数理解析研究所講究録
1021 巻 1997 年 187-205

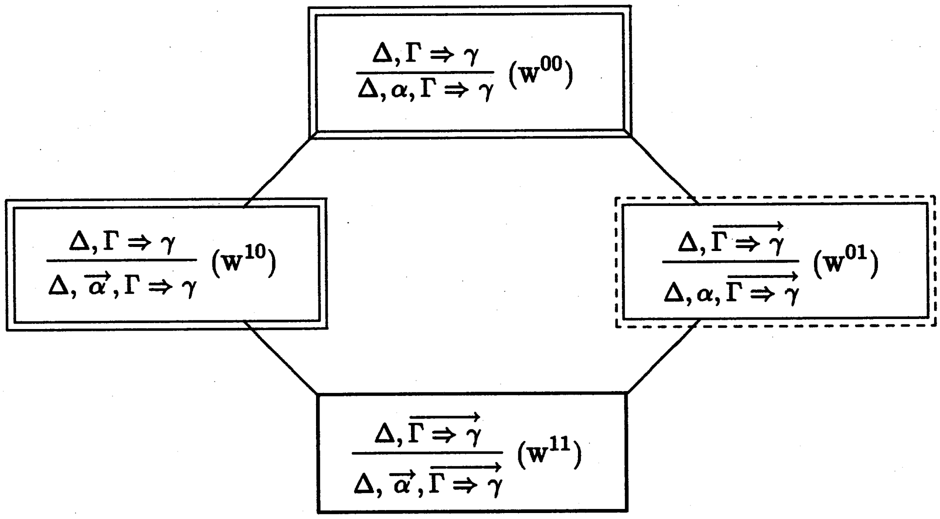
A Study on Substructural Logics with Restricted Exchange Rules, (2)

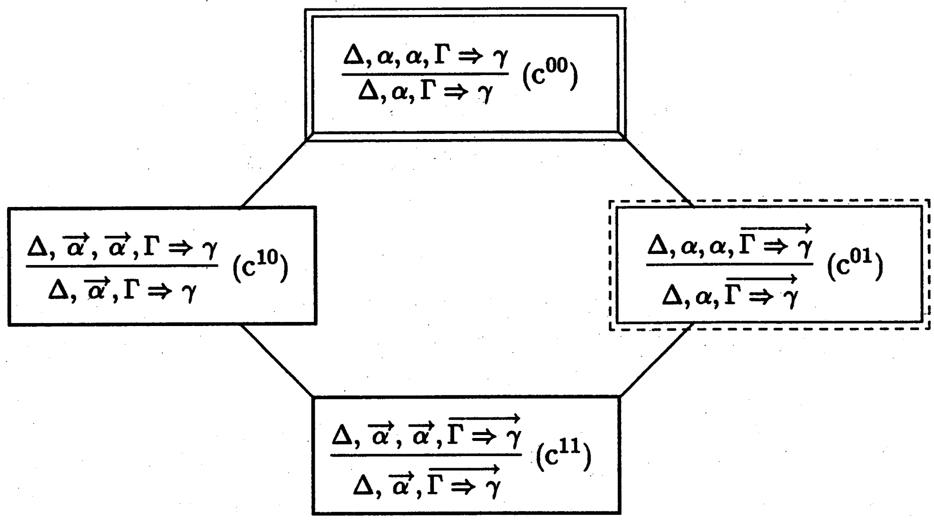
北陸先端科学技術大学院大学 情報科学研究科　鹿島 亮 (Ryo Kashima)
北陸先端科学技術大学院大学 情報科学研究科　上出 哲広 (Norihiro Kamide)

In the former paper [1] we have made investigations on the systems which have restricted exchange rules. In this sequel we introduce restricted weakening rules and restricted contraction rules, and prove the cut-elimination theorems for the systems based on $FL_{\rightarrow}+(e^{*1*})$. These include new cut-elimination results for the well-known relevance logics $E_{\rightarrow}$ and $S4_{\rightarrow}$. The detailed proofs of the cut-elimination and other theorems appear in the authors' research report [2].

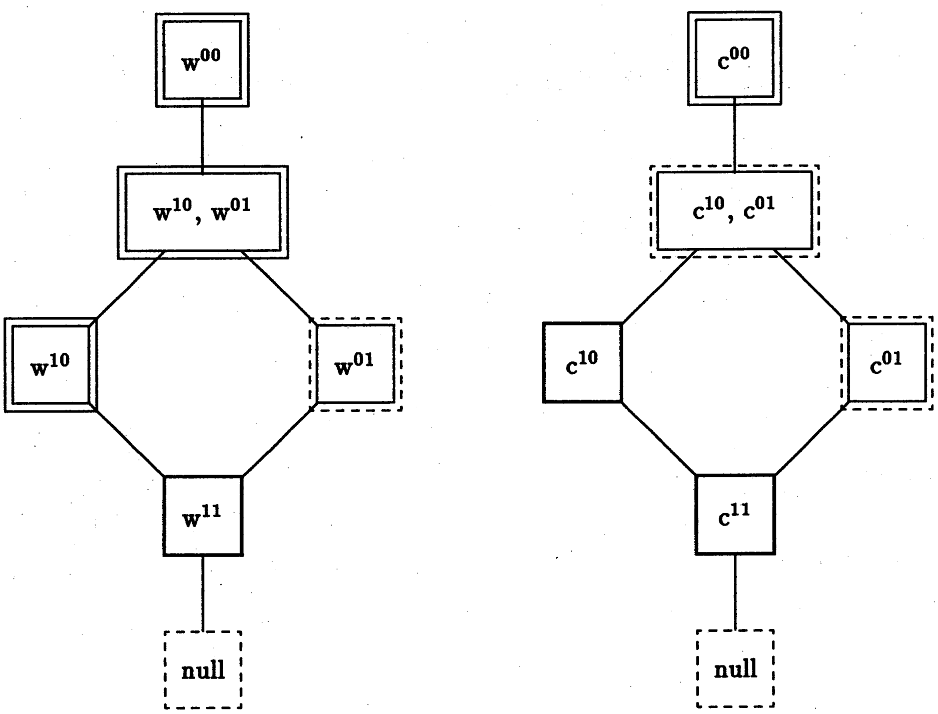
6 Restricted weakening and contraction

We introduce restricted weakening rules and restricted contraction rules as follows. (In those figures, the difference between the lines around (the combinations of) the rules shows equivalence explained below.)





We have six combinations (including the null combinations) of the weakenings and six combinations of the contractions:



It is known that $FL_{\rightarrow}$+(e^{*1*})+(c^{00}) is a system for the relevance logic $E_{\rightarrow}$, and $FL_{\rightarrow}$+(e^{*1*})+(w^{01})+(c^{00}) is a system for the the relevance logic $S4_{\rightarrow}$ (see [2]).

Theorem 6.1 *The rule* (c^{10}) *is derivable in* $FL_{\rightarrow}$+ (e^{111}) + (c^{11}) *and in* $FL_{\rightarrow}$+ (w^{11}) + (c^{11}). *Therefore the rules* (c^{11}) *and* (c^{10}) *are rule-equivalent over* $FL_{\rightarrow}$+ (e^{111}) *and over* $FL_{\rightarrow}$+ (w^{11}); *and if* (e^{111}) *or* (w^{11}) *is derivable in a system L, then* *the two systems L + (c^{11}) and L + (c^{10}) are theorem-equivalent and the two systems L + (c^{01}) and L + (c^{10}) + (c^{01}) are theorem-equivalent.*

Proof Let $\alpha \equiv \alpha_1 \to \alpha_2$. The sequent $\alpha, \alpha_2 \to \alpha_2 \Rightarrow \alpha$ is provable in $FL_{\to}$+ (e^{111}) (by Lemma 3.1 in [1]) and in $FL_{\to}$+ (w^{11}) (by one application of (w^{11}) to an initial sequent). Then the derivability of

$$\frac{\Gamma, \vec{\alpha}, \vec{\alpha} \Rightarrow p}{\Gamma, \vec{\alpha} \Rightarrow p} \ (c^{10})$$

is shown as follows.

$$\dfrac{\begin{matrix} \vdots \\ \Rightarrow \alpha_2 \to \alpha_2 \end{matrix} \qquad \dfrac{\dfrac{\vec{\alpha}, \alpha_2 \to \alpha_2 \Rightarrow \vec{\alpha} \quad \Gamma, \vec{\alpha}, \vec{\alpha} \Rightarrow p}{\Gamma, \vec{\alpha}, \vec{\alpha}, \alpha_2 \to \alpha_2 \Rightarrow p} \ (\text{cut})}{\Gamma, \vec{\alpha}, \alpha_2 \to \alpha_2 \Rightarrow p} \ (c^{11})}{\Gamma, \vec{\alpha} \Rightarrow p.} \ (\text{cut})$$

∎

Theorem 6.2 *Suppose* (e^{100}) *or* (w^{10}) *is derivable in a system L. Then the rule* (e^{000}) *is derivable in L* + (e^{111})*, the rule* (w^{00}) *is derivable in L* + (w^{11})*, and the rule* (c^{00}) *is derivable in L* + (c^{11})*. In other words, the existence of* (e^{100}) *or* (w^{10}) *makes the restrictions ineffective.*

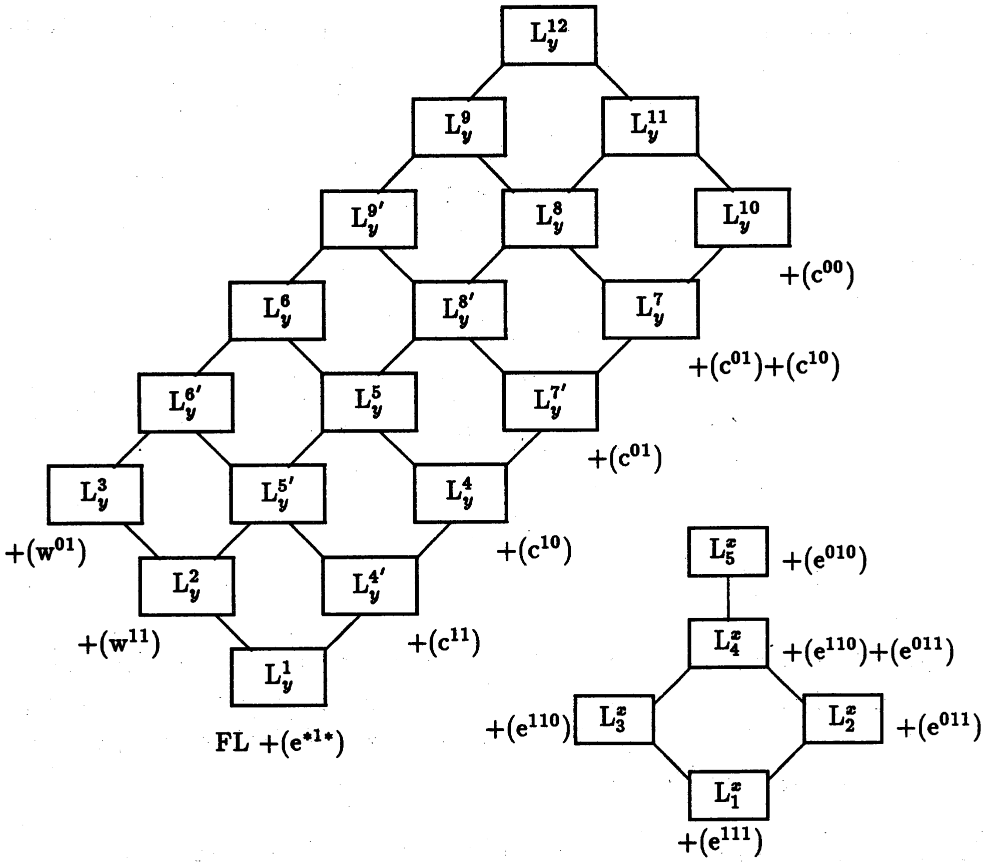
Proof Let $\tilde{p} \equiv (p \to p) \to p$. The sequents $p \Rightarrow \tilde{p}$ and $\tilde{p} \Rightarrow p$ are provable in L. That is, each propositional variable is equivalent to an implication in L. Then, the nonrestricted structural rules are derivable by using the restricted rules and the cut rule. ∎

7 Cut-elimination for $E_{\to}$, $S4_{\to}$, and their subsystems

In this section, we make thorough investigations on the cut-elimination property of the systems $FL_{\to} + e + w + c$ where

$e \in \{(e^{111}), (e^{011}), (e^{110}), (e^{110}) + (e^{011}), (e^{010})\}\ (= (e^{*1*}))$,
$w \in \{\text{null}, (w^{11}), (w^{01})\}$, and
$c \in \{\text{null}, (c^{11}), (c^{10}), (c^{01}), (c^{01}) + (c^{10}), (c^{00})\}$.

We will name them L^x_y, in which x denotes a combination of the weakening and contraction rules and y denotes a combination of the exchange rules as displayed in the following figure.



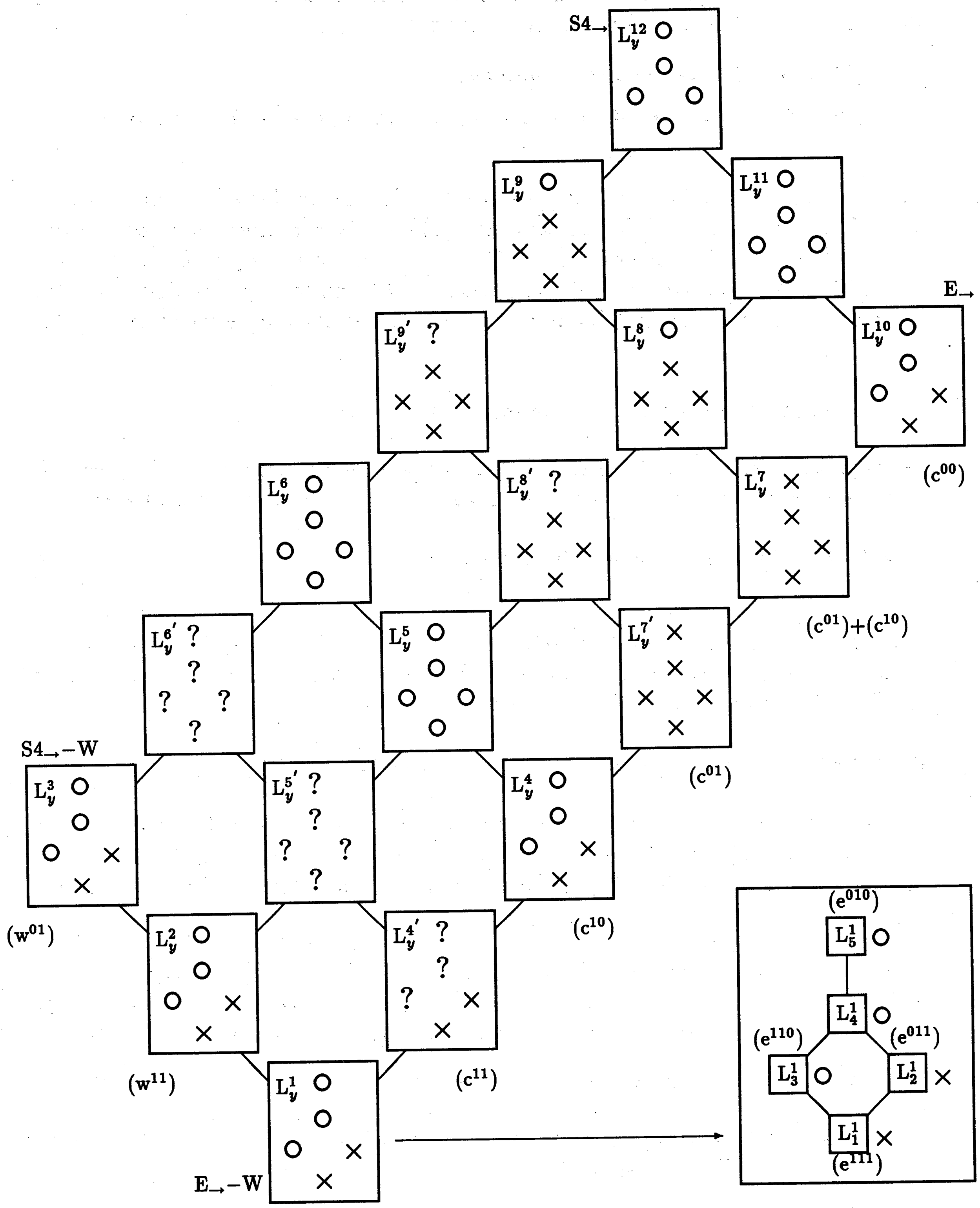
For example, $L^1_1 = FL_\rightarrow + (e^{111})$, $L^{7'}_4 = FL_\rightarrow + (e^{110}) + (e^{011}) + (c^{01})$, and $L^{12}_5 = FL_\rightarrow + (e^{010}) + (w^{01}) + (c^{00})$. Note that, for each $x \in \{1, ..., 12\}$, the five or ten systems $\{L^x_y, L^{x'}_y\}$ $(y = 1, ..., 5)$ are theorem-equivalent (by Theorem 3.2 in [1] and Theorem 6.1). L^1_y, L^3_y, L^{10}_y, and L^{12}_y are systems for the relevance logics $E_\rightarrow$−W, $S4_\rightarrow$−W, $E_\rightarrow$, and $S4_\rightarrow$ respectively. If we add (w^{10}) to those systems, the restriction on the inferences becomes ineffective (Theorem 6.2). Therefore those are all the considerable systems for $E_\rightarrow$, $S4_\rightarrow$ and their subsystems in our setting.

Our results on the cut-elimination property are summarized as follows.

Cut-elimination holds (denoted by ○):
L^1_{3-5}, L^2_{3-5}, L^3_{3-5}, L^4_{3-5}, L^5_{1-5}, L^6_{1-5}, L^8_5, L^9_5, L^{10}_{3-5}, L^{11}_{1-5}, L^{12}_{1-5}.

Cut-elimination does not hold (denoted by ×):
$L^1_{1,2}$, $L^2_{1,2}$, $L^3_{1,2}$, $L^{4,4'}_{1,2}$, $L^{7,7'}_{1-5}$, $L^{8,8'}_{1-4}$, $L^{9,9'}_{1-4}$, $L^{10}_{1,2}$

Unknown (denoted by ?):
$L^{4'}_{3-5}$, $L^{5'}_{1-5}$, $L^{6'}_{1-5}$, $L^{8'}_5$, $L^{9'}_5$.


S4→
E→
S4→−W
E→−W
(c00)
(c01)+(c10)
(c01)
(c10)
(c11)
(w01)
(w11)
(e010)
(e110)
(e011)
(e111)

Before the proofs of the cut-eliminations, we note a fact which will be implicitly used below. Let L and L^+ be systems such that

(1) L and L^+ are theorem-equivalent;

(2) L^+ is "stronger" than L; that is, each proof in L is also a proof in L^+.

Then the cut-elimination for L implies the cut-elimination for L^+: *Suppose a sequent S is provable in L^+. By the condition* (1) *and the cut-elimination for L, there is a cut-free proof P of S in L. Then P is also a cut-free proof in L^+ by the condition* (2). For example, the cut-elimination for L_3^1 implies the cut-elimination for L_4^1 and L_5^1, and failure in cut-elimination for L_4^9 implies the failure for L_y^9 and $\mathrm{L}_y^{9'}$ for $y = 1, .., 4$.

Now we start proving the cut-elimination theorems.

Lemma 7.1 (Inversion Lemma) *Let $L = \mathrm{L}_y^x$ (x and y are arbitrarily fixed). If $\Gamma \Rightarrow \alpha \to \beta$ is cut-free provable in L, then also $\Gamma, \alpha \Rightarrow \beta$ is cut-free provable in L.*

Proof By induction on the cut-free proof in L. ∎

Lemma 7.2 (Atomic Cut-Elimination) *Let $L = \mathrm{L}_y^x$ (x and y are arbitrarily fixed). For any propositional variable p, the rule* (p-cut) *(i.e., the cut rule whose cut-formula is p) is admissible in cut-free L.*

Proof By induction on the left upper subproof of (p-cut). ∎

We first show the cut-elimination for the systems L_5^8 and L_5^9, for which the cut-elimination fails if (e^{010}) is replaced by "weaker" exchange rules. In the cut-elimination procedure, the following lemma plays an important role like Lemma 3.6 in [1].

Lemma 7.3 (Key Lemma for L_5^8 and L_5^9) *Let $L = \mathrm{L}_5^8$ or L_5^9. If there is a cut-free proof P of $\Phi, \Psi \Rightarrow \psi$ in L and if $\Psi \Rightarrow \psi$ is an implication, then there are a sequence Φ^- and a proof P^- which satisfy the following conditions.*

(1) *P^- is a cut-free proof of $\Phi^-, \Psi \Rightarrow \psi$ in L.*

(2) *Φ^- is a (possibly empty) sequence of implications. If Φ does not contain an implication, then Φ^- is empty.*

(3) *The rule of inference*

$$\frac{\overrightarrow{\Gamma, \Phi^-, \Delta \Rightarrow \alpha}}{\Gamma, \Phi, \Delta \Rightarrow \alpha} \; (\mathcal{A}_{\Phi}^{\Phi^-})$$

is cut-free derivable in L. That is, for any sequence (Γ, Δ, α), if $\Delta \Rightarrow \alpha$ is an implication, then there is a cut-free derivation from $\Gamma, \Phi^-, \Delta \Rightarrow \alpha$ to $\Gamma, \Phi, \Delta \Rightarrow \alpha$ in L.

(The sequences Φ and (Ψ, ψ), which are components of the last sequent of the given proof P, will be called respectively a redex and an invariant.)

(Note: The rule $\mathcal{R}_{\Phi}^{\Phi^-}$ in Lemma 3.6 in [1] is stronger than $\mathcal{A}_{\Phi}^{\Phi^-}$, and it will be appear in Lemma 7.11 as $\mathcal{C}_{\Phi}^{\Phi^-}$.)

Proof Similar to the proof of Lemma 3.6. Here we show a case of (w^{11}): P is of the form

$$\dfrac{\begin{array}{c}\vdots\, Q\\ \Pi, \Sigma_1, \overrightarrow{\Sigma_2 \Rightarrow \psi}\end{array}}{\Pi, \overrightarrow{\beta}, \Sigma_1, \Sigma_2 \Rightarrow \psi,}\ (\mathrm{w}^{11})$$

and the redex Φ is (Π, β, Σ_1). In this case, the required proof P^- is

$$\begin{array}{c}\vdots\, Q^-\\ \Lambda^-, \Sigma_2 \Rightarrow \psi\end{array}$$

and the required sequence Φ^- is Λ^- where Q^- is a proof obtained by the induction hypothesis for Q in which the redex is $\Lambda \equiv (\Pi, \Sigma_1)$. The condition (2) is obviously satisfied by the induction hypothesis, and (3) — derivability of $\mathcal{A}_{\Pi,\beta,\Sigma_1}^{\Lambda^-}$ — is shown by

$$\dfrac{\dfrac{\Gamma, \Lambda^-, \overrightarrow{\Delta \Rightarrow \alpha}}{\Gamma, \Pi, \Sigma_1, \overrightarrow{\Delta \Rightarrow \alpha}}\ (\mathcal{A}_{\Pi,\Sigma_1}^{\Lambda^-})\ \text{(ind. hyp.)}}{\Gamma, \Pi, \overrightarrow{\beta}, \Sigma_1, \Delta \Rightarrow \alpha.}\ (\mathrm{w}^{11})$$

Note that the condition "$\Delta \Rightarrow \alpha$ is an implication" on $\mathcal{A}_{\Pi,\Sigma_1}^{\Lambda^-}$ is necessary for the application of (w^{11}) if Σ_1 is empty. ∎

Now we show the cut-elimination for L_5^8 and L_5^9. The "atomic cut" is eliminable by Lemma 7.2, and then we will show the "non-atomic cut-elimination". For this, we introduce a rule named (mix) which is of the form

$$\dfrac{\Phi_1 \Rightarrow \overrightarrow{\phi} \quad \cdots \quad \Phi_n \Rightarrow \overrightarrow{\phi} \quad \Psi_0, \overrightarrow{\phi}, \Psi_1, ..., \overrightarrow{\phi}, \Psi_n \Rightarrow \psi}{\Psi_0, \Phi_1, \Psi_1, ..., \Phi_n, \Psi_n \Rightarrow \psi}\ (\mathrm{mix})$$

where $n \geq 0$. Note that the "mix formula" ϕ must be an implication.

Lemma 7.4 (Mix-Elimination for L_5^8 and L_5^9) *Let $L = \mathrm{L}_5^8$ or L_5^9. The rule* (mix) *is admissible in cut-free L.*

Proof Let P be a proof

$$\dfrac{\begin{matrix}\vdots\, Q_1 \\ \Phi_1 \Rightarrow \overrightarrow{\phi}\end{matrix} \quad \cdots \quad \begin{matrix}\vdots\, Q_n \\ \Phi_n \Rightarrow \overrightarrow{\phi}\end{matrix} \quad \begin{matrix}\vdots\, R \\ \Psi \Rightarrow \psi\end{matrix}}{\Psi^\circ \Rightarrow \psi}\,(\text{mix})$$

where Q_i and R are cut-free proofs in L, and Ψ° denotes the sequence obtained from Ψ by replacing certain occurrences of ϕ by $\Phi_1, ..., \Phi_n$. (The superscript $\circ$ will be used similarly.) We define the *grade* g of this mix to be the length of the formula ϕ and the *rank* r of this mix to be the length of the proof R. If $n = 0$, we define $g = 0$. We prove, by double induction on the grade and rank of this mix, that there is a cut-free proof of $\Psi^\circ \Rightarrow \psi$ in L. We distinguish cases according to the form of R, and here we show some nontrivial cases concerning the weakening and contraction. (The other cases are easy; we use the Inversion Lemma 7.1 for the case of ($\rightarrow$left) and use the Key Lemma 7.3 for the case of (e^{010}) similarly to the proof of Theorem 3.8 in [1].)

(Case 1): The last inference of R is (w^{11}), and P is of the form

$$\dfrac{\begin{matrix}\vdots\, Q_1 \\ \Phi_1 \Rightarrow \overrightarrow{\phi}\end{matrix} \quad \cdots \quad \begin{matrix}\vdots\, Q_n \\ \Phi_n \Rightarrow \overrightarrow{\phi}\end{matrix} \quad \dfrac{\begin{matrix}\vdots\, R_0 \\ \Gamma, \overrightarrow{\Delta \Rightarrow \psi}\end{matrix}}{\Gamma, \overrightarrow{\phi}, \Delta \Rightarrow \psi}\,(w^{11})}{\Gamma^\circ, \Phi_k, \Delta^\circ \Rightarrow \psi.}\,(\text{mix})$$

(Subcase 1-1): $\Delta^\circ \Rightarrow \psi$ is an implication. We apply the Key Lemma 7.3 to Q_k in which the redex is Φ_k, and we get a sequence Φ_k^- of implications and cut-free derivability of the rule $\mathcal{A}^{\Phi_k^-}_{\Phi_k}$. Then, by the induction hypothesis, the required proof is obtained from the proof

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\, Q_1 \\ \Phi_1 \Rightarrow \phi\end{matrix} \quad \cdots \quad (Q_k \text{ is deleted}) \quad \cdots \quad \begin{matrix}\vdots\, Q_n \\ \Phi_n \Rightarrow \phi\end{matrix} \quad \begin{matrix}\vdots\, R_0 \\ \Gamma, \Delta \Rightarrow \psi\end{matrix}}{\Gamma^\circ, \overrightarrow{\Delta^\circ \Rightarrow \psi}}\,(\text{mix})}{\begin{matrix}\vdots\, (w^{11}) \\ \Gamma^\circ, \Phi_k^-, \overrightarrow{\Delta^\circ \Rightarrow \psi}\end{matrix}}}{\Gamma^\circ, \Phi_k, \Delta^\circ \Rightarrow \psi.}\,(\mathcal{A}^{\Phi_k^-}_{\Phi_k})$$

(Subcase 1-2): (Δ°, ψ) is a single atom. In this case, P is of the form

$$\dfrac{\begin{matrix}\vdots\, Q_1 \\ \Phi_1 \Rightarrow \overrightarrow{\phi}\end{matrix} \quad \cdots \quad \begin{matrix}\vdots\, Q_k \\ \Phi_k \Rightarrow \overrightarrow{\phi}\end{matrix} \quad \begin{matrix}\vdots\, Q_{k+1} \\ \Rightarrow \overrightarrow{\phi}\end{matrix} \quad \cdots \quad \begin{matrix}\vdots\, Q_{k+m} \\ \Rightarrow \overrightarrow{\phi}\end{matrix} \quad \dfrac{\begin{matrix}\vdots\, R_0 \\ \Gamma, \Delta \Rightarrow p\end{matrix}}{\Gamma, \overrightarrow{\phi}, \Delta \Rightarrow p}\,(w^{11})}{\Gamma^\circ, \Phi_k \Rightarrow p}\,(\text{mix})$$

where $\Delta \equiv \overbrace{\phi, ..., \phi}^{m}$ and $m \geq 1$. Then the required proof is obtained from the proof

$$\dfrac{\begin{matrix} \vdots\, Q_1 \\ \Phi_1 \Rightarrow \overrightarrow{\phi} \end{matrix} \quad \cdots \quad \begin{matrix} \vdots\, Q_k \\ \Phi_k \Rightarrow \overrightarrow{\phi} \end{matrix} \quad \begin{matrix} \vdots\, Q_{k+1} \\ \Rightarrow \overrightarrow{\phi} \end{matrix} \quad \cdots \quad \begin{matrix} \vdots\, Q_{k+m-1} \\ \Rightarrow \overrightarrow{\phi} \end{matrix} \qquad \begin{matrix} \vdots\, R_0 \\ \Gamma, \Delta \Rightarrow p \end{matrix}}{\Gamma^\circ, \Phi_k \Rightarrow p} \text{(mix)}$$

by the induction hypothesis.

(Case 2): The last inference of R is (w^{11}), and P is of the form

$$\dfrac{\begin{matrix} \vdots\, Q_1 \\ \Phi_1 \Rightarrow \phi \end{matrix} \quad \cdots \quad \begin{matrix} \vdots\, Q_n \\ \Phi_n \Rightarrow \phi \end{matrix} \quad \dfrac{\begin{matrix} \vdots\, R_0 \\ \Gamma, \overrightarrow{\Delta \Rightarrow \psi} \end{matrix}}{\Gamma, \overrightarrow{\alpha}, \Delta \Rightarrow \psi} (\mathrm{w}^{11})}{\Gamma^\circ, \alpha, \Delta^\circ \Rightarrow \psi.} \text{(mix)}$$

If $\Delta^\circ \Rightarrow \psi$ is an implication, the required proof is easily obtained by the induction hypothesis. If (Δ°, ψ) is a single atom, P is of the form

$$\dfrac{\begin{matrix} \vdots\, Q_1 \\ \Phi_1 \Rightarrow \overrightarrow{\phi} \end{matrix} \quad \cdots \quad \begin{matrix} \vdots\, Q_k \\ \Phi_k \Rightarrow \overrightarrow{\phi} \end{matrix} \quad \begin{matrix} \vdots\, Q_{k+1} \\ \Rightarrow \overrightarrow{\phi} \end{matrix} \quad \cdots \quad \begin{matrix} \vdots\, Q_{k+m} \\ \Rightarrow \overrightarrow{\phi} \end{matrix} \quad \dfrac{\begin{matrix} \vdots\, R_0 \\ \Gamma, \Delta \Rightarrow p \end{matrix}}{\Gamma, \overrightarrow{\alpha}, \Delta \Rightarrow p} (\mathrm{w}^{11})}{\Gamma^\circ, \alpha \Rightarrow p} \text{(mix)}$$

where $\Delta \equiv \overbrace{\phi, ..., \phi}^{m}$ and $m \geq 1$. We apply (w^{11}) to Q_{k+1}, and we get the following proof.

$$\dfrac{\begin{matrix} \vdots\, Q_1 \\ \Phi_1 \Rightarrow \overrightarrow{\phi} \end{matrix} \quad \cdots \quad \dfrac{\begin{matrix} \vdots\, Q_{k+1} \\ \Rightarrow \overrightarrow{\phi} \end{matrix}}{\overrightarrow{\alpha} \Rightarrow \overrightarrow{\phi}} (\mathrm{w}^{11}) \quad \cdots \quad \begin{matrix} \vdots\, Q_{k+m} \\ \Rightarrow \overrightarrow{\phi} \end{matrix} \quad \begin{matrix} \vdots\, R_0 \\ \Gamma, \Delta \Rightarrow p \end{matrix}}{\Gamma^\circ, \alpha \Rightarrow p.} \text{(mix)}$$

Then the required proof is obtained by the induction hypothesis.

(Case 3): The last inference of R is (c^{01}), and P is of the form

$$\dfrac{\mathcal{Q} \quad \begin{matrix} \vdots\, Q_k \\ \Phi_k \Rightarrow \overrightarrow{\phi} \end{matrix} \quad \mathcal{Q}' \quad \dfrac{\begin{matrix} \vdots\, R_0 \\ \Gamma, \overrightarrow{\phi}, \overrightarrow{\phi}, \overrightarrow{\Delta \Rightarrow \psi} \end{matrix}}{\Gamma, \overrightarrow{\phi}, \Delta \Rightarrow \psi} (\mathrm{c}^{01})}{\Gamma^\circ, \Phi_k, \Delta^\circ \Rightarrow \psi} \text{(mix)}$$

where $\mathcal{Q}$ and $\mathcal{Q}'$ are sequences of cut-free proofs of $\Phi_i \Rightarrow \phi$ $(i = 1, ..., (k-1), (k+1), ..., n)$. We apply the Key Lemma 7.3 to Q_k in which the redex is Φ_k, and we get a sequence Φ_k^- of implications, a cut-free proof Q_k^- of $\Phi_k^- \Rightarrow \phi$, and cut-free derivability of the rule $\mathcal{A}_{\Phi_k}^{\Phi_k^-}$.

(Subcase 3-1): Φ_k^- is empty. By the induction hypothesis, the required proof is obtained from the proof

$$\dfrac{\mathcal{Q}\quad \begin{array}{c}\vdots\, Q_k\\ \Phi_k \Rightarrow \phi\end{array} \quad \begin{array}{c}\vdots\, Q_k^-\\ \Rightarrow \phi\end{array} \quad \mathcal{Q}'\quad \begin{array}{c}\vdots\, R_0\\ \Gamma, \phi, \phi, \Delta \Rightarrow \psi\end{array}}{\Gamma^\circ, \Phi_k, \Delta^\circ \Rightarrow \psi.}\,(\mathrm{mix})$$

(Subcase 3-2): $\Delta^\circ \Rightarrow \psi$ is an implication. By the induction hypothesis, the required proof is obtained from the proof

$$\begin{array}{c}\dfrac{\mathcal{Q}\quad \begin{array}{c}\vdots\, Q_k^-\\ \Phi_k^- \Rightarrow \phi\end{array} \quad \begin{array}{c}\vdots\, Q_k^-\\ \Phi_k^- \Rightarrow \phi\end{array} \quad \mathcal{Q}'\quad \begin{array}{c}\vdots\, R_0\\ \Gamma, \phi, \phi, \Delta \Rightarrow \psi\end{array}}{\Gamma^\circ, \Phi_k^-, \Phi_k^-, \overrightarrow{\Delta^\circ \Rightarrow \psi}}\,(\mathrm{mix})\\ \vdots\, (\mathrm{e}^{010}),\ (\mathrm{c}^{01})\\ \dfrac{\Gamma^\circ, \Phi_k^-, \overrightarrow{\Delta^\circ \Rightarrow \psi}}{\Gamma^\circ, \Phi_k, \Delta^\circ \Rightarrow \psi.}\,(\mathcal{A}_{\Phi_k}^{\Phi_k^-})\end{array}$$

(Subcase 3-3): Φ_k^- is not empty and (Δ°, ψ) is a single atom. The condition (2) in the Key Lemma 7.3 implies the fact that Φ_k contains an implication, say $\overrightarrow{\alpha}$. Then we apply (w^{11}) to Q_k^-, and we get the following proof.

$$\begin{array}{c}\dfrac{\mathcal{Q}\quad \begin{array}{c}\vdots\, Q_k^-\\ \Phi_k^- \Rightarrow \overrightarrow{\phi}\end{array} \quad \dfrac{\begin{array}{c}\vdots\, Q_k^-\\ \Phi_k^- \Rightarrow \overrightarrow{\phi}\end{array}}{\Phi_k^-, \overrightarrow{\alpha} \Rightarrow \overrightarrow{\phi}}\,(\mathrm{w}^{11}) \quad \mathcal{Q}'\quad \begin{array}{c}\vdots\, R_0\\ \Gamma, \overrightarrow{\phi}, \overrightarrow{\phi}, \Delta \Rightarrow \psi\end{array}}{\Gamma^\circ, \Phi_k^-, \Phi_k^-, \overrightarrow{\alpha} \Rightarrow \psi}\,(\mathrm{mix})\\ \vdots\, (\mathrm{e}^{010}),\ (\mathrm{c}^{01})\\ \dfrac{\Gamma^\circ, \Phi_k^-, \overrightarrow{\alpha} \Rightarrow \psi}{\Gamma^\circ, \Phi_k, \overrightarrow{\alpha} \Rightarrow \psi}\,(\mathcal{A}_{\Phi_k}^{\Phi_k^-})\\ \vdots\, (\mathrm{e}^{010}),\ (\mathrm{c}^{10})\\ \Gamma^\circ, \Phi_k \Rightarrow \psi.\end{array}$$

The required proof is obtained from this by the induction hypothesis.

(Case 4): The last inference of R is (c^{01}), and P is of the form

$$\dfrac{\begin{array}{c}\vdots\, Q_1\\ \Phi_1 \Rightarrow \phi\end{array} \quad \cdots \quad \begin{array}{c}\vdots\, Q_n\\ \Phi_n \Rightarrow \phi\end{array} \quad \dfrac{\begin{array}{c}\vdots\, R_0\\ \Gamma, \alpha, \alpha, \overrightarrow{\Delta \Rightarrow \psi}\end{array}}{\Gamma, \alpha, \Delta \Rightarrow \psi}\,(\mathrm{c}^{01})}{\Gamma^\circ, \alpha, \Delta^\circ \Rightarrow \psi.}\,(\mathrm{mix})$$

If α or $\Delta^\circ \Rightarrow \psi$ is an implication, the required proof is easily obtained by the induction hypothesis and (c^{10} or c^{01}). Suppose α is atomic and (Δ°, ψ) is a single atom. In this case, P is of the form

$$\dfrac{\begin{matrix} \vdots Q_1 \\ \Phi_1 \Rightarrow \overrightarrow{\phi} \end{matrix} \quad \cdots \quad \begin{matrix} \vdots Q_k \\ \Phi_k \Rightarrow \overrightarrow{\phi} \end{matrix} \quad \begin{matrix} \vdots Q_{k+1} \\ \Rightarrow \overrightarrow{\phi} \end{matrix} \quad \cdots \quad \begin{matrix} \vdots Q_{k+m} \\ \Rightarrow \overrightarrow{\phi} \end{matrix} \quad \dfrac{\begin{matrix} \vdots R_0 \\ \Gamma, p, p, \Delta \Rightarrow q \end{matrix}}{\Gamma, p, \Delta \Rightarrow q}\,(c^{01})}{\Gamma^\circ, p \Rightarrow q}\,(\text{mix})$$

where $\Delta \equiv \overbrace{\phi, ..., \phi}^{m}$ and $m \geq 1$. Now we consider two cases:

(Case A): Γ° contains an implication, say $\overrightarrow{\beta}$. We apply (w^{11}) to Q_{k+1}, and we get the following proof.

$$\dfrac{\begin{matrix} \vdots Q_1 \\ \Phi_1 \Rightarrow \overrightarrow{\phi} \end{matrix} \quad \cdots \quad \dfrac{\begin{matrix} \vdots Q_{k+1} \\ \Rightarrow \overrightarrow{\phi} \end{matrix}}{\overrightarrow{\beta} \Rightarrow \overrightarrow{\phi}}\,(w^{11}) \quad \cdots \quad \begin{matrix} \vdots Q_{k+m} \\ \Rightarrow \overrightarrow{\phi} \end{matrix} \quad \begin{matrix} \vdots R_0 \\ \Gamma, p, p, \Delta \Rightarrow q \end{matrix}}{\dfrac{\Gamma^\circ, p, p, \overrightarrow{\beta} \Rightarrow q}{\begin{matrix} \Gamma^\circ, p, \overrightarrow{\beta} \Rightarrow q \\ \vdots\, (e^{010})(c^{01}) \\ \Gamma^\circ, p \Rightarrow q \end{matrix}}\,(c^{01})}\,(\text{mix})$$

Then the required proof is obtained by the induction hypothesis.

(Case B): Γ° does not contain implications. Consider the proof

$$\dfrac{\begin{matrix} \vdots Q_1 \\ \Phi_1 \Rightarrow \phi \end{matrix} \quad \cdots \quad \begin{matrix} \vdots Q_{k+m} \\ \Rightarrow \phi \end{matrix} \quad \begin{matrix} \vdots R_0 \\ \Gamma, p, p, \Delta \Rightarrow q \end{matrix}}{\Gamma^\circ, p, p \Rightarrow q.}\,(\text{mix})$$

Then, by the induction hypothesis, there is a cut-free proof P' of $\Gamma^\circ, p, p \Rightarrow q$ in L. This sequent consists of atomic formulas, and therefore the only possible inferences in P' are (w^{01}) and (c^{01}). If $L = L_5^8$, then L does not have (w^{01}) and there is no such proof in L. This means that Case B never happen for L_5^8. If $L = L_5^9$, then the fact $p \equiv q$ is easily verified by the form of P', and we get the required proof

$$\begin{matrix} p \Rightarrow p \\ \vdots\, (w^{01}) \\ \Gamma^\circ, p \Rightarrow p. \end{matrix}$$

∎

Theorem 7.5 (Cut-Elimination for L_5^8 and L_5^9) *The rule* (cut) *is admissible in cut-free* L_5^8 *and cut-free* L_5^9.

Proof By Lemmas 7.2 (for atomic cut) and 7.4 (for non-atomic cut). ∎

Note that the rules (e^{010}), (w^{11}) and (c^{10}) are used in the Cases 3 and 4 in the proof of the above Lemma 7.4. Therefore this procedure does not work for the systems L_5^7, L_4^8, L_4^9, $L_5^{8'}$, and $L_5^{9'}$. Indeed we will show that the cut-elimination fails for L_5^7, L_4^8, and L_4^9. (The authors do not know whether the cut-elimination holds for $L_5^{8'}/L_5^{9'}$.)

Let S be a sequent $\alpha^1, ..., \alpha^n \Rightarrow \alpha^0$ where $n \geq 0$, $\alpha^i \equiv \alpha_1^i \rightarrow \cdots \rightarrow \alpha_{f(i)}^i \rightarrow p_i$, $f(i) \geq 0$, and p_i are propositional variables ($i = 0, ..., n$). We say that a propositional variable v *occurs badly in* S if the following conditions are satisfied.

(1) $p_i \equiv v$ for some $i \geq 1$.

(2) If $p_0 \equiv v$, then $p_i \equiv p_j \equiv v$ for some $i > j \geq 1$.

(3) v does not occur in α_j^i for any i, j.

Lemma 7.6 *Let $L = L_y^x$ where $x \in \{1, 4, 4', 7, 7', 10\}$ and y is arbitrary (i.e., L is a system which has no weakening rule). If a sequent S is cut-free provable in L, then no propositional variable occurs badly in S.*

Proof By induction on the cut-free proof of S in L. ∎

In the following, α^+ will denote a nonempty sequence of α.

Theorem 7.7 (Failure of Cut-Elimination for L_5^7) *There is a sequent which is provable in L_5^7 but not cut-free provable in L_5^7.*

Proof Let $S \equiv p \rightarrow p \rightarrow I \rightarrow q, p \Rightarrow q$ where $I \equiv r \rightarrow r$ and p, q, r are mutually distinct propositional variables. We have $L_5^7 \vdash S$:

$$\dfrac{\begin{array}{c}\vdots \\ \Rightarrow I\end{array} \quad \dfrac{\begin{array}{c}\vdots \\ p \rightarrow p \rightarrow I \rightarrow q, p, p, I \Rightarrow q\end{array}}{p \rightarrow p \rightarrow I \rightarrow q, p, I \Rightarrow q}\,(c^{01})}{p \rightarrow p \rightarrow I \rightarrow q, p \Rightarrow q.}\,(\text{cut})$$

We will show that S is not cut-free provable. Suppose there is a cut-free proof P of S in L_5^7. By Lemma 7.6, the last inference in P must be either ($\rightarrow$left) or (e^{010}) (contraction of $p \rightarrow p \rightarrow I \rightarrow q$ never happens). In the former case, two candidates for the pair of upper sequents of this ($\rightarrow$left) contain non-tautologies $\Rightarrow p$ and $p \rightarrow I \rightarrow q \Rightarrow q$; therefore this cannot happen. In the latter case, Lemma 7.6 implies that P must be of the form

$$\dfrac{\begin{array}{c}\vdots \\ p, p \rightarrow p \rightarrow I \rightarrow q \Rightarrow q\end{array}}{p \rightarrow p \rightarrow I \rightarrow q, p \Rightarrow q.}\,(e^{010}) \quad \text{where the elided part is} \quad \dfrac{\begin{array}{c}\vdots \\ \Rightarrow p\end{array} \quad \begin{array}{c}\vdots \\ p^+, p \rightarrow I \rightarrow q \Rightarrow q\end{array}}{p^+, p \rightarrow p \rightarrow I \rightarrow q \Rightarrow q}\,(\rightarrow\text{left}) \;\vdots\,(c^{01})$$

However, this cannot happen because $\Rightarrow p$ is not provable. ∎

This counterexample also shows the following.

Theorem 7.8 (Failure of Cut-Elimination for L_4^8 and L_4^9) *There is a sequent which is provable in L_4^8 and in L_4^9 but neither in cut-free L_4^8 nor cut-free L_4^9.*

Proof Take the same sequent S as Theorem 7.7. We show that S is not cut-free provable in L_4^9. Suppose there is a cut-free proof P of S in L_4^9. Then, since $p \Rightarrow q$ is not an initial sequent, P must be of the form

$$\begin{array}{l} \dfrac{\qquad}{(p{\to}p{\to}I{\to}q)^+, p \Rightarrow q}\,(\to\text{left}) \\ \quad\vdots\ (\text{e}^{011}),(\text{e}^{110}),(\text{w}^{11}),(\text{w}^{01}),(\text{c}^{10}),(\text{c}^{01});\ \text{for } p{\to}p{\to}I{\to}q \\ p{\to}p{\to}I{\to}q, p \Rightarrow q. \end{array}$$

However, this cannot happen because all the candidates for the pair of upper sequents of this ($\to$left) contain non-tautologies. ∎

Next we show the cut-elimination for the systems L_3^x for $x = 1, ..., 6, 10, 11, 12$. Consider the following proof in L_3^3.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Rightarrow I\end{array}}{p \Rightarrow I}\,(\text{w}^{01}) \qquad \dfrac{\dfrac{I \Rightarrow I \qquad \dfrac{\alpha \Rightarrow \alpha \quad q \Rightarrow q}{\alpha{\to}q, \alpha \Rightarrow q}}{I{\to}\alpha{\to}q, I, \alpha \Rightarrow q}}{I{\to}\alpha{\to}q, \alpha, I \Rightarrow q}\,(\text{e}^{110})}{I{\to}\alpha{\to}q, \alpha, p \Rightarrow q}\,(\text{cut})$$

where $I \equiv r{\to}r$ and α is an implication. To get a cut-free proof of this sequent, we must move the application of (w^{01}) to an ancestor of the right upper sequent of the cut:

$$\dfrac{\begin{array}{c}\vdots\\ \Rightarrow I\end{array} \qquad \dfrac{\dfrac{\alpha \Rightarrow \alpha}{\alpha, p \Rightarrow \alpha}\,(\text{w}^{01}) \quad q \Rightarrow q}{\alpha{\to}q, \alpha, p \Rightarrow q}}{I{\to}\alpha{\to}q, \alpha, p \Rightarrow q.}$$

Such transformation is not described in the cut-elimination procedure for L_5^8 and L_5^9, and then we need some preparations for L_3^x ($x = 1, ..., 6, 10, 11, 12$).

Lemma 7.9 (Weakening Lemma for (w^{11})) *Let $L = L_y^x$ where $x \in \{2, 5, 5', 8, 8', 11\}$ and y is arbitrary (i.e., L is a system which has (w^{11})). Then, the inference*

$$\dfrac{\Gamma, \overrightarrow{\alpha}, \Delta \Rightarrow \beta}{\Gamma, \overrightarrow{\alpha}, \overrightarrow{\gamma}, \Delta \Rightarrow \beta}\,(\mathcal{B}_1)$$

is admissible in cut-free L.

Proof ($\mathcal{B}_1$) is an instance of (w^{11}) if $\Delta \Rightarrow \beta$ is an implication. Therefore we prove, by induction on the cut-free proof of $\Gamma, \vec{\alpha} \Rightarrow p$, that there is a cut-free proof of $\Gamma, \vec{\alpha}, \vec{\gamma} \Rightarrow p$. The only nontrivial case is that the proof is of the form

$$\dfrac{\begin{matrix}\vdots P_1 & \vdots P_2 \\ \Rightarrow \alpha_1 & \Gamma, \alpha_2 \Rightarrow p\end{matrix}}{\Gamma, \alpha_1 \to \alpha_2 \Rightarrow p.}\ (\to\text{left})$$

In this case, α_1 is an implication because P_1 is a cut-free proof. Then the required proof is

$$\dfrac{\dfrac{\begin{matrix}\vdots P_1 \\ \Rightarrow \vec{\alpha_1}\end{matrix}}{\vec{\gamma} \Rightarrow \vec{\alpha_1}}\ (w^{11}) \qquad \begin{matrix}\vdots P_2 \\ \Gamma, \alpha_2 \Rightarrow p\end{matrix}}{\Gamma, \alpha_1 \to \alpha_2, \vec{\gamma} \Rightarrow p.}\ (\to\text{left})$$

∎

Lemma 7.10 (Weakening Lemma for (w^{01})) *Let $L = L_y^x$ where $x \in \{3, 6, 6', 9, 9', 12\}$ and y is arbitrary (i.e., L is a system which has (w^{01})). Then, the inference*

$$\dfrac{\Gamma, \vec{\alpha}, \Delta \Rightarrow \beta}{\Gamma, \vec{\alpha}, \gamma, \Delta \Rightarrow \beta}\ (\mathcal{B}_0)$$

is admissible in cut-free L.

Proof Similar to the previous Lemma 7.9. ∎

Lemma 7.11 (Key Lemma for L_3^1,L_3^2,L_3^3,L_3^4,L_3^5,L_3^6,L_3^{10},L_3^{11} and L_3^{12}) *Lemma 7.3 (Key Lemma for L_5^8 and L_5^9) holds for $L = L_3^x$ where $x \in \{1, ..., 6, 10, 11, 12\}$. Moreover, the sequence Φ^- satisfies the following conditions in addition to the conditions* (1)–(3).

(4) *The rule of inference*

$$\dfrac{\Gamma, \vec{\theta}, \Phi^-, \Delta \Rightarrow \alpha}{\Gamma, \vec{\theta}, \Phi, \Delta \Rightarrow \alpha}\ (\mathcal{B}_\Phi^{\Phi^-})$$

is admissible in cut-free L. That is, for any sequence $(\Gamma, \theta, \Delta, \alpha)$, if the sequent $\Gamma, \theta, \Phi^-, \Delta \Rightarrow \alpha$ is cut-free provable in L and if θ is an implication, then also $\Gamma, \theta, \Phi, \Delta \Rightarrow \alpha$ is cut-free provable in L.

(5) *If Φ^- is not empty, then the rule of inference*

$$\dfrac{\Gamma, \Phi^-, \Delta \Rightarrow \alpha}{\Gamma, \Phi, \Delta \Rightarrow \alpha}\ (\mathcal{C}_\Phi^{\Phi^-})$$

is admissible in cut-free L.

(Note: $\mathcal{A}^{\Phi^-}_{\Phi}$: a condition is imposed After Φ^-. $\mathcal{B}^{\Phi^-}_{\Phi}$: a condition is imposed Before Φ^-. $\mathcal{C}^{\Phi^-}_{\Phi}$: no Condition is imposed. Each instance of $\mathcal{B}_1$ and $\mathcal{B}_0$ (Lemmas 7.9 and 7.10) is an instance of $\mathcal{B}^{\Phi^-}_{\Phi}$ where Φ^- is empty and Φ is a formula.)

Proof The construction of the required proof P^- and the required sequence Φ^- is the same as that in the proof of Lemma 3.6 in [1] and Lemma 7.3. Then, to prove this lemma, we add proofs of the conditions (4) and (5) to each cases. Here we show some critical cases.

(Case 2-2 in Lemma 3.6): Admissibility of $\mathcal{B}^{\Pi^-,\beta\to\gamma,\Lambda_1^-}_{\Pi,\beta\to\gamma,\Lambda_1}$ and $\mathcal{C}^{\Pi^-,\beta\to\gamma,\Lambda_1^-}_{\Pi,\beta\to\gamma,\Lambda_1}$ is shown by

$$\dfrac{\dfrac{\Gamma,(\overrightarrow{\theta},)\ \Pi^-,\beta\to\gamma,\Lambda_1^-,\Delta\Rightarrow\alpha}{\Gamma,(\theta,)\ \Pi^-,\beta\to\gamma,\Lambda_1,\Delta\Rightarrow\alpha}\ (\mathcal{B}^{\Lambda_1^-}_{\Lambda_1})\ \text{(ind. hyp.)}}{\Gamma,(\theta,)\ \Pi,\beta\to\gamma,\Lambda_1,\Delta\Rightarrow\alpha.}\ (\mathcal{A}^{\Pi^-}_{\Pi})\ \text{(ind. hyp.)}$$

(Case 4-2 in Lemma 3.6): Admissibility (derivability) of $\mathcal{B}^{\Pi^-,\gamma}_{\Pi,\gamma}$ and $\mathcal{C}^{\Pi^-,\gamma}_{\Pi,\gamma}$ is shown by using $\mathcal{A}^{\Pi^-}_{\Pi}$.

(The case described in the proof of Lemma 7.3): When Σ_1 is not empty, admissibility of $\mathcal{B}^{\Lambda^-}_{\Pi,\beta,\Sigma_1}$ and $\mathcal{C}^{\Lambda^-}_{\Pi,\beta,\Sigma_1}$ is shown by

$$\dfrac{\dfrac{\Gamma,(\overrightarrow{\theta},)\ \Lambda^-,\Delta\Rightarrow\alpha}{\Gamma,(\theta,)\ \Pi,\overrightarrow{\Sigma_1,\Delta\Rightarrow\alpha}}\ (\mathcal{B}^{\Lambda^-}_{\Pi,\Sigma_1})\text{ or }(\mathcal{C}^{\Lambda^-}_{\Pi,\Sigma_1})\ \text{(ind. hyp.)}}{\Gamma,(\theta,)\ \Pi,\overrightarrow{\beta},\Sigma_1,\Delta\Rightarrow\alpha.}\ (\mathrm{w}^{11})$$

When Σ_1 is empty, that is shown by

$$\dfrac{\dfrac{\Gamma,(\overrightarrow{\theta},)\ \Lambda^-,\Delta\Rightarrow\alpha}{\Gamma,(\theta,)\ \Lambda^-,\overrightarrow{\beta},\Delta\Rightarrow\alpha}\ (\mathcal{B}_1)\ \text{(Weakening Lemma 7.9)}^{\dagger}}{\Gamma,(\theta,)\ \Pi,\overrightarrow{\beta},\Delta\Rightarrow\alpha.}\ (\mathcal{A}^{\Lambda^-}_{\Pi})\ \text{(ind. hyp.)}$$

(† We use another Weakening Lemma 7.10 in the case that the last inference of P is (w^{01}).) ∎

Lemma 7.12 (Mix-Elimination for $L_3^1,L_3^2,L_3^3,L_3^4,L_3^5,L_3^6,L_3^{10},L_3^{11}$ and L_3^{12}) *Let* $L = L_3^x$ *where* $x\in\{1,\dots,6,10,11,12\}$. *The rule* (mix) *is admissible in cut-free* L.

Proof Similar to the proof of Lemma 7.4 (mix-elimination procedure for L_5^8 and L_5^9). Here we show some nontrivial cases which are different from those in Lemma 7.4.

(Case 1): The last inference of R is (e^{110}), and P is of the form

$$\dfrac{\begin{matrix}\vdots\ Q_1\\ \Phi_1\Rightarrow\overrightarrow{\phi}\end{matrix}\quad\cdots\quad\begin{matrix}\vdots\ Q_n\\ \Phi_n\Rightarrow\overrightarrow{\phi}\end{matrix}\quad\dfrac{\begin{matrix}\vdots\ R_0\\ \Gamma,\overrightarrow{\phi},\overrightarrow{\beta},\Delta\Rightarrow\psi\end{matrix}}{\Gamma,\overrightarrow{\beta},\overrightarrow{\phi},\Delta\Rightarrow\psi}\ (\mathrm{e}^{110})}{\Gamma^{\circ},\beta,\Phi_k,\Delta^{\circ}\Rightarrow\psi.}\ (\mathrm{mix})$$

We apply the Key Lemma 7.11 to Q_k in which the redex is Φ_k, and we get a sequence Φ_k^- of implications, a cut-free proof Q_k^- of $\Phi_k^- \Rightarrow \phi$, and cut-free admissibility of the rule $\mathcal{B}_{\Phi_k}^{\Phi_k^-}$. Then, by the induction hypothesis, the required proof is obtained from the proof

$$\dfrac{\dfrac{\dfrac{\begin{matrix} \vdots\, Q_1 & & \vdots\, Q_k^- & & \vdots\, Q_n & \vdots\, R_0 \\ \Phi_1 \Rightarrow \overrightarrow{\phi} & \cdots & \Phi_k^- \Rightarrow \overrightarrow{\phi} & \cdots & \Phi_n \Rightarrow \overrightarrow{\phi} & \Gamma, \overrightarrow{\phi}, \overrightarrow{\beta}, \Delta \Rightarrow \psi \end{matrix}}{\Gamma^\circ, \Phi_k^-, \overrightarrow{\beta}, \Delta^\circ \Rightarrow \psi}\,(\text{mix})}{\vdots\,(\text{e}^{110})}}{\dfrac{\Gamma^\circ, \overrightarrow{\beta}, \Phi_k^-, \Delta^\circ \Rightarrow \psi}{\Gamma^\circ, \beta, \Phi_k, \Delta^\circ \Rightarrow \psi.}\,(\mathcal{B}_{\Phi_k}^{\Phi_k^-})}$$

(Case 2) The last inference of R is (c^{10}), and P is of the form

$$\dfrac{\begin{matrix} & \vdots\, Q_k & & \dfrac{\vdots\, R_0 \quad \Gamma, \overrightarrow{\phi}, \overrightarrow{\phi}, \Delta \Rightarrow \psi}{\Gamma, \overrightarrow{\phi}, \Delta \Rightarrow \psi}\,(\text{c}^{10}) \\ \mathcal{Q} & \Phi_k \Rightarrow \overrightarrow{\phi} & \mathcal{Q}' & \end{matrix}}{\Gamma^\circ, \Phi_k, \Delta^\circ \Rightarrow \psi.}\,(\text{mix})$$

where $\mathcal{Q}$ and $\mathcal{Q}'$ are sequences of cut-free proofs of $\Phi_i \Rightarrow \phi$ $(i = 1, ..., (k-1), (k+1), ..., n)$. We apply the Key Lemma 7.11 to Q_k in which the redex is Φ_k, and we get a sequence Φ_k^- of implications, a cut-free proof Q_k^- of $\Phi_k^- \Rightarrow \phi$, and cut-free admissibility of the rule $\mathcal{C}_{\Phi_k}^{\Phi_k^-}$ if Φ_k^- is nonempty.

(Subcase 2-1): Φ_k^- is empty. This is the same as Subcase 3-1 in Lemma 7.4.

(Subcase 2-2): Φ_k^- is not empty. By the induction hypothesis, the required proof is obtained from the proof

$$\dfrac{\dfrac{\dfrac{\begin{matrix} & \vdots\, Q_k^- & \vdots\, Q_k^- & & \vdots\, R_0 \\ \mathcal{Q} & \Phi_k^- \Rightarrow \phi & \Phi_k^- \Rightarrow \phi & \mathcal{Q}' & \Gamma, \phi, \phi, \Delta \Rightarrow \psi \end{matrix}}{\Gamma^\circ, \Phi_k^-, \Phi_k^-, \Delta^\circ \Rightarrow \psi}\,(\text{mix})}{\vdots\,(\text{e}^{110}), (\text{c}^{10})}}{\dfrac{\Gamma^\circ, \Phi_k^-, \Delta^\circ \Rightarrow \psi}{\Gamma^\circ, \Phi_k, \Delta^\circ \Rightarrow \psi.}\,(\mathcal{C}_{\Phi_k}^{\Phi_k^-})}$$

∎

Theorem 7.13 (Cut-Elimination for $\mathbf{L}_3^1$, $\mathbf{L}_3^2$, $\mathbf{L}_3^3$, $\mathbf{L}_3^4$, $\mathbf{L}_3^5$, $\mathbf{L}_3^6$, $\mathbf{L}_3^{10}$, $\mathbf{L}_3^{11}$ and $\mathbf{L}_3^{12}$) *Let $L = \mathrm{L}_3^x$ where $x \in \{1,...,6, 10, 11, 12\}$. The rule* (cut) *is admissible in cut-free L.*

Proof By Lemmas 7.2 (for atomic cut) and 7.12 (for non-atomic cut). ∎

This cut-elimination theorem can be extended to the systems L_1^x if (w^{11}) and (c^{10}) exist:

Theorem 7.14 (Cut-Elimination for L_1^5,L_1^6,L_1^{11} and L_1^{12}) *Let* $L = L_1^x$ *where* $x \in \{5, 6, 11, 12\}$. *The rule* (cut) *is admissible in cut-free* L.

Proof The following proof shows the fact that the rule (e^{110}) is admissible in cut-free L.

$$\cfrac{\cfrac{\cfrac{\Gamma, \vec{\alpha}, \vec{\beta} \Rightarrow p}{\Gamma, \vec{\alpha}, \vec{\beta}, \vec{\alpha} \Rightarrow p}\ (\mathcal{B}_1 \text{ or } \mathcal{B}_0)\ (\text{Weakening Lemma 7.9 or 7.10})}{\Gamma, \vec{\beta}, \vec{\alpha}, \vec{\alpha} \Rightarrow p}\ (e^{111})}{\Gamma, \vec{\beta}, \vec{\alpha} \Rightarrow p.}\ (c^{10})$$

Now suppose a sequent is provable in L_1^x. It is also provable in L_3^x, and then the cut-elimination for L_3^x (Theorem 7.13) and the above fact imply that it is cut-free provable in L_1^x. ∎

On the other hand, we cannot extend Theorem 7.13 if the system lacks (w^{11}) or (c^{10}):

Theorem 7.15 (Failure of Cut-Elimination for L_2^1,L_2^2,L_2^3,L_2^4 and L_2^{10}) *Let* $L = L_2^x$ *where* $x \in \{1, 2, 3, 4, 10\}$. *There is a sequent which is provable in* L *but not cut-free provable in* L.

Proof Let $S \equiv p{\rightarrow}q, (p{\rightarrow}q){\rightarrow}r \Rightarrow r$ where p, q, r are mutually distinct propositional variables. We have $L \vdash S$ (Theorem 3.2 in [1]). Here we show that S is not cut-free provable in L. Since L_2^3 has no contraction rule, it is easily verified that S is not cut-free provable in L_2^3 (L_2^2, L_2^1). Now suppose there is a cut-free proof P of S in L_2^{10} (L_2^4). By Lemma 7.6, P must be of the form

$$\cfrac{}{(p{\rightarrow}q)^+, (p{\rightarrow}q){\rightarrow}r \Rightarrow r}\ ({\rightarrow}\text{left})$$
$$\vdots\ (c^{00}) \text{ and } (e^{011}), \text{ for } p{\rightarrow}q$$
$$p{\rightarrow}q, (p{\rightarrow}q){\rightarrow}r \Rightarrow r.$$

However, this cannot happen because all the candidates for the pair of upper sequents of this (→left) contain non-tautologies. ∎

By using our cut-elimination theorems, we can separate the twelve logics.

Theorem 7.16 (Separation of L^1,...,L^{12}) *The twelve classes* L_y^1,...,L_y^{12} *are completely separated. That is, there are sequents* $S_1, \ldots, S_5$ *which satisfy the following (y is arbitrary).*

(1) $L_y^3 \nvdash S_1$, *and* $FL_{\rightarrow} +$ (c^{11}) $\vdash S_1$ *(therefore* $L_y^4 \vdash S_1$*).*

(2) $L_y^6 \nvdash S_2$, *and* $FL_{\rightarrow} +$ (c^{01}) $\vdash S_2$ *(therefore* $L_y^7 \vdash S_2$*).*

(3) $L_y^9 \nvdash S_3$, *and* $FL_{\rightarrow}+$ $(c^{00}) \vdash S_3$ *(therefore* $L_y^{10} \vdash S_3$*).*

(4) $L_y^{10} \nvdash S_4$, *and* $FL_{\rightarrow}+$ $(w^{11}) \vdash S_4$ *(therefore* $L_y^2 \vdash S_4$*).*

(5) $L_y^{11} \nvdash S_5$, *and* $FL_{\rightarrow}+$ $(w^{01}) \vdash S_5$ *(therefore* $L_y^3 \vdash S_5$*).*

Proof Let p, q, r, s be mutually distinct propositional variables.

(1) Take $S_1 \equiv (p\to q)\to(p\to q)\to r\to s, p\to q, r \Rightarrow s$. Since L_y^3 has no contraction rule, it is easily verified that there is no cut-free proof of S_1 in L_3^3.

(2) Take $S_2 \equiv S$ which appears in the proofs of Theorems 7.7 and 7.8. $L_3^6 \nvdash S$ is shown similarly to Theorem 7.8.

(3) Take $S_3 \equiv p\to p\to q, p \Rightarrow q$. We need a preparation to show $L_5^9 \nvdash S_3$. If a sequent is of the form $\Gamma, \overrightarrow{\alpha} \Rightarrow v$ (v is a propositional variable), then we say this sequent is *bad.* We have the following fact: *If a sequent* S *is cut-free provable in* L_5^9 *and if* S *consists of only subformulas of* $p\to p\to q$*, then* S *is not bad.* This is proved by induction on the cut-free proof P of S. Then we show $L_5^9 \nvdash S_3$. Suppose there is a cut-free proof P of S_3 in L_5^9. By the above fact, P cannot contain a bad sequent, and P must be of the form

$$\dfrac{}{(p\to p\to q)^+, p \Rightarrow q}\ (\to\text{left})$$
$$\vdots\ (e^{010}),(w^{01}),(c^{10}),(c^{01});\ \text{for } p\to p\to q$$
$$p\to p\to q, p \Rightarrow q.$$

However, this cannot happen because all the candidates for the pair of upper sequents of this $(\to$left$)$ contain non-tautologies.

(4) Take $S_4 \equiv p\to q, r \Rightarrow r$. Suppose there is a cut-free proof P of S_4 in L_3^{10}. Then P must be of the form

$$\dfrac{}{(p\to q)^+, r^+ \Rightarrow r}\ (\to\text{left})$$
$$\vdots\ (e^{110}),(c^{00})$$
$$p\to q, r \Rightarrow r.$$

However, this cannot happen because all the candidates for the pair of upper sequents of this $(\to$left$)$ contain non-tautologies.

(5) Take $S_5 \equiv p, q \Rightarrow q$. There is no cut-free proof of S_5 in L_y^{11} because atomic formulas cannot arise by (w^{11}) and $p^+, q^+ \Rightarrow q$ is not an initial sequent. ∎

We can also separate the logics from $FL_{\rightarrow}+$ (e^{001}).

Theorem 7.17 (Separation of L^{12} from (e^{001})) *There is a sequent which is provable in* $FL_{\rightarrow}+$ (e^{001}) *but not provable in* L_y^{12}.

Proof Let $S \equiv p, p\to I\to q \Rightarrow q$ where $I \equiv r\to r$ and p, q, r are mutually distinct propositional variables. $FL_{\rightarrow}+$ $(e^{001}) \vdash S$ is shown in Theorem 4.4 in [1], and here we show $\mathrm{L}_y^{12} \not\vdash S$. Suppose there is a cut-free proof P of S in L_3^{12}. Then P must be of the form

$$\dfrac{\dfrac{}{p^*, (p{\rightarrow}I{\rightarrow}q)^+ \Rightarrow q}\,(\rightarrow\text{left})}{p, p{\rightarrow}I{\rightarrow}q \Rightarrow q}\;\vdots\;(\mathrm{e}^{110}),(\mathrm{w}^{01}),(\mathrm{c}^{00})$$

where p^* denotes either p^+ or the empty sequence. This cannot happen because all the candidates for the pair of upper sequents of this ($\rightarrow$left) contain non-tautologies. ∎

References

[1] R.Kashima and N.Kamide, A Study on Substructural Logics with Restricted Exchange Rules, 京都大学数理解析研究所講究録 1010 (1997).

[2] R.Kashima and N.Kamide, A Family of Substructural Implicational Logics [1], Research Report IS-RR-97-0037F, Japan Advanced Institute of Science and Technology (1997).

[1]dvi-file compressed by 'gzip' is available:
`ftp://logic.jaist.ac.jp/pub/papers/kashima/kashima1.dvi.gz`