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Decision Making in Partially Interactive Games I: Game Theoretic Developments

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Abstract

From the *ex ante* point of view, we consider decision making in partially interactive games. In a partially interactive game, players may not be required to predict all the others' decisions such as in Prisoner's Dilemma. In some games, some players can ignore some other players' decision making but need to predict decisions of some different ones. In this case, the common knowledge of the entire structure of a game may be unnecessary, but some different form of the knowledge of the payoff functions of some players as well as of their behavioral postulates may be required. We develop some generalizations of Nash equilibrium and dominant strategies in Part I, and then investigate epistemic aspects of decision making in games in the game logic framework in Parts II and III. One new concept, an interaction structure, plays an essential role in both developments. We give a full characterization of the knowledge structure necessary for decision making in the games whose Nash equilibria satisfies a certain weakening of interchangeability. This characterization tells when the solution necessarily involves the common knowledge or not.

In Part I, we develop game theoretical notions, and in Parts II and III, we will investigate the knowledge structure required for decision making in games in the game logic framework.

1. Introduction

1.1. Problems

We consider decision making in a game from the *ex ante* deductive point of view, i.e., individual decision making before the actual play of the game. There have been quite a few arguments on such decision making as well as on necessary epistemic conditions for it. Here, we consider two possibly contradictory argument. The one is that for the players to be able to make decisions, the game structure such as payoff functions should be common knowledge among the players and resulting decisions are Nash equilibrium strategies. This is explicitly formulated in the game logic framework in Kaneko-Nagashima [6] and Kaneko [3]. The other is that a dominant strategy equilibrium such as in Prisoner's Dilemma (Table 1.1) does not require the common knowledge assumption but it suffices for each player to know only his own payoff function. These arguments are mutually exclusive, but it seems reasonable to interpret them as applied to different games. Nevertheless, there is no general framework to discuss the problems of which games need such a common knowledge assumption, which can avoid it and what are different in the consequences of the above arguments applied to the same games. The purpose of this paper is to develop such a general framework.

In fact, there is a *great spectrum* of games between the classes of games to which the above two arguments can be directly applied. In this subsection, we start with considering the dominant strategy argument in Prisoner's Dilemma, and then go to a slightly more complex game. In these considerations, one aspect would emerge: the *behavioral postulate* for each individual player may be required to be known to some other players. Then we will consider a game for which the Nash equilibrium argument is needed. Finally, we will consider a 4-person game for which both dominant strategy and Nash equilibrium arguments are involved. By doing so, we would like to convey two different points: there is a great spectrum of games between the classes of games to which the above two clear-cut arguments are applied, and players' knowledge on the behavioral postulates of some others are involved in quite entangled manners for their decision making.

Example 1.1. The following are 2-person games with two pure strategies for each player, both of which have the same, unique Nash equilibrium $(\mathbf{s}_{12}, \mathbf{s}_{22})$. Table 1.2 is obtained from 1.1 by changing just player 2's payoff 6 to 2. Table 1.2 has a dominant strategy, $\mathbf{s}_{12}$, for player 1, but no dominant strategy for 2.

	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$
$\mathbf{s}_{11}$	(5,5)	(1,6)
$\mathbf{s}_{12}$	(6,1)	(3,3)*

Prisoner's Dilemma

Table 1.1

	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$
$\mathbf{s}_{11}$	(5,5)	(1,2)
$\mathbf{s}_{12}$	(6,1)	(3,3)*

Table 1.2

In Prisoner's Dilemma, each player i thinks about his own payoff function and finds that $\mathbf{s}_{i2}$ is the dominant strategy (is optimal regardless of whatever the other chooses) for him. Then he *can* ignore the other player's choice. Here no interactions are involved.

In the game of Table 1.2, $\mathbf{s}_{12}$ is a dominant strategy for 1 but not $\mathbf{s}_{22}$ for 2. For this game, the dominant strategy argument is not applied to player 2, but is modified as follows: Player 2 predicts that player 1 chooses $\mathbf{s}_{12}$ anyway since it is a dominant strategy for 1, and under this *prediction*, he chooses strategy $\mathbf{s}_{22}$. If player 2 applies the dominant strategy argument to player 1, then 2 may think that 1 does ignore 2' choice. Here the interaction is one-direction. This is an argument motivating the notion of dominance-solvability (cf., Moulin [8]).

Notice that the above argument for the game of Table 1.2 assumes that player 2 knows (or believes) that player 1 follows the dominant strategy argument as his behavioral postulate. In fact, player 2's behavioral postulate is based upon (includes) his knowledge of 1's behavior postulate. Here we assume that player 2 knows his own postulate, too. Then it would be parallel to assume that player 1 himself also knows his own behavioral postulate (but may ignore 2's).

Prisoner's Dilemma is too simple to be cautious of the problem of the knowledge of their behavioral postulates, since no player needs to think about the other's decision and to be conscious of himself. Nevertheless, if player 1 is assumed to be conscious of his own behavior, he himself knows his behavioral postulate. Hence the same problem is involved in Prisoner's Dilemma, though it may be ignored. In this paper, we assume that each player is at least conscious of his own behavioral postulate.

When a game has no dominant strategies, the above argument is no longer applicable. Instead, we need the Nash equilibrium argument. We can construct this argument in a manner comparable with the above. Let us regard of Table 1.3 as the payoff matrix of a 2-person game, which is obtained from Table 1.1 by adding one strategy for each player and has no dominant strategies. The argument runs as follows:

(1): player 1 chooses a strategy maximizing his payoff under the prediction of (2);

(2): player 2 chooses a strategy maximizing his payoff under the prediction of (1).

The prediction part of (1) requires player 1 to know (believe) 2's behavioral postulate, i.e., (2), and simultaneously the prediction part of (2) requires player 2 to know (1). Each of (1) and (2) requires the corresponding player to know the other. This yields meet an infinite regress of the knowledge (or belief) over the behavioral postulates. Ignoring the knowledge structure of the infinite regress and focussing only on the behavioral consequence, however, the resulting outcome must be Nash equilibrium strategies.

The infinite regress is, in fact, the common knowledge of the behavioral postulates, which was discussed in a nonformalized manner in Johansen [1] and in a formalized manner in Kaneko-Nagashima [6] and Kaneko [3]. It is proved in [6] and [3] that the solution of the infinite regress is the common knowledge of Nash equilibrium if a game is solvable in Nash's [9] sense.

Here it is important to notice a similarity as well as a difference in the dominant strategy and Nash equilibrium arguments. In the both arguments, the knowledge of the behavioral postulates are inseparable from the behavioral postulates themselves, while their knowledge structures take a form of a finite hierarchy in the former and a form of an infinite regress.

In some games, the different aspects of dominant strategy and Nash equilibrium arguments are involved in an entangled manner, which is exemplified here.

Example 1.2. Consider the 4-person game $g = (g_1, .., g_4)$ with strategy space $\Sigma_i = \{\mathbf{s}_{i1}, \mathbf{s}_{i2}, \mathbf{s}_{i3}\}$ for player $i = 1, ..., 4$. Suppose that the payoffs of 1 and 2 depend only upon their own strategies, i.e., g_1 and g_2 are (regarded as) defined on $\Sigma_1 \times \Sigma_2$ which are given as Table 1.3. Then let g_3 and g_4 depend upon the strategy choices of 3 and 4 as well as those of 1 and 2, i.e., they are defined on $\Sigma_1 \times \Sigma_2 \times \Sigma_3 \times \Sigma_4$ which are given as Table 1.3 (with the replacements of players 1, 2 with 3, 4) and Table 1.4: if 1 and 2 play (s_{12}, s_{22}), the payoffs for 3 and 4 are given in Table 1.3, and otherwise, they are given in Table 1.4, which is obtained from 1.3 by permuting the first and second strategies for 3 and 4.

	s_{21}	s_{22}	s_{23}
s_{11}	5, 5	1, 6	3, 0
s_{12}	6, 1	3, 3*	0, 2
s_{13}	0, 3	2, 0	2, 2

Players 1 and 2

3' and 4' payoffs if (s_{12}, s_{22}) are played

Table 1.3

	s_{41}	s_{42}	s_{43}
s_{31}	3, 3	6, 1	0, 2
s_{32}	1, 6	5, 5	3, 0
s_{33}	2, 0	0, 3	2, 2

otherwise

Table 1.4

In this game, we could regard players 1 and 2 as playing simply the 2-person game of Table 1.3 if they ignore the choices of 3 and 4. However, players 3 and 4 need to predict 1's and 2's decisions to maximize their payoffs. In this game, the first argument is applied to the "partial" game of 1 and 2 and to the game of 3 and 4 conditional upon the predictions on the decisions in the game of 1 and 2.

We find a similarity between this game and that of Table 1.2 by regarding $\{1, 2\}$ and $\{3, 4\}$ as players 1 and 2 of Table 1.2. Also, the above Nash equilibrium argument is applied between 1 and 2, and the common knowledge of their payoffs as well as their behavioral postulates are involved. Between 3 and 4, the situation is more complicated: the same NE argument is applied to between 3 and 4, and furthermore, the argument for 1 and 2 itself is commonly known to 3 and 4, since they should predict 1 and 2's decisions.

From the above arguments and examples, we have found the necessity of a new general treatment of the knowledge of players on their behavior postulates and that of a unified treatment of dominant strategies and Nash equilibrium including hybrids such as the game of Example 1.2. In the next subsection, we will give brief descriptions of the developments of our theory of these problems.

1.2. Game Theoretical Developments and Game Logic Considerations

In the above subsection, we argued that for an individual player's decision making, his knowledge of his and/or others' behavioral postulates as well as of payoff functions play important roles, and that there is a great spectrum of games which seem to require different treatments of the knowledge of the behavioral postulates and payoff functions. To encompass such a spectrum as well as the seemingly different arguments, we will take two basic research steps.

The first step is to develop new game theoretical concepts to unify the above different concepts. Then the second step is to discuss the knowledge of the players over their behavioral postulates as well as their payoff functions. The first step will be developed in Part I of this paper. The second will be discussed in Parts II and III in the game logic framework developed in Kaneko-Nagashima [6], [7] and Kaneko [2].

In Part I, first, we introduce the concept a *partial Nash equilibrium*, which is a generalization of both Nash equilibrium and dominant strategy. Then we consider an *interaction structure*, which tells whose decision making, each player should think about. By these generalizations, we can discuss the above examples in a unified way, and will take another substep of associating a graph with an interaction structure. We call this associated graph the *skeleton*, which consists of a partition of players and a partial ordering on the partition. This concept will play crucial roles in Parts I, II and III. In Part I, we give a condition on a game for the skeleton to have a node having multiple players. In Part II, it is proved that the players in one node need to share common knowledge of their behavioral postulates and payoff functions for their decision making. We also prove in Part I that if the skeleton consists of singleton sets, the (generic) game is dominance-solvable. In this case, any form of common knowledge can be avoided, which will be proved in Part II.

The introduction of an interaction structure leads to two types of interchangeability conditions of equilibria, which are given in Section 5. In Part II, the epistemic axiomatization of individual decision making will be given under these conditions. When one of these interchangeability conditions is violated, some coordination in addition to the knowledge of the behavioral postulates and payoff functions would be needed for decision making. This problem will be discussed in Part III.

2. Partial Nash Equilibria and Interaction Structures

2.1. Partial Nash Equilibria

Consider an n-person noncooperative game $g = (g_1, ..., g_n)$. The *player set* is given as $N = \{1, ..., n\}$, and each player $i \in N$ has ℓ_i (pure) strategies $\mathbf{s}_{i1}, ..., \mathbf{s}_{i\ell_i}$. We assume throughout Parts I and II that $\ell_i \geq 2$ for all $i \in N$ and that the players do *not* play mixed strategies. Player i's *strategy space* $\{\mathbf{s}_{i1}, ..., \mathbf{s}_{i\ell_i}\}$ is denoted by Σ_i, and his *payoff function* is a real-valued function g_i on $\Sigma := \Sigma_1 \times \cdots \times \Sigma_n$ for $i \in N$. We call an element $(a_1, ..., a_n) \in \Sigma$ a *strategy profile.*

A strategy profile $a = (a_1, ..., a_n)$ is called a *Nash equilibrium* (*NE*) iff for all $i \in N$,

$$g_i(a) \geq g_i(x_i, a_{-i}) \text{ for all } x_i \in \Sigma_i,$$

where $a_{-i} = (a_1, ..., a_{i-1}, a_{i+1}, ..., a_n)$ and $(b_i, a_{-i}) = (a_1, ..., a_{i-1}, b_i, a_{i+1}, ..., a_n)$.

We now provide a generalization of Nash equilibrium, which is rather an auxiliary concept to be used in later developments in this paper. Let S be a nonempty subset of N, and T a (possibly empty) subset of N with $S \cap T = \emptyset$. We say that $a_S = (a_i)_{i \in S} \in \Sigma_S := \prod_{i \in S} \Sigma_i$ is an *S-partial NE conditional upon*

$b_T \in \Sigma_T$ iff for all $y_{-S-T} \in \Sigma_{-S-T} := \prod_{i \in N-(S \cup T)} \Sigma_i$ and for all $i \in S$,

$$g_i(a_S, b_T, y_{-S-T}) \geq g_i(x_i, a_{S-i}, b_T, y_{-S-T}) \text{ for all } x_i \in \Sigma_i. \tag{2.1}$$

Here a_{S-i} is the vector $(a_j)_{j \in S-\{i\}}$. When T is empty, we call $a_S = (a_i)_{i \in S}$ simply an *S-partial NE.* Note that an N-partial NE is a NE itself, and *vice versa.* Also, d is a NE if and only if d_i is a $\{i\}$-partial NE conditional upon d_{-i} for all $i \in N$. It should be emphasized that this is an auxiliary concept and will be used in later developments.

A partial NE is also a generalization of a dominant strategy. That is, d_i is called a *dominant strategy* iff it is a $\{i\}$-partial NE. We call $d = (d_1, ..., d_n)$ a *dominant strategy equilibrium* iff d_i is a dominant strategy for all $i \in N$. Thus, $d = (d_1, ..., d_n)$ is a dominant strategy equilibrium if and only if d_i is a $\{i\}$-partial NE for all $i \in N$.

In Prisoner's Dilemma, $\mathbf{s}_{i2}$ is a unique $\{i\}$-partial NE for each $i = 1, 2$. In the game of Table 1.2, $\mathbf{s}_{12}$ is a $\{1\}$-partial NE, while $\mathbf{s}_{22}$ is a $\{2\}$-partial NE conditional upon $\mathbf{s}_{12}$. The 2-person game of Table 1.3 has no proper partial NE, while $(\mathbf{s}_{12}, \mathbf{s}_{22})$ is a NE. In the 4-person game of Example 1.2, $(\mathbf{s}_{12}, \mathbf{s}_{22})$ is a $\{1,2\}$-partial NE, and $(\mathbf{s}_{32}, \mathbf{s}_{42})$ is a $\{3,4\}$-partial NE conditional upon $(\mathbf{s}_{12}, \mathbf{s}_{22})$. Thus, our generalization of Nash equilibrium differentiates these games.

2.2. Interaction Structures

The examples of Section 1 suggest that decision making for some players are conditional upon the knowledge or inference on the choice of some other players. To describe this idea, we introduce the concept of an interaction structure.

We say that an n-tuple $(J_1, ..., J_n)$ of subsets of N is an *interaction structure* iff

(Reflexivity): $J_i \ni i$ for all $i = 1, ..., n$;

(Transitivity): $J_i \ni j$ and $J_j \ni k$ imply $J_i \ni k$.

An interaction structure $(J_1, ..., J_n)$ will be used to describe, later, the idea that $J_i - \{i\}$, denoted as $J_i - i$ in the following, represents the set of players whose choices player i *may* need to infer. Following this interpretation, Reflexivity means that he himself is conscious of his own decision, and Transitivity means that if player i needs to infer j's choice and player j needs k's choice, player i needs to infer k's choice. Transitivity is written also as

$$j \in J_i \text{ implies } J_j \subseteq J_i. \tag{2.2}$$

Of course, $\mathcal{N} = (N, ..., N)$ is an interaction structure, which we call the *full* interaction structure.

Let $\mathcal{J} = (J_1, ..., J_n)$ be an interaction structure. We say that a subset J of N is *$\mathcal{J}$-closed* iff $\bigcup_{j \in J} J_j = J$. By Transitivity, each J_i is $\mathcal{J}$-closed, and by Reflexivity of $\mathcal{J}$, so is N. Since our primary objective is to consider individual decision making, we would like to focus on each J_i. However, Example 2.1

given below shows that we may not decentralize our consideration and that the consideration of a $\mathcal{J}$-closed set may be essential.

We consider the following condition for a game g, a $\mathcal{J}$-closed set J and a profile d_J in Σ_J:

$$\text{for each } i \in J, \ d_i \text{ is an } \{i\}\text{-partial NE conditional upon } d_{J_i - i}. \tag{2.3}$$

Profile d_J in (2.3) is an J-partial NE, but not *vice versa.* We say that an interaction structure $\mathcal{J} = (J_1, ..., J_n)$ is *feasible for* J in game g iff there is a profile d_J in Σ_J for which (2.3) holds. When $\mathcal{J} = (J_1, ..., J_n)$ is feasible for N, a strategy profile d in (2.3) is a NE. Conversely, if g has a NE d, the full interaction structure $\mathcal{N} = (N, ..., N)$ is feasible for N.

Prisoner's Dilemma has four feasible interaction structures for N:

$$\mathcal{J}_1 = (\{1\},\{2\}),\ \mathcal{J}_2 = (\{1,2\},\{2\}),\ \mathcal{J}_3 = (\{1\},\{1,2\}),\ \mathcal{J}_4 = (\{1,2\},\{1,2\}).$$

In the game of Table 1.2, neither $\mathcal{J}_1$ nor $\mathcal{J}_2$ is feasible for N, but $\mathcal{J}_3, \mathcal{J}_4$ are feasible N. The 2-person game defined by Table 1.3 has the unique feasible interaction structure $\mathcal{J}_4$ – we will later call such games fully interactive games.

The game of Example 1.2 has only two feasible interaction structures for N:

$$(\{1,2\},\{1,2\},N,N) \text{ and } (N,N,N,N).$$

In this game, players 3 and 4 need to infer the choices of 1 and 2. When player 1 (or 2) wants to infer 3's and 4's decisions, then 2 (or 1) also needs to infer their decision. Therefore we obtain the feasible interaction structure (N, N, N, N) for N. In (N, N, N, N), the inferences of players 1 and 2 on the choices of 3 and 4 could be redundant for 1's and 2's decision making unless either 1 and 2 wants to know 3's and 4's decisions.

In this game, $(\{1,2\},\{1,2\},\{1,2,3\},\{1,2,4\})$ is feasible for $\{1,2\}$, but for neither $\{1,2,3\}$ nor $\{1,2,4\}$.

A partial ordering $\leq$ over the interaction structures is defined by $\mathcal{J} = (J_1, ..., J_n) \leq \mathcal{J}' = (J_1', ..., J_n')$ iff $J_i \subseteq J_i'$ for all $i \in N$. Then we can consider minimal and maximal feasible structures for N (or J) in a game by this ordering $\leq$ if the game has feasible interaction structures for N (or J).

The following are simple observations on feasible interaction structures.

Proposition 2.1.(1). Game g has a dominant strategy equilibrium if and only if $(\{1\}, ..., \{n\})$ is a feasible interaction structure for N in g.

(2): If $\mathcal{J} \leq \mathcal{J}'$ and $\mathcal{J}$ is feasible for N, then $\mathcal{J}'$ is also feasible for N.

(3): g has a NE if and only if $\mathcal{N}$ is feasible for N.

It follows from (1) and (2) that if game g has a dominant strategy equilibrium, then any interaction structure can be feasible for N. Here we allow an interaction structure to include some redundancy in the inferences for decision making. In the present paper, we will not consider the further choice of an interaction structure, since more structures should be developed before such a discussion.

Now we give a game which has a feasible interaction structure for a proper subset J but not for N – some players could make decisions but some not. Also, it implies that we cannot necessarily decentralize our consideration into each J_i.

Example 2.1. Consider the 5-person game $(g_1, .., g_5)$ with $\Sigma_i = \{\mathbf{s}_{i1}, \mathbf{s}_{i2}\}$ for $i \in N$. Let g_5 take always 0 over Σ. The payoff functions g_1 and g_2 over $\Sigma_1 \times \Sigma_2 \times \Sigma_5$ are given as Table 1.1 (Prisoner's Dilemma) if player 5 chooses $\mathbf{s}_{51}$, and as Table 2.1 (Matching Penny) if 5 chooses $\mathbf{s}_{52}$. The payoff functions g_3 and g_4 defined over $\Sigma_3 \times \Sigma_4 \times \Sigma_5$ are given as Table 2.1 if 5 chooses $\mathbf{s}_{51}$, and as Table 1.1 if 5 chooses $\mathbf{s}_{52}$.

	$\mathbf{s}_{41}$	$\mathbf{s}_{42}$
$\mathbf{s}_{31}$	$1, -1$	$-1, 1$
$\mathbf{s}_{32}$	$-1, 1$	$1, -1$

Matching Pennies

Players 1 and 2 : $(\mathbf{s}_{12}, \mathbf{s}_{22})$

Table 2.1

Consider the interaction structure $\mathcal{J} = (\{1,2,5\}, \{1,2,5\}, \{3,4,5\}, \{3,4,5\}, \{5\})$. This $\mathcal{J}$ is feasible for $J_1 = J_2$ with $(\mathbf{s}_{12}, \mathbf{s}_{22}, \mathbf{s}_{51})$, and it is feasible for $J_3 = J_4$ with $(\mathbf{s}_{32}, \mathbf{s}_{42}, \mathbf{s}_{52})$. However, $\mathcal{J}$ is not feasible for N, since there are no $\{1,2\}$-partial NE's conditional upon $\mathbf{s}_{52}$ and no $\{3,4\}$-partial NE's conditional upon $\mathbf{s}_{51}$.

One condition for decentralization is given in the following proposition, and a condition on a game for decentralization will be given in Section 5.

Proposition 2.2. Let J be a $\mathcal{J}$-closed subset of N. Suppose that J has a partition consisting of sets in $\mathcal{J}$. Then in game g, $\mathcal{J}$ is feasible for J_i for each $i \in J$ if and only if $\mathcal{J}$ is feasible for J.

3. Skeletons of Interaction Structures

Graph theoretic considerations of interaction structures are useful for the further developments of our theory. The graph directly associated with an interaction structure $\mathcal{J} = (J_1, ..., J_n)$ is the directed graph $(N, \{(j,i) : j \in J_i\})$, but this is not convenient for our later purposes. Instead, we consider the skeleton of this graph by treating each maximal cycle in $(N, \{(j,i) : j \in J_i\})$ as one point. The skeleton will be used to investigate partial Nash equilibrium and to characterize the knowledge structure defined by $\mathcal{J}$ in Part II.

Let $\mathcal{J} = (J_1, ..., J_n)$ be an interaction structure. We call a subset $\{i_1, ..., i_\ell\}$ of N a *cycle* iff

$$J_{i_t} \ni i_{t+1} \text{ for all } t \text{ (mode } \ell).$$

Note that any singleton set $\{i\}$, i.e., $\ell = 1$, is a cycle. We denote the class of maximal cycles by $\mathcal{C}_{\mathcal{J}}$. Then $\mathcal{C}_{\mathcal{J}}$ is a partition of N.

We introduce a binary relation $\succ$ over $\mathcal{C}_{\mathcal{J}}$ as follows: for any distinct $S, T \in \mathcal{C}_{\mathcal{J}}$,

$$S \succ T \text{ if and only if } J_i \ni j \text{ for some } i \in S \text{ and } j \in T. \tag{3.1}$$

It follows from Transitivity for $\mathcal{J} = (J_1, ..., J_n)$ that $S \succ T$ if and only if $J_i \ni j$ for all $i \in S$ and all $j \in T$. Then it is easy to see that $\succ$ satisfies Transitivity and Anti-Symmetry, which implies the following lemma.

Lemma 3.1 The relation $\succ$ is a partial ordering over $\mathcal{C}_{\mathcal{J}}$.

We call the graph $(\mathcal{C}_{\mathcal{J}}, \succ)$ the *skeleton* of the interaction structure $\mathcal{J}$. Since $\mathcal{C}_{\mathcal{J}}$ is a finite set, we can use inductive arguments over $\mathcal{C}_{\mathcal{J}}$ from the minimal components in $\mathcal{C}_{\mathcal{J}}$ with respect to $\succ$. For inductive arguments as well as graphical representations, it would be useful to introduce the *immediate part* $\succ^I$ of $\succ$ over $\mathcal{C}_{\mathcal{J}}$: for any $S, T \in \mathcal{C}_{\mathcal{J}}, S \succ^I T$ if and only if $S \succ T$ but $S \succ R \succ T$ for no $R \in \mathcal{C}_{\mathcal{J}}$. We also write $S \succeq T$ iff $S \succ T$ or $S = T$.

It follows from the definition of $\mathcal{C}_{\mathcal{J}}$ and Transitivity that

$$J_i = J_j \text{ if } i, j \in S \in \mathcal{C}_{\mathcal{J}}. \tag{3.2}$$

Hence we can denote J_i $(i \in S)$ by $J(S)$. Of course, $J(S)$ is $\mathcal{J}$-closed. Since $J(S) - S$ is $\bigcup_{T \prec^I S} J(T)$, $J(S) - S$ is also $\mathcal{J}$-closed.

In Prisoner's Dilemma, there are four feasible interaction structures for N. The skeletons of these structures are described ($\succ^I$ is denoted by $\leftarrow$ ("is affected by") in the diagrams):

	$\{2\}$	$\{1\}$	
	$\uparrow$	$\uparrow$	
$\{1\}$ $\{2\}$	$\{1\}$	$\{2\}$	$\{1,2\}$
$\mathcal{J}_1 = (\{1\},\{2\})$	$\mathcal{J}_2 = (\{1\},\{1,2\})$	$\mathcal{J}_3 = (\{1,2\},\{2\})$	$\mathcal{J}_4 = (\{1,2\},\{1,2\})$

The game of Example 1.2 has two feasible interaction structures for N whose skeletons are:

$\{3,4\}$	
$\uparrow$	
$\{1,2\}$	$\{1,2,3,4\}$
$(\{1,2\},\{1,2\},N,N)$	(N,N,N,N)
Diagram 3.1	Diagram 3.2

The skeleton of $\mathcal{J} = (\{1,2,5\},\{1,2,5\},\{3,4,5\},\{3,4,5\},\{5\})$ of Example 2.1 is given as Diagrams 3.3:

$$\{1,2\} \quad \nwarrow \quad \{5\} \quad \nearrow \quad \{3,4\}$$

Diagram 3.3.

Thus the skeletons permit much simpler graphical representations of interaction structures.

The following example shows that the skeleton may have a cycle.

Example 3.1. Consider the following 4-person game where each player has two pure strategies. The payoff function g_1 depends upon Σ_1, which is given by Table 3.1, and the payoff functions g_2 and g_3 depending upon $\Sigma_1 \times \Sigma_2$ and $\Sigma_1 \times \Sigma_3$, respectively, are given by Tables 3.2 and 3.3. Finally, g_4 depends upon $\Sigma_2 \times \Sigma_3 \times \Sigma_4$ given by Tables 3.4 and 3.5.

s_{11}	1
s_{12}	0

Table 3.1

	s_{21}	s_{22}
s_{11}	1	0
s_{12}	0	1

Table 3.2

	s_{31}	s_{32}
s_{11}	1	0
s_{12}	0	1

Table 3.3

s_{41}	1
s_{42}	0

2's and 3's strategies : (s_{21}, s_{31})

Table 3.4

s_{41}	0
s_{42}	1

otherwise

Table 3.5

In this game, $(\{1\}, \{1,2\}, \{1,3\}, \{1,2,3,4\})$ is a unique minimal feasible interaction structure for N with $d = (s_{11}, s_{21}, s_{31}, s_{41})$. The skeleton of this structure is described as:

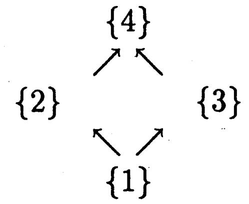

Diagram 3.4

Using the skeleton of an interaction structure, condition (2.3) is written as follows.

Proposition 3.2. Let $\mathcal{J}$ be an interaction structure, $(\mathcal{C}_{\mathcal{J}}, \succ)$ its skeleton and J a $\mathcal{J}$-closed subset. Also, let $d_J \in \Sigma_J$. Then (2.3) for all $i \in J$ if and only if d_T is a T-partial NE conditional upon $d_{J(T)-T}$ for all $T \in \mathcal{C}_{\mathcal{J}}$ with $T \subseteq J$.

Let us apply this proposition to the 4-person game of Example 1.2 with the interaction structure $\mathcal{J} = (\{1,2\}, \{1,2\}, N, N)$. The skeleton of $\mathcal{J}$ is given as Diagram 3.1. Then

(i): (s_{12}, s_{22}) is a $\{1,2\}$-partial NE;

(ii): (s_{32}, s_{42}) is a $\{3,4\}$-partial NE conditional upon (s_{12}, s_{22}).

These should be interpreted as having two sets of statements, instead of two independent statements. One consists of simply (i), which means that players

1 and 2 think that (s_{12}, s_{22}) is a partial NE for them, and the other consists of (i) and (ii). That is, players 3 and 4 predict (i) for 1' and 2' decisions and think about (ii) under this prediction – the conditional of (ii). This interpretation is still implicit. The fully explicit treatment will be given in the game logic framework in Part II.

To avoid multiple statements like (i) and (ii) above, we give a name to them. Let J be a $\mathcal{J}$-closed subset of N. Then we say that $d_J \in \Sigma_J$ is a *$\mathcal{J}$-NE$_J$* iff

$$\begin{array}{c} \text{for all } T \in \mathcal{C}_{\mathcal{J}} \text{ with } T \subseteq J, \\ d_T \text{ is a } T\text{-partial NE conditional upon } d_{J(T)-T}\,. \end{array} \tag{3.3}$$

For the full interaction structure $\mathcal{N} = (N, ..., N)$, d is a $\mathcal{N}$-NE$_N$ if and only if it is simply a NE. In the above 4-person game example with $\mathcal{J} = (\{1,2\}, \{1,2\}, N, N)$, (s_{12}, s_{22}) is a $\mathcal{J}$-NE$_{\{1,2\}}$ and $(s_{12}, s_{22}, s_{32}, s_{42})$ is a $\mathcal{J}$-NE$_N$.

The latter part of Proposition 3.2 is simply that d_S is a $\mathcal{J}$-NE$_S$. Thus we have the following proposition.

Proposition 3.3. For a $\mathcal{J}$-closed subset J of N and a game $g = (g_1, ..., g_n)$, $\mathcal{J}$ is feasible for J if and only if there is a $\mathcal{J}$-NE$_J$ d_J.

It holds that if d is a $\mathcal{J}$-NE$_N$, then d is a NE, but the converse is not necessarily true.

4. Games with and without Mutual Interactions

We say that an interaction structure $\mathcal{J} = (J_1, ..., J_n)$ has *mutual interactions* iff the skeleton $(\mathcal{C}_{\mathcal{J}}, \succ)$ of $\mathcal{J}$ has at least one component consisting at least two players. It will be shown in Part II that a feasible interaction structure has a mutual interaction if and only if the individual decision making involves some common knowledge. Hence it is important to give some conditions for a game to separate interaction structures with from ones without mutual interactions.

4.1. Games with Mutual Interactions

Let g be a game, J a subset of N and $i, j \in J$. We say that player i is *immediately dependent upon* player j in a J-partial NE d_J in g iff there are $y_j \in \Sigma_j$ and $x_i \in \Sigma_i$ such that

$$g_i(d_i, y_j, d_{J-\{i,j\}}, z_{-J}) < g_i(x_i, y_j, d_{J-\{i,j\}}, z_{-J}) \text{ for some } z_{-J} \in \Sigma_{-J}.$$

This means that for player i's utility maximization, he needs to infer j's choice d_j. We say that player i is *dependent upon* j in d_J iff there is a sequence $i_1 = i, i_2, ..., i_k = j$ in J such that i_t is immediately dependent upon i_{t+1} for $t = 1, ..., k-1$ in d_J. We say that i and j are *mutually dependent* iff i is dependent upon j and *vice versa*. The dependence relation is transitive, and the mutual dependence relation is an equivalence relation.

Lemma 4.1. Let $\mathcal{J} = (J_1, ..., J_n)$ be an interaction structure feasible for a $\mathcal{J}$-closed set J with d_J, and $i, j \in J$. If i is dependent upon j, then $j \in J_i$.

Lemma 4.1 implies the following proposition.

Proposition 4.2. Let $(J_1, ..., J_n)$ be an interaction structure feasible for a $\mathcal{J}$-closed set J with d_J in game g.

(1): If players i and j in J are mutually dependent, then $J_i \ni j$ and $J_j \ni i$;

(2): If any two players in $J = N$ are mutually dependent in g, then $(N, ..., N)$ is the only feasible interaction structure for N.

We say that a game g is *fully interactive* iff $(N, ..., N)$ is only a feasible interaction structure for N. It will be shown in Part II that in a fully interactive game, the payoff functions should be common knowledge for each player to have a final decision.

In the following example, each player is immediately dependent upon only one player, but is dependent upon the others. Hence it is a fully interactive game by Proposition 4.2.(2).

4.2. Games without Mutual Interactions

Here we consider a game with a feasible interaction $\mathcal{J} = (J_1, ..., J_n)$ for N where every component of the skeleton $(\mathcal{C}_{\mathcal{J}}, \succ)$ is singleton. It will be shown in Part II that in such a game, the common knowledge of the payoff functions can be avoided for decision making. In this subsection, we look at the status of these games in the literature of game theory.

The following proposition follows a known result (cf., Osborne-Rubinstein [10], p.94), but is proved for completeness. Note that a 2-person game with an NE has a unique minimum feasible interaction structure.

Proposition 4.3. Let g be a 2-person game with two strategies for each player. Assume that the game has a unique NE. Then $J_1 = \{1\}$ or $J_2 = \{2\}$ for the minimal feasible interaction structure (J_1, J_2) for $\{1, 2\}$.

Next we consider the notion of dominance-solvability. We say that for strategies $a_i, b_i \in \Sigma_i$, a_i is *dominated* by b_i iff $g_i(a_i, a_{-i}) \leq g_i(b_i, a_{-i})$ for all $a_{-i} \in \Sigma_{-i}$ and $g_i(a_i, a_{-i}) < g_i(b_i, a_{-i})$ for some $a_{-i} \in \Sigma_{-i}$. A strategy is called a *dominated strategy* iff it is dominated by some other strategy. We say that a game g is *dominance-solvable* iff there is a finite sequence $g^0 = g, g^1, ..., g^m$ of games such that

(1): for $t = 0, ..., m-1$, g^{t+1} is obtained from g^t by eliminating dominated strategies in g^t;

(2): for all $i \in N$, $g_i^m(a_i, a_{-i}) = g_i^m(b_i, a_{-i})$ for any a_i, b_i and a_{-i} in g^m.

Dominance-solvability describes "after finitely many elimination rounds all strategies of any player are equivalent to him but not necessarily to other players" (Moulin [8], p.51). The following proposition states that it is a sufficient condition for dominance-solvable that the skeleton of a feasible interaction structure for a generic game consists of singleton sets.

Proposition 4.4. Assume that for $i = 1, ..., n$, $g_i(a_i, a_{-i}) \neq g_i(b_i, a_{-i})$ for any distinct $a_i, b_i \in \Sigma_i$ and $a_{-i} \in \Sigma_{-i}$. Suppose that g has a feasible interaction structure $\mathcal{J} = (J_1, ..., J_n)$ for N where every component of $\mathcal{C}_{\mathcal{J}}$ is singleton. Then g is dominance-solvable.

5. Conditional Interchangeability and Concatenation

Nash [9] demarcates between the games with interchangeable NE's and the ones with non-interchangeable NE's, where the NE's of a game g are said to be *interchangeable* iff

$$\text{if } a \text{ and } b \text{ are NE's, then } (a_i, b_{-i}) \text{ is also a NE for all } i \in N. \tag{5.1}$$

Interchangeability is an extension of uniqueness and avoids the problem of an individual choice from the multiple NE strategies. If game g has the interchangeable NE's, then each individual player can choose any of his NE strategies, but if not, he needs to worry about the double-cross such as in the game of Battle of the Sexes:

	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$
$\mathbf{s}_{11}$	$(2,1)^*$	$(0,0)$
$\mathbf{s}_{12}$	$(0,0)$	$(1,2)^*$

Battle of the Sexes

Table 5.1

When we introduce an interaction structure $\mathcal{J}$, we may meet another type of interchangeability as well as the type of (5.1). In this section, we give these two types of interchangeabilities. These conditions would play essential roles in the epistemic axiomatization of decision making in Part II where we take explicitly the knowledge structure into account. If at least one of these conditions is violated, some additional coordination, e.g., communication, would be required to have final decisions, which is the subject of Part III.

Let an interaction structure $\mathcal{J} = (J_1, ..., J_n)$, its skeleton $(\mathcal{C}_{\mathcal{J}}, \prec)$ and a $\mathcal{J}$-closed subset J of N be given. The two conditions for J are: for all $T \in \mathcal{C}_{\mathcal{J}}$ with $T \subseteq J$,

(I-1)(Conditional Interchangeability): if $(d_T, d_{J(T)-T})$ and $(c_T, d_{J(T)-T})$ are $\mathcal{J}$-NE$_{J(T)}$, so is $(c_i, d_{T-i}, d_{J(T)-T})$ for all $i \in T$.

(I-2)(Concatenation): if $(d_T, d_{J(T)-T})$ is a $\mathcal{J}$-NE$_{J(T)}$ and if $c_{J(T)-T}$ is a $\mathcal{J}$-NE$_{J(T)-T}$, then $(d_T, c_{J(T)-T})$ is also a $\mathcal{J}$-NE$_{J(T)}$.

Conditional Interchangeability, I-1, is a restriction of (5.1) over T conditional upon $d_{J(T)-T}$, which is based on the same reasoning as that of (5.1). Concatenation I-2 guarantees that the T-part of any $\mathcal{J}$-NE$_{J(T)}$ can be concatenated with any $\mathcal{J}$-NE$_{J(T)-T}$. In the decision making of the players in $J(T) - T$, they ignore decision making of T. When there are multiple $\mathcal{J}$-NE$_{J(T)-T}$, this multiplicity may generates no difficulty for $J(T) - T$, but it may affect T's choices. Condition I-2 rules this possibility out.

These conditions are satisfied by the examples previously given, except Examples 2.1, 3.2 and Battle of the Sexes. In Battle of the Sexes, the full interaction structure $\mathcal{N} = (\{1,2\}, \{1,2\})$ is only feasible. Both $(\mathbf{s}_{11}, \mathbf{s}_{21})$ and $(\mathbf{s}_{12}, \mathbf{s}_{22})$

are $\mathcal{N}$-NE$_N$, and thus, I-1 is violated. In Example 3.2, $(\mathbf{s}_{12}, \mathbf{s}_{22}, \mathbf{s}_{32}, \mathbf{s}_{42})$ is a unique $\mathcal{J}$-NE$_N$ for $\mathcal{J} = (\{1,2\}, \{1,2\}, N, N)$. Hence both I-1 and I-2 are fulfilled. For $\mathcal{N} = (N, ..., N)$, $(\mathbf{s}_{12}, \mathbf{s}_{22}, \mathbf{s}_{32}, \mathbf{s}_{42})$ remains to be a $\mathcal{N}$-NE$_N$ but there is another $\mathcal{N}$-NE$_N$ $(\mathbf{s}_{13}, \mathbf{s}_{23}, \mathbf{s}_{33}, \mathbf{s}_{43})$. Hence I-1 is violated but not I-2.

In Example 2.1 with $\mathcal{J} = (\{1,2,5\}, \{1,2,5\}, \{3,4,5\}, \{3,4,5\}, \{5\})$, whose skeleton is

$$\begin{array}{ccc} \{1,2\} & & \{3,4\} \\ & \nwarrow \ \nearrow & \\ & \{5\} & \end{array}$$

Diagram 5.1

$(\mathbf{s}_{12}, \mathbf{s}_{22}, \mathbf{s}_{51})$ is a unique $\mathcal{J}$-NE$_{J_1}$ and $(\mathbf{s}_{32}, \mathbf{s}_{42}, \mathbf{s}_{52})$ is also a unique $\mathcal{J}$-NE$_{J_3}$. Both $\mathbf{s}_{51}$ and $\mathbf{s}_{52}$ are $\mathcal{J}$-NE$_{J_5}$. However, neither $(\mathbf{s}_{12}, \mathbf{s}_{22}, \mathbf{s}_{52})$ nor $(\mathbf{s}_{32}, \mathbf{s}_{42}, \mathbf{s}_{51})$ is a $\mathcal{J}$-NE$_{J_1}$ or $\mathcal{J}$-NE$_{J_3}$. Thus I-2 is violated. This causes the infeasibility of $\mathcal{J}$ for N.

When the skeleton $(\mathcal{C}_{\mathcal{J}}, \succ)$ consists of singleton sets, I-1 is fulfilled in general since each decision making is made by a single player. Condition I-2 is fulfilled for a generic game, since each player's choice is unique and no concatenation problem occurs.

Proposition 5.1. Let $\mathcal{J}$ be an interaction structure feasible for a $\mathcal{J}$-closed set J in a game g. Suppose that $\{j\} \in \mathcal{C}_{\mathcal{J}}$ for all $j \in J$.

(1): Condition I-1 holds for J.

(2): Assume that for all $i \in J$, $g_i(a_i, a_{-i}) \neq g_i(b_i, a_{-i})$ for any distinct $a_i, b_i \in \Sigma_i$ and $a_{-i} \in \Sigma_{-i}$. Then I-2 holds for J.

Conditions I-1 and I-2 imply the following, rather direct generalization of (5.1), but the converse is not true. Example 5.1 is a counterexample for the converse.

Proposition 5.2. Let $\mathcal{J}$ be an interaction structure, and J a $\mathcal{J}$-closed subset of N. If I-1 and I-2 hold for J in game g, then

(I-1*): if d_J and c_J are $\mathcal{J}$-NE$_J$, then so is (c_i, d_{J-i}) for all $i \in J$.

The following proposition states that I-2 guarantees that we can decentralize our consideration as far as feasibility is concerned.

Proposition 5.3. Let $\mathcal{J}$ be an interaction structure, and J a $\mathcal{J}$-closed subset of N. Suppose that I-2 holds for J. Then if $\mathcal{J}$ is feasible for J_i for all $i \in J$, then $\mathcal{J}$ is feasible also for J.

6. Concluding Remark

We should give one remark on the relationship between the feasibility of an interaction structure $\mathcal{J}$ and conditions I-1 and I-2, since they are not related in this Part I. Feasibility for a $\mathcal{J}$-closed set J is, in fact, a necessary condition for the players to have final decisions. In fact, its point is the negative part: it will be shown in Part II that $\mathcal{J}$ is not feasible for J in game g if and only if the players cannot have final decisions. On the other hand, I-1 and I-2 are sufficient conditions for these players to have final decisions without coordinations in addition to individual thinking on decision making. It will be shown in Part II that if $\mathcal{J}$ is feasible for J in game g and if I-1 and I-2 hold, then they can have final decisions under appropriate knowledge of their payoff functions but without further coordinations. The subject of Part III is the consideration of necessary coordinations when at least one of I-1 and I-2 are violated but feasibility is, of course, assumed.

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