{\ulcorner A\urcorner})$.

On the other hand, $\vdash A$ implies $\vdash \neg\exists x\, \mathbf{P}(x, \overline{\ulcorner A\urcorner})$. This contradicts the consistency of G.

(ii) Proof of $\nvdash \neg A$. The result $\nvdash A$ implies that $\mathrm{prov_F}(m, \ulcorner A\urcorner)$ for no m. By numeralwise expressibility, $\vdash \neg\mathbf{P}(\overline{m}, \overline{\ulcorner A\urcorner})$ for all m. Since G is ω-consistent, $\nvdash \neg\forall x\, \neg\mathbf{P}(x, \overline{\ulcorner A\urcorner})$, i. e. $\nvdash \exists x\, \mathbf{P}(x, \overline{\ulcorner A\urcorner})$. Hence $\nvdash \neg A$ by the definition of A.

Case 2: The predicate prov_F is not numeralwise expressible. Because prov_F is Π_1, there exists a total recursive predicate R such that

$$\text{prov}_F(a,b) \sim \forall x\, R(a,b,x).$$

Since R is numeralwise expressible, there exists a formula $\mathbf{R}(u,v,w)$ numeralwise expressing R. Since prov_F is not numeralwise expressed by $\forall x\, \mathbf{R}(u,v,x)$, there exists a and b satisfying

either $\quad \text{prov}_F(a,b) \quad \text{and} \quad \nvdash \forall x\, \mathbf{R}(\bar{a},\bar{b},x)$

or $\quad \neg\text{prov}_F(a,b) \quad \text{and} \quad \nvdash \neg\forall x\, \mathbf{R}(\bar{a},\bar{b},x).$

If the latter holds, there exists a c such that $\neg R(a,b,c)$. Therefore $\vdash \neg\mathbf{R}(\bar{a},\bar{b},\bar{c})$ by numeralwise expressibility of R, hence $\vdash \neg\forall x\, \mathbf{R}(\bar{a},\bar{b},x)$, a contradiction. Therefore we have

$$\text{prov}_F(a,b) \quad \text{and} \quad \nvdash \forall x\, \mathbf{R}(\bar{a},\bar{b},x)$$

for some a and b. Hence $R(a,b,c)$ for every c, hence $\vdash \mathbf{R}(\bar{a},\bar{b},\bar{c})$ for every c. By the ω-consistency of G, this implies

$$\nvdash \neg\forall x\, \mathbf{R}(\bar{a},\bar{b},x).$$

□

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